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ON A CHARACTERISTIC PROPERTY OF THE SPHERE

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ABSTRACT. We are studying a characteristic property of the disc in the Euclidean plane $\mathbb{R}^2$ with the help of an intersection property of the semicircles with end-points in the boundary of the disc and a characteristic property of the ball in the Euclidean space $\mathbb{R}^3$.

1. Introduction

The reader unfamiliar with the theory of convex sets is referred to the books of Bonnesen and Fenchel (1934), Blaschke (1956), Valentine (1968), Gruber and Wills (1993), Webster (1994), and Boltyanski *et al.* (1997). Let M be a set in the n-dimensional Euclidean space $\mathbb{R}^n$. In the following we shall denote by $\operatorname{card} M$, $\operatorname{int} M$, $\operatorname{cl} M$, ϑM, $\operatorname{conv} M$, the cardinal number, the interior, the closure, the boundary and the convex hull of the set M, respectively. With $d(x,y)$ we denote the Euclidean distance of the points x and y and with $L(x,y)$ the line determined by the points x and y. The diameter $\operatorname{diam} M$ of a set M is $\operatorname{diam} M = \sup\{d(x,y) : x,y \in M\}$. For two distinct points a and b in $\mathbb{R}^2$, we denote with $C(a,b)$ the circle going through the points a and b with diameter $d(a,b)$ and with $C_1(a,b)$ and $C_2(a,b)$ the two semicircles determined on the circle $C(a,b)$ by the line $L(a,b)$.

A convex body in the space $\mathbb{R}^n$ is a compact convex set with a nonempty interiorn (see Bonnesen and Fenchel 1934; Valentine 1968). A convex body in $\mathbb{R}^n$ is smooth if in every boundary point there is exactly one supporting hyperplane. A convex body in $\mathbb{R}^n$ is rotund if every supporting hyperplane has at most one contact point with the convex body. A survey of rotundity of convex bodies is the paper of Cudia (1963).

We define now two intersection properties for compact convex sets with a nonempty interior in the space $\mathbb{R}^2$:

Definition 1. *A compact convex set $K \subset \mathbb{R}^2$ with a nonempty interior has the* intersection property (ip) *if*
(ip) *for any two distinct boundary points a and b of the set K we have $C_i(a,b) \cap \operatorname{int} K = \emptyset$ for at least one $i \in \{1,2\}$.*

Definition 2. *A compact convex set $K \subset \mathbb{R}^2$ with a nonempty interior has the* small intersection property (sip) *if* *(sip)* *there is a constant $k < \operatorname{diam} K$ such that, for any two distinct boundary points a and b of the set K with $d(a,b) < k$, we have $C_i(a,b) \cap \operatorname{int} K = \emptyset$ for at least one $i \in \{1,2\}$.*

Definition 3. *Let $K \subset \mathbb{R}^2$ be a compact convex set with a nonempty interior and satisfying the small intersection property (sip). We define now the following characteristic constant:*

$$\rho(K) := \frac{\sup \mathscr{T}_k}{\operatorname{diam} K},$$

where $\mathscr{T}_K$ is the set of all numbers $k \leq \operatorname{diam} K$ such that for any two distinct boundary points x and y of the set K with $d(x,y) < k$ we have $C_i(x,y) \cap \operatorname{int} K = \emptyset$ for at least one $i \in \{1,2\}$.

2. Main Results

We have the following characterization of a disc in the Euclidean plane $\mathbb{R}^2$:

Theorem 1. *Let $K \subset \mathbb{R}^2$ be a compact convex set with a nonempty interior. The set K is a disc if and only if K has the intersection property (ip).*

Proof. The 'if'-part of the theorem follows immediately. If K is a disc with center c and radius r and two arbitrary points $a \in \vartheta K, b \in \vartheta K$ we have $C_i(a,b) \cap \operatorname{int} K = \emptyset$ verified for exactly one $i \in \{1,2\}$ if the points a,b and c are not collinear, and for both i's if the points a,b and c are collinear.

To prove the 'only if'-part of the theorem, let K be a compact convex set in the Euclidean plane which has inner points and verifies (ip).
First we show that the set K is *strictly convex*. Let us suppose the contrary, i.e. the boundary ϑK contains a line segment ab. Let c be the midpoint of the segment ab and L_c the line perpendicular in c on the line segment ab and d the point $d \in L_c \cap \vartheta K$ different from c. Let $C_1(c,d)$ and $C_2(c,d)$ be the two semicircles determined by the line L_c on the circle $C(c,d)$. Then we have evidently $C_1(c,d) \cap \operatorname{int} K \neq \emptyset$ and $C_2(c,d) \cap \operatorname{int} K \neq \emptyset$ in contradiction with the intersection property (ip) of the set K. Thereby we have proved the strictly convexity of the set K.

From now on a and b will always denote two points of the compact convex set K such that

$$d(a,b) = sup\{d(x,y) : x,y \in K\}.$$

By definition, it immediately follows that a and b are boundary points of the set K, i.e. $a \in \vartheta K$ and $b \in \vartheta K$. We also set $r = d(a,b)/2$.

By the strictly convexity of the set K it follows that all points of the segment ab different from the points a and b are inner points of the convex set K. Denote with c the midpoint of the segment ab and with L_c the line going through the point c, perpendicular on the segment ab. Denote with d and e the boundary points of the set K lying on the line L_c.

Let us now suppose that we have $d(d,e) < d(a,b) = 2r$. Consider then the circle $C(d,e)$ going through the points d and e with diameter $d(d,e)$ and the two semicircles $C_1(d,e)$ and $C_2(d,e)$ of the circle $C(d,e)$ determined by the line $L(d,e)$. Let D_0 be the disc with the boundary $C(d,e)$. Because $d(d,e) < d(a,b) = 2r$ the points a and b must be in the exterior of the disc D_0. We have then of course $C_1(d,e) \cap \operatorname{int} K \neq \emptyset$ and also

$C_2(d,e) \cap \text{int} K \neq \emptyset$ in contradiction to the intersection property (ip) of the set K. It follows that $d(d,e) = d(a,b) = 2r$.

