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ON UNIFORMLY RESOLVABLE $\{K_{1,2}, K_{1,3}\}$-DESIGNS

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ABSTRACT. Given a collection of graphs $\mathscr{H}$, a uniformly resolvable $\mathscr{H}$*-design* of order v is a decomposition of the edges of K_v into isomorphic copies of graphs from $\mathscr{H}$ (also called *blocks*) in such a way that all blocks in a given parallel class are isomorphic to the same graph from $\mathscr{H}$. We consider the case $\mathscr{H} = \{K_{1,2}, K_{1,3}\}$ and prove that the necessary conditions for the existence of such designs are also sufficient.

1. Introduction

Given a collection of graphs $\mathscr{H}$, an $\mathscr{H}$*-design* of order v (also called an $\mathscr{H}$-decomposition of K_v) is a decomposition of the edges of K_v into isomorphic copies of graphs from $\mathscr{H}$, the copies of $H \in \mathscr{H}$ in the decomposition are called *blocks*. An $\mathscr{H}$-design is called *resolvable* if it is possible to partition the blocks into *classes* $\mathscr{P}_i$ such that every vertex of K_v appears exactly once in some block of each $\mathscr{P}_i$.

A resolvable $\mathscr{H}$-decomposition of K_v is sometimes also referred to as an $\mathscr{H}$*-factorization of* K_v, a class can be called an $\mathscr{H}$*-factor of* K_v. When $\mathscr{H} = \{K_2\}$ we speak of 1*-factorization of* K_v and it is well known to exist if and only if v is even. A single class of a 1-factorization is also known as a 1*-factor* or a *perfect matching*. A resolvable $\mathscr{H}$-design is called *uniform* if every block of the class is isomorphic to the same graph from $\mathscr{H}$. Of particular note is the result of Rees (1987) who gives necessary and sufficient conditions for the existence of uniformly resolvable $\{K_2, K_3\}$-designs of order v. Uniformly resolvable decompositions of K_v have also been studied by Danziger *et al.* (2009), Dinitz *et al.* (2009), Schuster (2009a,b), Schuster and Ge (2010), Gionfriddo and Milici (2013), Milici (2013), Schuster (2013), Gionfriddo and Milici (2014), Milici and Tuza (2014), Küçükçifçi *et al.* (2015a,b), Lo Faro *et al.* (2015), and Gionfriddo *et al.* (2016). In what follows, we will denote by $[a; a_1, \ldots, a_k]$, $k \geq 2$, the k-star $K_{1,k}$ having vertex set $\{a, a_1, \ldots, a_k\}$ and edge set $\{\{a, a_1\}, \{a, a_2\}, \ldots, \{a, a_k\}\}$. A resolvable $K_{1,2}$-design (i.e., P_3-design) of order v exists if and only if $v \equiv 9 \pmod{12}$ (see Horton 1985), while there exists no $K_{1,3}$-design (see Küçükçifçi *et al.* 2015a). Denoted by $(K_{1,2}, K_{1,3})$-URD$(v; r, s)$ a uniformly resolvable decomposition of K_v into r classes containing only copies of 2-stars $K_{1,2}$ and s classes containing only copies of 3-stars $K_{1,3}$, here we study the existence problem when r and s are positive integers and so, from now on, we assume $r, s > 0$ and, necessarily, $v \equiv 0 \pmod{12}$. Let URD$(v; K_{1,2}, K_{1,3})$ be the set of all pairs (r,s) such that there exists a $(K_{1,2},K_{1,3})$-URD$(v;r,s)$, and given $v \equiv 0 \pmod{12}$, let

$$J(v) = \left\{ \left(6+9x, 2+\frac{2(v-12)}{3} - 8x\right) : \; x = 0,1,\ldots,\frac{v-12}{12} \right\}$$

in this paper we characterize the existence of uniformly resolvable $\{K_{1,2},K_{1,3}\}$-designs, by proving the following result:

Main Theorem. *A $(K_{1,2},K_{1,3})$-URD$(v;r,s)$ exists if and only if $v \equiv 0 \pmod{12}$ and URD$(v;K_{1,2},K_{1,3})$=$J(v)$.*