Next, we want to show that $d(e,c) = d(c,d) = r$. Let us suppose that $d(e,c) \neq d(c,d)$. Denote with f the midpoint of the segment ed and consider then the line L_f perpendicular in the point f on the segment ed. Consider on the line L_f the boundary points $g = L_f \cap \vartheta K$ and $h = L_f \cap \vartheta K$. As above we can show that we have $d(g,h) = d(a,b) = 2r$. As the segments ab and gh both of length $2r$ are on parallel lines, one of the diagonals of the parallelogram $\text{conv}\{a,b,h,g\}$ is longer than the side ab in contradiction to the definition of the points a and b. It follows that we must have $d(e,c) = d(c,d) = r$. Thereby we have proved the existence of a square $\vartheta\text{conv}\{a,b,d,e\}$ (with the center c) inscribed into the boundary ϑK of the strictly convex compact set K.

Consider then the disc $D(c,r)$ which has the circle $C(a,b)$ as its boundary. We have then: $K \subset D(c,r)$. To prove this let us suppose that there is a point $p \in K$ such that $p \notin D(c,r)$. Denote with p_0 the boundary point of the set K lying on the halfline with origin c going through the point p and with t the other boundary point of K on the line $L(c,p_0)$. From the convexity of the set K it immediately follows $\text{conv}\{a,b,d,e\} \subset K$. Denote with q the intersection point of the circle $C(a,b)$ and the line segment $\text{conv}\{c,p_0\}$, i.e. $q = C(a,b) \cap \text{conv}\{c,p_0\}$. From the convexity of the set K it follows that $q \in \text{int} K$. As the points a,b,d and e are boundary points of the set K it follows that the point q cannot coincide with anyone of these points. The points a,d,b and e are dividing the circle $C(a,b)$ in 4 arcs $\alpha(b,e), \alpha(e,a), \alpha(a,d), \alpha(d,b)$ each of length $r\pi/2$. Without loss of generality we can suppose that the point q is between the points e and b on the arc $\alpha(b,e)$. We have of course the inequality: $d(p_0,t) \leq d(a,b) = 2r$. Let c_0 be the center of the segment tp_0. Because $d(c,p_0) > r$ it follows that the point c_0 is on the line segment cp_0. Then we have $\text{conv}\{a,c\} \cap C(t,p_0) \neq \emptyset$ and $\text{conv}\{d,c\} \cap C(t,p_0) \neq \emptyset$. Denote with $c_a = \text{conv}\{a,c\} \cap C(t,p_0)$ and with $c_d = \text{conv}\{d,c\} \cap C(t,p_0)$. We have then $c_a \in \text{int} K$ and $c_d \in \text{int} K$. This means that we have also: $C_1(p_0,t) \cap \text{int} K \neq \emptyset$ and $C_2(p_0,t) \cap \text{int} K \neq \emptyset$ where $C_1(p_0,t)$ and $C_2(p_0,t)$ are the two semicircles determined on the circle $C(p_0,t)$ by the line $L(p_0,t)$. Because t and p_0 are boundary points of K we have got a contradiction to the property (ip) of the compact convex set K. Thereby we have proved: $K \subset D(c,r)$.

Consider now an arbitrary line L going through the center c of this square and denote with $x = L \cap \vartheta K$ and $y = L \cap \vartheta K$ the two intersection points of the line L with the boundary of the set K. Let us suppose that we have $d(x,y) < d(a,b) = 2r$. Consider then the circle $C(x,y)$ containing the points x and y and with diameter $d(x,y)$. Because $x \in K, y \in K$ and $K \subset D(c,r)$ it follows that for the circle $C(x,y)$ we have: $C(x,y) \subset D(c,r)$.

We distinguish now the following two cases:

- (i) Both points x and y are in the interior of the disc $D(c,r)$. In this case the circle $C(x,y)$ is also completely in the interior of the disc $D(c,r)$. The circle $C(x,y)$ must then intersect each of the line segments ac, bc, dc, ec in inner points of the set K. It follows that we have: $C_1(x,y) \cap \text{int} K \neq \emptyset$ and $C_2(x,y) \cap \text{int} K \neq \emptyset$, where $C_1(x,y)$ and $C_2(x,y)$ are the semicircles determined on the circle $C(x,y)$ by the line $L(x,y)$. But this is in contradiction to (ip), our hypothesis relative to the set K.
- (ii) One point for instance x is in the interior of the disc $D(c,r)$ and y is a boundary point of $D(c,r)$. Without loss of generality we can suppose that the point y is

on the circle $C(a,b)$ between the points b and e. The circle $C(x,y)$ has then to intersect in inner points the line segments ca and cd. This means that we have: $C_1(x,y) \cap \operatorname{int} K \neq \emptyset$ and $C_2(x,y) \cap \operatorname{int} K \neq \emptyset$, where $C_1(x,y)$ and $C_2(x,y)$ are the semicircles determined on the circle $C(x,y)$ by the line $L(x,y)$. But this contradicts again our hypothesis (ip) for the set K.

Thereby our supposition $d(x,y) < 2r$ is false and we have $d(x,y) = 2r$. □

Let us now consider the analog problem for the 3-dimensional Euclidean space $\mathbb{R}^3$. For 3 nonlinear points x,y,z in $\mathbb{R}^3$ we shall denote with $C(x,y,z)$ the circle circumscribed to the triangle Δxyz in the plane $P(x,y,z)$ determined by the 3 points x,y,z, and with $c(x,y,z)$ and $r(x,y,z)$ the center and the radius of the circle $C(x,y,z)$. We shall denote with $S(x,y,z)$ the sphere with center $c(x,y,z)$ and radius $r(x,y,z)$ and with $B(x,y,z)$ the ball having $S(x,y,z)$ as its boundary. With $S_1(x,y,z)$ and $S_2(x,y,z)$ we denote the two closed hemispheres of the sphere $S(x,y,z)$ determined by the plane $P(x,y,z)$.

We have then:

Lemma 2. *Let K be a convex body in the space $\mathbb{R}^3$ such that:*
$(ip3)$ for any three distinct nonlinear boundary points x,y,z of the set K we have

$$S_j(x,y,z) \cap \operatorname{int} K = \emptyset \qquad \text{for at least one } j \in \{1,2\}.$$

Then K is a strictly convex body.

Proof. Let us suppose the contrary, i.e. the boundary ϑK contains a line segment a_1a_2. Let b be the midpoint of the segment a_1a_2 and P_b the plane perpendicular in b on the line $L(a_1,a_2)$. Denote with H_1 and H_2 the closed halfspaces determined by the plane P_b containing the point a_1 and respectively a_2. As K is a convex body there is a point $c_i \in P_b \cap \operatorname{int} K$. Let c be the point $c \in L(b,c_i) \cap \vartheta K$ different from b. Consider now in the plane P_b the line L_i going through the point c_i perpendicular on the line $L(b,c_i)$. Denote with d one of the two points from $L_i \cap \vartheta K$. Let $C(b,c,d)$ be the circle circumscribed to the triangle Δbcd in the plane P_b, and let o and r be the center and the radius of this circle. Let $B(b,c,d)$ be the ball with center o and radius r and $S(b,c,d)$ its boundary sphere. Denote with $S_1(b,c,d) = H_1 \cap S(b,c,d)$ and $S_2(b,c,d) = H_2 \cap S(b,c,d)$ the closed hemispheres of the sphere $S(b,c,d)$ determined by the plane P_b contained in the halfspace H_1 and respectively H_2. Consider the tetrahedrons:

$T = \operatorname{conv}\{a_1,a_2,c,d\}$, $T_1 = \operatorname{conv}\{a_1,b,c,d\}$ and $T_2 = \operatorname{conv}\{a_2,b,c,d\}$.

We have then $T = T_1 \cup T_2$, $S_1(b,c,d) \cap \operatorname{int} T_1 \neq \emptyset$ and $S_2(b,c,d) \cap \operatorname{int} T_2 \neq \emptyset$. As $\operatorname{int} T_1 \subset \operatorname{int} K$ and $\operatorname{int} T_2 \subset \operatorname{int} K$ we have $S_1(b,c,d) \cap \operatorname{int} K \neq \emptyset$ and $S_2(b,c,d) \cap \operatorname{int} K \neq \emptyset$ in contradiction with our hypothesis $(ip3)$. Thereby K is a strictly convex body. □

Lemma 3. *Let K be a strictly convex body in the Euclidean space $\mathbb{R}^3$ and P a plane in $\mathbb{R}^3$ such that $P \cap K \neq \emptyset$. Each interior point p of the set $P \cap K$ in the 2-dimensional space P is also an interior point of the set K in $\mathbb{R}^3$.*

Proof. If the set $P \cap K$ has an interior point p in the 2-dimensional space P let $D = D(p,r)$ be a disc with center p and radius r such that $D \subset P \cap K$. Consider now the two open halfspaces H_1 and H_2 determined by the plane P in the space $\mathbb{R}^3$. From the strict-convexity of the set K and $D \subset K$ it follows that we cannot have $D \subset \vartheta K$ i.e. P cannot be a supporting plane for K in the point p and therefore there are two points p_1 and p_2 such that $p_1 \in H_1 \cap K$ and $p_2 \in H_2 \cap K$. Consider now the convex hulls $C_i = \operatorname{conv}\{p_i, D\}, i = 1,2$. As the convex sets C_1 and C_2 have nonempty interiors in the space $\mathbb{R}^3$, the point p is an *interior point* of the set $C = C_1 \cup C_2$ and it follows then from $C \subset K$ that p is also an *interior point* of the set K. □

We have then the following:

Theorem 4. *A convex body K in $\mathbb{R}^3$ is a ball if and only if Property $(ip3)$ holds.*

Proof. The "if"-part of the theorem follows immediately. If we have a ball $B(c,r)$ with center c and radius r and three arbitrary points $x \in \vartheta B, y \in \vartheta B, z \in \vartheta B$ we have $(ip3)$ verified for exactly one $i \in \{1,2\}$, if the points x,y,z and the center c of the ball are not coplanar, and for both i if the points x,y,z and c are coplanar.

To prove the "only if"-part of the theorem let K be a convex body in the Euclidean space $\mathbb{R}^3$ verifying $(ip3)$. By Lemma 2, K is a *strictly convex* body. Let a ad b be two points in the strictly convex body K such that $d(a,b) = \operatorname{diam} K$ and $\mathscr{P}$ be an arbitrary plane containing the line $L(a,b)$. Let o be the midpoint of the line segment ab. Consider in the plane $\mathscr{P}$ the circle $C_{\mathscr{P}}(o,r)$ with center o and radius $r = d(a,b)/2$ and the disc $D_{\mathscr{P}}(o,r) = \operatorname{conv} C_{\mathscr{P}}(o,r)$. We shall now prove that the set $K_P = K \cap \mathscr{P}$ must coincide with the disc $D_{\mathscr{P}}(o,r)$.

Let us suppose the contrary, i.e. $K_P \neq D_{\mathscr{P}}(o,r)$. Because K is a strictly convex set, the segment ab cannot be contained in the boundary ϑK of the set K. Therefore, the segment ab must contain a point $c \in \operatorname{int} K$. In the two-dimensional Euclidean space $\mathscr{P}$ the point c is then also an interior point of the set K_P. But then K_P is a compact convex set with a nonempty interior in the two-dimensional Euclidean space $\mathscr{P}$. Because K_P is not a disc, there exists then by Theorem 1 two boundary points x and y of the set K_P such that we have the following in the plane $\mathscr{P}$ for the circle $C(x,y)$ with the center in the midpoint m of the segment $\operatorname{conv}\{x,y\}$ and radius $r_m = d(x,y)/2$ and the two semicircles $C_1(x,y)$ and $C_2(x,y)$ of the circle $C(x,y)$ determined in the plane $\mathscr{P}$ by the line $L(x,y)$:

(4) $\qquad C_i(x,y) \cap \operatorname{int} K_P \neq \emptyset$ for $i = 1$ and $i = 2$.