2. Preliminaries and necessary conditions

In this section we will introduce some useful definitions, results and give necessary conditions for the existence of a uniformly resolvable decomposition of K_v into r classes of $K_{1,2}$ and s classes of $K_{1,3}$. For missing terms or results that are not explicitly explained in the paper, the reader is referred to the handbook of Colbourn and Dinitz (2007) and its online updates. For some results below, we also cite this handbook instead of the original papers. A (resolvable) $\mathscr{H}$-decomposition of the complete multipartite graph with u parts each of size g is known as a resolvable group divisible design $\mathscr{H}$-RGDD of type g^u, the parts of size g are called the groups of the design. When $\mathscr{H} = \{K_n\}$ we will call it an n-(R)GDD. A $(K_{1,2},K_{1,3})$-URGDD (r,s) of type g^u is a uniformly resolvable decomposition of the complete multipartite graph with u parts each of size g into r classes containing only copies of $K_{1,2}$ and s classes containing only copies of $K_{1,3}$.

If the blocks of an $\mathscr{H}$-GDD of type g^u can be partitioned into partial parallel classes, each of which contain all vertices except those of one group, we refer to the decomposition as a *frame*. When $\mathscr{H} = \{K_n\}$ we will call it an *n-frame* and it is easy to deduce that the number of partial factors missing a specified group G is $\frac{|G|}{n-1}$.

An incomplete resolvable $(K_{1,2},K_{1,3})$-decomposition of K_{v+h} with a hole of size h is an $(K_{1,2},K_{1,3})$-decomposition of $K_{v+h} \setminus K_h$ in which there are two types of classes, *partial* classes which cover every vertex except those in the hole (the vertices of K_h are referred to as the *hole*) and *full* classes which cover every vertex of K_{v+h}. Specifically, a $(K_{1,2},K_{1,3})$-IURD$(v+h,h;[r_1,s_1],[\bar{r}_1,\bar{s}_1])$ is a uniformly resolvable $(K_{1,2},K_{1,3})$-decomposition of $K_{v+h} \setminus K_h$ with r_1 and s_1 partial classes of $K_{1,2}$ and $K_{1,3}$, respectively, and $\bar{r}_1$ and $\bar{s}_1$ full classes of $K_{1,2}$ and $K_{1,3}$, respectively.

We now recall some results that can be used to produce the main result.

Theorem 2.1. (Milici and Tuza 2014) *Let $v \equiv 0 \pmod{3}$, $v \geq 9$. The union of any two edge-disjoint parallel classes of 3-cycles of K_v can be decomposed into three parallel classes of $K_{1,2}$.*

We also need the following definitions. Let (s_1,t_1) and (s_2,t_2) be two pairs of non-negative integers. Define $(s_1,t_1)+(s_2,t_2) = (s_1+s_2,t_1+t_2)$. If X and Y are two sets of pairs of non-negative integers, then $X+Y$ denotes the set $\{(s_1,t_1)+(s_2,t_2) : (s_1,t_1) \in X, (s_2,t_2) \in Y\}$. If X is a set of pairs of non-negative integers and h is a positive integer, then $h*X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

Lemma 2.2. *Let* $v \equiv 0 \pmod{12}$. *If there exists a* $(K_{1,2}, K_{1,3})$*-URD*$(v; r, s)$*, then* $(r, s) \in J(v)$.

Proof. The condition $v \equiv 0 \pmod{12}$ is trivial. Let D be a $(K_{1,2}, K_{1,3})$-URD$(v; r, s)$ of K_v. Counting the edges of K_v that appear in D we obtain

$$\frac{2rv}{3} + \frac{3sv}{4} = \frac{v(v-1)}{2},$$

and hence that

$$8r + 9s = 6(v-1). \tag{1}$$

Since $v \equiv 0 \pmod{12}$, Equation (1) implies $8r \equiv 3 \pmod 9$, $9s \equiv 2 \pmod 8$, and so $r \equiv 6 \pmod 9$, $s \equiv 2 \pmod 8$. Letting now $r = 6 + 9x$, the equation (1) yields $9s = 6(v-1) - 48 - 72x$. Then we obtain $s = 2 + \frac{2(v-12)}{3} - 8x$, where $8x \leq \frac{2(v-12)}{3}$ since s is a positive integer. This completes the proof. □

3. Small cases

Lemma 3.1. *URD*$(12; K_{1,2}, K_{1,3}) = \{(6,2)\}$.