The line $L(x,y)$ determines in the plane $\mathscr{P}$ two closed half-planes P_1' and P_2' such that $C_i(x,y) \subset P_i'$ for $i = 1,2$ and we can consider two points t_1 and t_2 for which we have $t_1 \in C_1(x,y) \cap \operatorname{int} K_P \subset P_1'$ and $t_2 \in C_2(x,y) \cap \operatorname{int} K_P \subset P_2'$. The points t_1 and t_2 are by Lemma 3 also interior points of the strictly convex body K in the 3-dimensional space $\mathbb{R}^3$.

Let $B(m,r_m)$ be the ball with the center m and the radius r_m and $S(m,r_m)$ its boundary sphere.

We distinguish the cases:

(i) $B(m,r_m) \setminus \{x,y\} \subset \operatorname{int} K$;
(ii) $B(m,r_m) \setminus \{x,y\} \not\subset \operatorname{int} K$.

In the case (i) we will first show that $\operatorname{card}\{a,b,x,y\} \geq 3$. Suppose, on the contrary, that $\operatorname{card}\{a,b,x,y\} = 2$. In that case we can consider without loss of generality that the point x coincides with the point a and the point y coincides with b. It follows immediately that the disc $D_{\mathscr{P}}(o,r)$ coincides with the disc $B(m,r_m) \cap \mathscr{P}$. From (i) it follows then $D_{\mathscr{P}}(o,r) \setminus \{a,b\} \subset \operatorname{int} K_P$. Denote in the plane $\mathscr{P}$ for a line L_1 going through the point o

and different from the line $L(a,b)$ with a_1 and a_2 the intersection points of the line L_1 with the circle $C_{\mathscr{P}}(o,r)$. Since a_1 and a_2 are in the plane $\mathscr{P}$ interior points of the compact convex set K_P, the line L_1 will intersect the boundary ϑK_P in two points a'_1 and a'_2 such that the points a'_1, a_1, a_2, a'_2 are in this sequence on the line L_1 and the points a'_1 and a'_2 are outside of the disc $D_{\mathscr{P}}(o,r)$. We have then of course $d(a'_1, a'_2) > d(a_1, a_2) = 2r = d(a,b) = \operatorname{diam} K$. As $a'_1 \in K_P \subset K$ and $a'_2 \in K_P \subset K$ we have got a contradiction. Thus there is at least one of the points a or b different from the points x and y. Let it be for the sequel the point a.

Consider then the sphere $S(x,y,a)$. The points x,y and a are boundary points of the set K and all three points are in the plane $\mathscr{P}$. Since $x \in S(x,y,a) \cap S(m,r_m)$, $y \in S(x,y,a) \cap S(m,r_m)$ and the centers of the spheres $S(x,y,a)$ and $S(m,r_m)$ are both in the plane $\mathscr{P}$, the set $S(x,y,a) \cap S(m,r_m)$ is a circle C_0 of diameter $d(x,y)$ containing the points x and y and C_0 is lying in a plane perpendicular on the plane $\mathscr{P}$. As $C_0 \setminus \{x,y\} \subset B(m,r_m) \setminus \{x,y\} \subset \operatorname{int} K$, it follows immediately that $S_i(x,y,a) \cap \operatorname{int} K \neq \emptyset$ for $i = 1,2$. Thus we got a contradiction with $(ip3)$.

In the case (ii) there is a point u such that $u \in B(m,r_m) \setminus \{x,y\}$ and $u \notin \operatorname{int} K$. We have then of course $d(u,m) \leq r_m$ and $u \neq x$ and $u \neq y$. From the strict convexity of the set K and from $x \in \vartheta K$ and $y \in \vartheta K$ it follows that each inner point of the line segment xy is also an inner point of K. Thereby we have $u \notin \operatorname{conv}\{x,y\}$ and $u \notin L(x,y)$.

We have now to distinguish the cases: $u \in \mathscr{P}$ or $u \notin \mathscr{P}$.

In the case $u \in \mathscr{P}$ it follows from $u \notin L(x,y)$ that u must lie in one of the open half-planes of the plane $\mathscr{P}$ determined by the line $L(x,y)$. Without loss of generality we can suppose that u is in the same open half-plane as the point t_1. We have to distinguish then the cases $d(u,m) = r_m$ and $d(u,m) < r_m$.

Let us suppose that $d(u,m) = r_m$. As $u \notin \operatorname{int} K$ we can have $u \in \vartheta K$ or $u \notin K$. In the case $u \in \vartheta K$ the set K has then 3 boundary points x,y and u on the circle $C(x,y)$ and we have also two inner points of K on $C(x,y)$, namely the points t_1 and t_2. Because the circle $C(x,y,u)$ coincides with the circle $C(x,y)$ this means that we have for the sphere $S(x,y,u)$ and the two closed hemispheres $S_1(x,y,u)$ and $S_2(x,y,u)$ of the sphere $S(x,y,u)$ determined by the plane $\mathscr{P}$ the following: $S_i(x,y,u) \cap \operatorname{int} K \neq \emptyset, i = 1,2$. This contradicts the property $(ip3)$ of the set K. In the case $u \notin K$ consider the arc $\alpha = arc\{t_1, u\}$ of the semicircle $C_1(x,y)$. The arc α connects the inner point t_1 with the point u which is not in K. It follows that the arc α must intersect the boundary ϑK. Denote with $w = \vartheta K \cap arc\{t_1, u\}$. The set K has then again 3 boundary points x,y and w on the circle $C(x,y)$ and we have also two inner points t_1 and t_2 of K on $C(x,y)$. Because the circle $C(x,y,w)$ coincides with the circle $C(x,y)$ this means that we have for the sphere $S(x,y,w)$ and the two closed hemispheres $S_1(x,y,w)$ and $S_2(x,y,w)$ of the sphere $S(x,y,w)$ determined by the plane $\mathscr{P}$ the following: $S_i(x,y,w) \cap \operatorname{int} K \neq \emptyset, i = 1,2$. This is in contradiction with the property $(ip3)$ of the set K.

Let us now suppose that $d(u,m) < r_m$. Denote with $L_h(m,u)$ the open half-line with the origin m and going through the point u and with v' the intersection point of $L_h(m,u)$ and the semicircle $C_1(x,y)$. If we suppose that $v' \in K$ we have $u \in \operatorname{int}\{\operatorname{conv}\{x,y,v'\}\} \subset \operatorname{int} K$ in contradiction to $u \notin \operatorname{int} K$. It follows that we must have $v' \notin K$. Consider then the arc $\alpha_1 = arc\{t_1, v'\}$ of the semicircle $C_1(x,y)$. The arc α_1 connects the inner point t_1 with the point v' which does not belong to K. It follows that the arc α_1 must intersect the boundary ϑK. Denote with $w' = \vartheta K \cap arc\{t_1, v'\}$. The set K has then 3 boundary points x,y and w' on the circle $C(x,y)$ and also two inner points t_1 and t_2 on $C(x,y)$. Because the circle $C(x,y,w')$ coincides with the circle $C(x,y)$ this means that we have for the sphere $S(x,y,w')$ and the two closed hemispheres $S_1(x,y,w')$ and $S_2(x,y,w')$ of the sphere $S(x,y,w')$ determined by the plane $\mathscr{P}$ the following: $S_i(x,y,w') \cap \text{int} K \neq \emptyset, i = 1,2$. This contradicts property $(ip3)$ of the set K.

If we have $u \notin \mathscr{P}$ then let $L'(m,u)$ be the half-line with origin in the point m and going through the point u. Let $q = L'(m,u) \cap S(m,r_m)$ be the intersection point of the half-line $L'(m,u)$ and the sphere $S(m,r_m)$. We have of course $q \notin \mathscr{P}$. As $m \in \text{int} K$ and $u \notin \text{int} K$ and K is a convex set it follows that we have also $q \notin \text{int} K$. We deduce now the existence of a point z different from the points x and y such that $z \in \vartheta K \cap S(m,r_m)$ and $z \notin \mathscr{P}$. If $q \in \vartheta K$ let be $z = q$. If $q \notin \vartheta K$, it means that the point q is exterior to the convex body K. Let $P_1 = P(m,q,t_1)$ be the plane containing the three points m,q and t_1. Denote with t_1' the other point, in which the line $L(t_1,m)$ intersects the circle $C(x,y)$ in the plane $\mathscr{P}$. For the circle $C(q,t_1,t_1')$ going through the three points q,t_1 and t_1' in the plane P_1 we have then $C(q,t_1,t_1') = S(m,r_m) \cap P_1$. Let $\beta = arc\{q,t_1\}$ be the shortest arc of the circle $C(q,t_1,t_1')$ joining the points q and t_1. As q is exterior to the convex body K and $t_1 \in \text{int} K$ there is a point $z \in \beta \cap \vartheta K$ and we have of course also $z \in S(m,r_m) \cap \vartheta K$. Because $q \notin \mathscr{P}$ and $t_1 \in \mathscr{P}$ it is clear that $z \notin \mathscr{P}$.

The plane $P(x,y,z)$ is intersecting the plane $\mathscr{P}$ in the line $L(x,y)$. As the points t_1 and t_2 are lying in different half-planes of the plane $\mathscr{P}$ determined by the line $L(x,y)$ it follows that these two points are lying in different half-spaces of $\mathbb{R}^3$ determined by the plane $P(x,y,z)$. The sphere $S(x,y,z)$ is identic with the sphere $S(m,r_m)$ with the center in m (the midpoint of the segment ab) and the radius $r = d(x,y)/2$ and we have also $x \in \vartheta K$, $y \in \vartheta K$ and $z \in \vartheta K$. For the two hemispheres $S_1(x,y,z)$ and $S_2(x,y,z)$ of the sphere $S(x,y,z)$ determined by the plane $P(x,y,z)$ we have then $t_1 \in S_1(x,y,z)$ and $t_2 \in S_2(x,y,z)$ and therefore $S_i(x,y,z) \cap \text{int} K \neq \emptyset$ for $i = 1,2$ in contradiction to $(ip3)$.

So we have proved:
For every plane $\mathscr{P}$ containing the line $L(a,b)$ the set $K_P = K \cap \mathscr{P}$ must coincide with the disc $D_{\mathscr{P}}(o,r)$ in the plane $\mathscr{P}$ with center o (the midpoint of the segment ab) and radius $r = d(a,b)/2$ and therefore the convex body K is a ball with center o and radius r. □

We will now study the small intersection property (sip) (see Definition 2 above) for some compact convex sets in the Euclidean space $\mathbb{R}^2$. In the study of such sets we need the following lemma:

Lemma 5. *A compact convex set $P_n = \text{conv}\{p_1, p_2, ..., p_n\}$ with inner points in the Euclidean plane $\mathbb{R}^2$, whose boundary ϑP_n is a polygon with n sides and which has at least one acute angle, does not have the small intersection property (sip).*