Proof. Let $V(K_{12}) = \{0, 1, \ldots, 11\}$ be the vertex set and the classes listed below:
$\{(0;1,2,3), (4;5,6,7), (8;9,10,11)\}$, $\{(1;8,2,3), (5;0,6,7), (9;4,10,11)\}$,
$\{(0;4,9), (1;5,6), (2;8,11), (3;7,10)\}$, $\{(4;1,8), (5;9,10), (6;0,3), (7;2,11)\}$,
$\{(8;0,5), (9;1,2), (10;4,7), (11;3,6)\}$,$\{(2;4,5), (6;7,10), (3;9,8), (11;0,1)\}$,
$\{(6;9,8), (10;11,2), (7;0,1), (3;4,5)\}$, $\{(10;0,1), (2;3,6), (11;4,5), (7;8,9)\}$.

□

Lemma 3.2. *There exists a* $(K_{1,2}, K_{1,3})$*-URGDD*(r, s) *of type* 12^2 *with* $(r, s) \in \{(9,0), (0,8)\}$.

Proof. The case $(9,0)$ corresponds to a $K_{1,2}$-factorization of $K_{12,12}$ which is known to exist (Ushio 1988). The case $(0,8)$ corresponds to a $K_{1,3}$-factorization of $K_{12,12}$ which is known to exist (Chen and Cao 2016). □

Lemma 3.3. *There exists a* $(C_3, K_{1,3})$*-URGDD*(r, s) *of type* 4^3 *with* $(r, s) \in \{(1,4), (4,0)\}$.

Proof. The case $(4,0)$ corresponds to a $3-$RGDD of type 4^3 which is known to exist (Rees and Stinson 1987). For the case $(1,4)$ take the groups to be $\{a_0, a_1, \ldots, a_3\}$, $\{b_0, b_1, \ldots, b_3\}$ and $\{c_0, c_1, \ldots, c_3\}$ and the classes listed below:
$\{(a_i; b_{i+1}, b_{i+2}, b_{i+3}), (b_i; c_{i+1}, c_{i+2}, c_{i+3}), (c_i; a_{i+1}, a_{i+2}, a_{i+3}), i \in Z_4\}$,
$\{(a_i, b_i, c_i), i \in Z_4\}$

□

Lemma 3.4. *There exists a* $(C_3, K_{1,3})$*-URGDD*(r, s) *of type* 12^3 *with* $(r, s) \in \{(12,0), (9,4), (6,8), (3,12), (0,16)\}$.

Proof. The case $(0,16)$ corresponds to a $K_{1,3}$-factorization of $K_{12,12,12}$ which is known to exist (Küçükçifçi *et al.* 2015b). For all the other cases take a 3-RGDD $\mathcal{D}$ of type 3^3 which is known to exist (Rees and Stinson 1987). Expand each vertex 4 times and for each block b of a given factor of $\mathcal{D}$ place on $b \times \{1,2,3,4\}$ a copy of a $(C_3, K_{1,3})$-URGDD(r, s) of type 4^3 with $(r, s) \in \{(1,4), (4,0)\}$, which exists by Lemma **3.3**. Since

$\mathscr{D}$ contains three factors, the result is a $(C_3, K_{1,3})$-URGDD(r,s) of type 12^3, for every $(r,s) \in 3 * \{(4,0),(1,4)\} = \{(12,0)),(9,4),(6,8),(3,12)\}$. □

Lemma 3.5. *There exists a* $(K_{1,2}, K_{1,3})$*-URGDD*(r,s) *of type* 12^3 *with* $(r,s) \in \{(18,0),(9,8),(0,16)\}$.

Proof. Take a $(C_3, K_{1,3})$-URGDD(r,s) of type 12^3 with $(r,s) \in \{(12,0),(6,8),(0,16)\}$. Since, by Theorem **2.1**, each two parallel classes of C_3 can be decomposed into three parallel classes of $K_{1,2}$ we obtain the result. □

Lemma 3.6. *URD*$(36; K_{1,2}, K_{1,3}) = \{(24,2),(15,10),(6,18)\}$.