Proof. Without loss of generality we can suppose that the acute angle $\alpha = \angle p_1 p_2 p_3$ is formed by the sides $p_1 p_2$ and $p_2 p_3$. Consider now a constant t such that

$$2t \leq min\{d(p_1,p_2), d(p_2,p_3), ..., d(p_n,p_1)\}$$

and two points p_1' on the side $p_1 p_2$ and p_3' on the side $p_2 p_3$ such that

$$d(p_2,p_1') = d(p_2,p_3') = t.$$

If we consider now the circle $C(p_1', p_3')$ with the diameter $d(p_1', p_3')$ and going through the points p_1' and p_3' it is easy to show that we have $C_i(p_1', p_3') \cap \operatorname{int} P_n \neq \emptyset$ for $i = 1$ and $i = 2$, because the angle $\alpha = \angle p_1 p_2 p_3 < 90°$. Thereby the set P_n does not have the small intersection property (sip). □

Definition 4. *For a natural number $n \geq 3$ we shall denote with $\mathscr{P}_n$ the family of all compact convex sets P_n in $\mathbb{R}^2$ having interior points and such that the boundary ϑP_n is a polygon with n sides.*

We have then the following:

Theorem 6. *For the members of the family $\mathscr{P}_n$ we have:*

(a) *There is no $K \in \mathscr{P}_3$ having the small intersection property (sip).*
(b) *The boundary ϑK of a set $K \in \mathscr{P}_4$ having the small intersection property (sip) is a rectangle.*
(c) *If $n \geq 5$ any set $K \in \mathscr{P}_n$ with all angles $\geq 90°$has the small intersection property (sip).*

Proof. Since each triangle has at least one acute angle, (a) follows immediately from Lemma 5.

It also follows from Lemma 5 that the boundary of quadrangle K of the family $\mathscr{P}_4$ having the small intersection property (sip) cannot have acute angles. But the only quadrangle without acute angles is a rectangle. Let then $p_1 p_2 p_3 p_4$ be a rectangle and $K = \operatorname{conv}\{p_1, p_2, p_3, p_4\}$ the convex set having this rectangle as its boundary. Consider a constant k such that $k = min\{d(p_1, p_2), d(p_2, p_3), d(p_3, p_4), d(p_4, p_1)\}$.

If x_1 and x_2 are two arbitrary points such that $x_i \in \vartheta K$ for $i = 1, 2$ and $d(x_1, x_2) < k$ then we have two possibilities: either x_1 and x_2 are on the same side of the rectangle, or they are on two different sides having a common endpoint. If they are on the same side then it is evident that for one of the two semicircles $C_i(x_1, x_2), i \in \{1, 2\}$ we have $C_i(x_1, x_2) \cap \operatorname{int} K = \emptyset$. If x_1 and x_2 are on two different sides of the rectangle with a common endpoint this common endpoint lies on one of the semicircles $C_i(x_1, x_2), i \in \{1, 2\}$ because the angle has 90°and thereby we have again for that semicircle $C_i(x_1, x_2) \cap \operatorname{int} K = \emptyset$.

The proof of (c) uses also Lemma 5. Let $P_n = p_1 p_2 ... p_n$ be a polygon with n sides such that all its angles are $\geq 90°$and $K = \operatorname{conv}\{p_1, p_2, ..., p_n\}$ the convex set having this polygon as its boundary. If x_1 and x_2 are two arbitrary points such that $x_i \in \vartheta K$ for $i = 1, 2$ and $d(x_1, x_2) \leq k$ then we have again two possibilities: either x_1 and x_2 are on the same side of the polygon P_n, or they are on 2 different sides having a common endpoint. If they are on the same side, then it is immediately that for one of the two semicircles $C_i(x_1, x_2), i \in \{1, 2\}$ we have $C_i(x_1, x_2) \cap \operatorname{int} K = \emptyset$. If x_1 and x_2 are on two different sides of the polygon P_n with a common endpoint, then this common endpoint lies on one of the semicircles $C_i(x_1, x_2), i \in \{1, 2\}$ or in the interior of one of the semidiscs determined by these semicircles, because the angle is $\geq 90°$and thereby we have again for that semicircle $C_i(x_1, x_2) \cap \operatorname{int} K = \emptyset$. □

We shall denote a circular arc with endpoints e_1 and e_2 in the sequel with $a_c(e_1, e_2)$. Theorem 7 identifies a class of subsets in the Euclidean plane $\mathbb{R}^2$ having the small intersection property (sip).

Theorem 7. *A compact smooth convex set K in the Euclidean plane $\mathbb{R}^2$ such that the boundary ϑK is formed by an alternating sequence of line segments and circular arcs has the small intersection property (sip).*

Proof. Consider in the space $\mathbb{R}^2$ the set of points $S_n = \{p_1, p_2, ..., p_{2n-1}, p_{2n}\}$. Let us suppose that the boundary ϑK of the set K is:

$$\vartheta K = \{\operatorname{conv}\{p_1, p_2\}, a_c(p_2, p_3), \operatorname{conv}\{p_3, p_4\}, a_c(p_4, p_5), ..., a_c(p_{2n}, p_1)\}.$$

Consider a constant k such that:

$$(i) \quad 2k < min\{d(p_1, p_2), d(p_2, p_3), ..., d(p_{2n-1}, p_{2n}), d(p_{2n}, p_1)\}$$

and two arbitrary points $x_i \in \vartheta K$ for $i = 1, 2$ with $d(x_1, x_2) \leq k$.

We must then distinguish the following three cases:

(1) x_1 and x_2 are on the same line segment of ϑK;
(2) x_1 and x_2 are on the same circular arc of ϑK;
(3) one point for instance x_1 is on a line segment of ϑK and x_2 is on a circular arc of ϑK and this line segment and the circular arc have a common endpoint. From the smoothness of K it follows that the line segment is tangent in the common endpoint to the circular arc.

In each of the three cases it is easy to see that for one of the semicircles $C_i(x_1, x_2), i \in \{1, 2\}$ we have $C_i(x_1, x_2) \cap \operatorname{int} K = \emptyset$, i.e. the set K has the small intersection property (sip). ☐

Lemma 8. *A compact smooth strictly convex set K in $\mathbb{R}^2$ can have at most one pair of boundary points $a_1 \in \vartheta K$ and $b_1 \in \vartheta K$ such that*

$$(ii) \quad C_i(a_1, b_1) \setminus \{a_1, b_1\} \subset \operatorname{int} K, \text{ for } i = 1, 2.$$

Proof. Let us suppose that there are two such pairs of points $\{a_1, b_1\} \subset \vartheta K$ and $\{a_1', b_1'\} \subset \vartheta K$ for which condition (ii) is verified.

Consider the circle $C(a_1, b_1)$ with diameter $d(a_1, b_1)$ going through the points a_1 and b_1 and the lines L_a tangent in the point a_1 and L_b tangent in the point b_1 to the circle $C(a_1, b_1)$. Denote with $B = \operatorname{conv}\{L_a, L_b\}$. We shall show first that we have the inclusion $K \subset B$. To prove this let us suppose the contrary i.e. there is a point p such that $p \in K$ and $p \notin B$.

We distinguish then the following cases:

(a) $p \in L((a_1, b_1)$ and $a_1 \in \operatorname{conv}\{p, b_1\}$. In this case we immediately get that

$$a_1 \in \operatorname{int}\{conv\{p \cup C(a_1, b_1)\}\} \subset \operatorname{int} K.$$

But this is in contradiction to $a_1 \in \vartheta K$.

(b) $p \in L((a_1, b_1)$ and $b_1 \in \operatorname{conv}\{p, a_1\}$. Similarly to the previous case, we get a contradiction.

(c) $p \notin L(a_1, b_1)$ and p is in the halfplane of $\mathbb{R}^2$ determined by the line L_a which does not contain the line L_b. In this case the line $L(p, a_1)$ must intersect the circle $C(a_1, b_1)$ also in a second point, let it be the point q, which is an interior point of the set K. There is then a constant r_q such that, for the circle $C(q, r_q)$ with the center q and the radius r_q, we have $\operatorname{conv} C(q, r_q) \subset \operatorname{int} K$. But then we have $a_1 \in \operatorname{int}\{\operatorname{conv}\{p, C(q, r_q)\}\} \subset K$ again in contradiction to $a_1 \in \vartheta K$.

(d) $p \notin L(a_1, b_1)$ and p is in the halfplane of $\mathbb{R}^2$ determined by the line L_b which does not contain the line L_a. Similarly to the previous case, we get a contradiction.

Thus the inclusion $K \subset B$ is proved.

Analogously, we can consider for the points $\{a'_1, b'_1\} \subset \vartheta K$ the circle $C(a'_1, b'_1)$ with diameter $d(a'_1, b'_1)$ going through a'_1 and b'_1 and the lines L'_a tangent in the point a'_1 and L'_b tangent in the point b'_1 to the circle $C(a'_1, b'_1)$. Denote with $B' = \operatorname{conv}\{L'_a, L'_b\}$. As before we can prove the inclusion: $K \subset B'$.

Since we have $\operatorname{conv}\{C_i(a_1, b_1)\} \subset K \subset B'$ and also $\operatorname{conv}\{C_i(a'_1, b'_1)\} \subset K \subset B$, it follows immediately that $d(L_a, L_b) = d(L'_a, L'_b)$, and that we must have $d(a_1, b_1) = d(a'_1, b'_1)$. Consequently, $B = B'$. Because of the inclusion $\{a_1, b_1, a'_1, b'_1\} \subset \vartheta K$ and by property (ii), the two circles $C_i(a_1, b_1)$ and $C_i(a'_1, b'_1)$ cannot coincide. Denote then with $a"_1 = C_i(a'_1, b'_1) \cap L_a$ and $b"_1 = C_i(a'_1, b'_1) \cap L_b$. It follows immediately that $a"_1 \in \vartheta K$ and $b"_1 \in \vartheta K$. Because the circle $C_i(a'_1, b'_1)$ has also the property (ii) there cannot be more than two boundary points of K on this circle. Thereby, we can assume without loss of generality that: $a"_1 = a'_1$ and $b"_1 = b'_1$. This means that for the line segment $\operatorname{conv}\{a_1, a'_1\}$ we have $\operatorname{conv}\{a_1, a'_1\} \subset \vartheta K$ in contradiction to the strict convexity of the set K. This completes the proof. □

Lemma 9. *A compact smooth strictly convex set K in $\mathbb{R}^2$ can have at most one pair of boundary points $a \in \vartheta K$ and $b \in \vartheta K$ such that:*

$$(iii) \quad C(a,b) \setminus \{a,b\} \subset \mathbb{R}^2 \setminus K.$$

Proof. Let us suppose there are two such pairs of points $\{a, b\} \subset \vartheta K$ and $\{a', b'\} \subset \vartheta K$ for which condition (iii) is verified.

Consider the circle $C(a,b)$ with diameter $d(a,b)$ going through the points a and b and denote with $D(a,b)$ the disc having $C(a,b)$ as its boundary. Because K is convex we have then $\operatorname{conv}\{a,b\} \subset K$. From the strict-convexity of K it follows that $\operatorname{conv}\{a,b\} \setminus \{a,b\} \subset \operatorname{int} K$ and also that $K \setminus \{a,b\} \subset \operatorname{int} D(a,b)$. But then the points a' and b' are also in the interior of the disc $D(a,b)$ and therefore $d(a',b') < d(a,b)$.

Consider now the circle $C(a',b')$ with diameter $d(a',b')$ going through the points a' and b' and denote with $D(a',b')$ the disc having $C(a',b')$ as its boundary. Because K is convex we have then $\operatorname{conv}\{a',b'\} \subset K$. From the strict-convexity of K it follows that $\operatorname{conv}\{a',b'\} \setminus \{a',b'\} \subset \operatorname{int} K$ and also that $K \setminus \{a',b'\} \subset \operatorname{int} D(a',b')$. But then the points a and b have to be in the interior of the disc $D(a',b')$ and so we must have $d(a,b) < d(a',b')$ in contradiction to the inequality $d(a',b') < d(a,b)$ obtained above. □

Theorem 10. *A compact smooth strictly convex set K in the Euclidean plane $\mathbb{R}^2$ has the small intersection property (sip).*

Proof. If the set K is a disc then by Theorem 1 K has the intersection property (ip) and thereby also the small intersection property (sip).