Proof. Start from a $(K_{1,2}, K_{1,3})$-URGDD(r,s) of type 12^3 with $(r,s) \in \{(18,0),(9,8),(0,16)\}$, which exists by Lemma **3.5**, and fill the three groups of size 12 with a copy of a $(K_{1,2}, K_{1,3})$-URD$(12;6,2)$, which exists by Lemma **3.1**.

□

Lemma 3.7. *There exists a* $(K_{1,2}, K_{1,3})$*-IURD*$(36,12;[6,2],[r,s])$ *for every* $(r,s) \in \{(18,0),(9,8),(0,16)\}$.

Proof. Start from a $(K_{1,2}, K_{1,3})$-URGDD(r,s) of type 12^3 with $(r,s) \in \{(18,0),(9,8),(0,16)\}$, which exists by Lemma **3.5**, and fill in two groups of size 12 with a copy of a $(K_{1,2}, K_{1,3})$-URD$(12;6,2)$, which exists by Lemma **3.1**. □

Lemma 3.8. *URD*$(60; K_{1,2}, K_{1,3}) = J(60) = \{(42,2),(33,10),(24,18),(15,26),(6,34)\}$.

Proof. For the case $(6,34)$ start from a $(K_{1,2}, K_{1,3})$-URGDD$(0,32)$, which is known to exist (Küçükçifçi *et al.* 2015b), and fill the five groups of size 12 with a copy of a $(K_{1,2}, K_{1,3})$-URD$(12;6,2)$, which exists by Lemma **3.1**. For all the other cases take a 3-RGDD $\mathscr{D}$ of type 3^5 which is known to exist (Rees and Stinson 1987). Expand each vertex 4 times and for each block b of a given factor of $\mathscr{D}$ place on $b \times \{1,2,3,4\}$ a copy of a $(C_3, K_{1,3})$-URGDD(r_1,s_1) of type 4^3 with $(r_1,s_1) \in \{(1,4),(4,0)\}$, which exists by Lemma **3.3**. This gives,since $\mathscr{D}$ contains six factors, a $(C_3, K_{1,3})$-URGDD(r_2,s_2) of type 12^5, for every $(r_2,s_2) \in 6 * \{(4,0),(1,4)\} = \{(24,0)),(21,4),(18,8),(15,12),(12,16),(9,20),(6,24)\}$. Applying Theorem **2.1** we obtain a $(K_{1,2}, K_{1,3})$-URGDD(r_3,s_3) of type 12^5, for every $(r_3,s_3) \in \{(36,0),(27,8),(18,16),(9,24)\}$. Fill in each group of size 12 with a copy of a $(K_{1,2}, K_{1,3})$-URD$(12;6,2)$, which exists by Lemma **3.1**. This gives a $(K_{1,2}, K_{1,3})$-URD$(60;r,s)$ for every $(r,s) \in \{(6,2)\} + \{(36,0),(27,8),(18,16),(9,24)\} = J(60)$.

□

4. Main results

Lemma 4.1. *For every* $v \equiv 0 \pmod{24}$, $J(v) \subseteq$ *URD*$(v; K_{1,2}, K_{1,3})$.

Proof. For $v \geq 24$ start with a 2-RGDD G of type $1^{\frac{v}{12}}$ (Colbourn and Dinitz 2007). Give weight 12 to each vertex of this 2-RGDD and place on each edge of a given resolution class the same $(K_{1,2}, K_{1,3})$-URGDD(r,s) of type 12^2, with $(r,s) \in \{(9,0),(0,8)\}$, which exists by Lemma **3.2**. Fill the groups of sizes 12 with the same $(K_{1,2}, K_{1,3})$-URD$(12;6,2)$, which exists by Lemma **3.1**. Since G contains $\frac{v-12}{12}$ resolution classes the result is a $(K_{1,2}, K_{1,3})$-URD$(v;r,s)$ of K_v for each $(r,s) \in \{(6,2)\} + \frac{v-12}{12} * \{(9,0),(0,8)\}$. This implies

$$URD(v;K_{1,2},K_{1,3}) \supseteq \{(6,2)\} + \frac{(v-12)}{12} * \{(9,0),(0,8)\}.$$

Since $\frac{v-12}{12} * \{(9,0),(0,8)\}$=$\left\{\left(9x, \frac{2(v-12)}{3} - 8x\right), x = 0,1,\ldots,\frac{v-12}{12}\right\}$, it easy to see that $\{(6,2)\} + \frac{(v-12)}{12} * \{(9,0),(0,8)\}$=$J(v)$. This completes the proof. □

Lemma 4.2. *For every* $v \equiv 12 \pmod{24}$, $v \neq 60$, $J(v) \subseteq URD(v;K_{1,2},K_{1,3})$.

Proof. For $v = 12, 36$ the conclusion follows from Lemmas **3.1** and **3.6**. For $v > 60$ start with a 2−frame F of type $2^{\frac{v-12}{24}}$ (Schuster and Ge 2010) with groups G_i, $i = 1,2,\ldots,\frac{v-12}{24}$. For $j = 1,2$ let $p_{i,j}$ be the partial parallel classes which miss the group G_i. Expand each vertex 12 times and add a set H of 12 ideal vertices $a_1, a_2, \ldots, a_{12}$. For each $i = 1,2,\ldots,\frac{v-12}{24}$, place on $G_i \times \{1,2,\ldots,12\} \cup H$ a copy $\mathscr{D}_i$ of a $(K_{1,2},K_{1,3})$-IURD$(36,12;[6,2],[r,s])$, with $(r,s) \in \{(18,0),(9,8),(0,16)\}$ (which exists by Lemma **3.7**). For each $b \in p_{i,j}$, place on $b \times \{1,2,\ldots,12\}$ a copy $\mathscr{D}^b_{i,j}$ of a $(K_{1,2},K_{1,3})$-URGDD(r_1,s_1) of type 12^2 with $(r_1,s_1) \in \{(9,0),(0,8)\}$, which exists by Lemma **3.2**. Now combine all together the factors of $\mathscr{D}^b_{i,j}$, $b \in p_{i,j}$, along with the factors of $\mathscr{D}_i$ so to obtain r_2 $K_{1,2}$-factors and s_2 $K_{1,3}$-factors with $(r_2,s_2) \in \{(18,0),(9,8),(0,16)\}$, on $H \cup (\cup_{i=1}^{u} G_i \times \{1,2,\ldots,12\})$. Fill the hole H with a copy $\mathscr{D}$ of $(K_{1,2},K_{1,3})$-URD$(12;6,2)$ and combine the factors of $\mathscr{D}$ with the partial factors of $\mathscr{D}_i$ so to obtain 6 $K_{1,2}$-factors and 2 $K_{1,3}$-factors on $H \cup (\cup_{i=1}^{u} G_i \times \{1,2,\ldots,12\})$. The result is a $(K_{1,2},K_{1,3})$-URD $(v;r,s)$ for each $(r,s) \in \{(6,2)\} + \frac{v-12}{24} * \{(18,0),(9,8),(0,16)\}$. This implies

$$URD(v;K_{1,2},K_{1,3}) \supseteq \{(6,2)\} + \frac{v-12}{24} * \{(18,0),(9,8),(0,16)\}.$$

Since $\frac{v-12}{24} * \{(18,0),(9,8),(0,16)\}$=$\left\{\left(9x, \frac{2(v-12)}{3} - 8x\right) : x = 0,1,\ldots,\frac{v-12}{12}\right\}$, it easy to see that $\{(6,2)\}$+$\frac{v-12}{24} * \{(18,0),(9,8),(0,16)\}$=$J(v)$. This completes the proof.

□

5. Conclusion

We are now in position to prove the main result of the paper.

Theorem 5.1. *For every* $v \equiv 0 \pmod{12}$, $URD(v;K_{1,2},K_{1,3}) = J(v)$.

Proof. Necessity follows from Lemma **2.2**. Sufficiency follows from Lemmas **3.8**, **4.1** and **4.2**. □

Remark. Note that the existence of uniformly resolvable $\{K_{1,k}, K_{1,k+1}\}$-designs with $k > 2$ is currently under investigation.

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