In the following we will consider the case when K is not a disc. Let us suppose that K does not have the small intersection property (sip) i.e. there does not exist a positive constant k such that for any two distinct boundary points a and b of the set K with $d(a,b) \leq k$ we have $C_i(a,b) \cap \operatorname{int} K = \emptyset$ for at least one $i \in \{1,2\}$.

This means that for each natural number n we can find two boundary points $p_1^n \in \vartheta K$ and

$p_2^n \in \vartheta K$ with $d(p_1^n, p_2^n) \leq 1/n$ such that

$$(iv) \quad C_i(p_1^n, p_2^n) \cap \operatorname{int} K \neq \emptyset \text{ for } i = 1,2.$$

By Lemma 8, we can have at most one pair of points $\{p_1^n, p_2^n\}$ of the infinite sequence $\mathscr{S} = \{\{p_1^n, p_2^n\}, n = 1,2,3,...\}$ with the property (ii) of Lemma 8. If there is such a pair of $\{p_1^n, p_2^n\}$ in the sequence $\mathscr{S}$ having the property (ii) we eliminate this pair of points from the sequence $\mathscr{S}$.

Consider then an arbitrary pair of points $\{p_1^n, p_2^n\}$ of the remaining sequence $\mathscr{S}$. There is then a point q_1 on the semicircle $C_1(p_1^n, p_2^n)$ such that $q_1 \in \operatorname{int} K$ and a point q_2 on the semicircle $C_2(p_1^n, p_2^n)$ such that $q_2 \in \operatorname{int} K$. Because the pair $\{p_1^n, p_2^n\}$ does not have the property (ii) of Lemma 8 there follows the existence of a point q such that

$$q \in C(p_1^n, p_2^n) \setminus \{p_1^n, p_2^n\} \qquad \text{and} \qquad q \in \mathbb{R}^2 \setminus \operatorname{int} K.$$

The point q belongs to one of the four circular arcs in which the circle $C_1(p_1^n, p_2^n)$ is subdivided by the points p_1^n, p_2^n, q_1 and q_2 and differs from their endpoints. Without loss of generality we can suppose that q is on the circular arc of $C_1(p_1^n, p_2^n)$ between the endpoints p_1^n and q_1. As $q \in \mathbb{R}^2 \setminus \operatorname{int} K$ the point q can be an boundary point of the set K or an external point of K. If q is an external point of K, between q and the intern point q_1 of K there must exist an other boundary point of K on this circular arc of $C_1(p_1^n, p_2^n)$. Thus we have deduced in both cases the existence of a third point $p_3^n \in \vartheta K \cap C(p_1^n, p_2^n)$. For this point we have of course: $\angle p_1^n p_3^n p_2^n = 90°$.

For the compact set ϑK there exists then an accumulation point $p_a \in \vartheta K$ of the infinite sequence $\{p_3^n, n = 1,2,3,...\}$ with the property that in any arbitrary small neighborhood of the point p_a there is a triplet of boundary points $\{p_1^n, p_2^n, p_3^n\} \subset K$ with $\angle p_1^n p_3^n p_2^n = 90°$. But this is in contradiction with the smoothness of the set K. □

It is clear that every compact smooth strictly convex set K in the Euclidean plane $\mathbb{R}^2$ is a rotund set. We will now show that we cannot simply substitute the hypothesis 'smooth strictly convex' in Theorem 10 with that of 'rotundity' of the set K. We will give an example of a rotund set which does not have the property (sip). Consider for $i = 1,2$ the circles $C_i(c_i, r)$ with center c_i and radius r with two common points a and b and the corresponding discs $D_1 = \operatorname{conv} C_1(c_1, r)$ and $D_2 = \operatorname{conv} C_2(c_2, r)$. Let $D = D_1 \cap D_2$ be the intersection of the discs D_1 and D_2. In $\mathbb{R}^2$, D is then a rotund convex set with a nonempty interior. Let L_1 be tangent line in the point a to the circle C_1 and L_2 the tangent in the point a to the circle C_2. Denote with L_1' that closed halfline of the line L_1 with origin a and with L_2' that closed halfline of the line L_2 with origin a for which we have $D \subset \operatorname{conv}\{L_1' \cup L_2'\}$. The distance $d(c_1, c_2)$ can be chosen so that the angle with the vertex a formed by the halflines L_1' and L_2' is an acute angle. Let now k be an arbitrary constant such that $k < \operatorname{diam} D$. We can then choose a point c on the line $L(a,b)$ such that $d(a,c) < d(c,b)$). The line L_0 perpendicular in the point c on the line $L(a,b)$ intersects then the boundary ϑD in two points $p_1 \in C_1(c_1, r)$ and $p_2 \in C_2(c_2, r)$. Let $q_1 = L_0 \cap L_1'$ and $q_2 = L_0 \cap L_2'$. We can choose the point c on the line $L(a,b)$ sufficiently near to a so that $d(p_1, p_2) < d(q_1, q_2) < k$. For the circle $C(p_1, p_2)$ going through the two boundary points p_1 and p_2 and of diameter $d(p_1, p_2)$ and the two closed semicircles $C_i(p_1, p_2)$ $i = 1,2$ determined on the circle $C(p_1, p_2)$ by the line $L(p_1, p_2)$ we have then $C_i(p_1, p_2) \cap \operatorname{int} D \neq \emptyset$ for $i \in \{1,2\}$. Thereby D does not have the property (sip).

3. Conclusion

The constant $\rho(K)$ can be considered as a sort of measure for the roundness of a compact convex set K with a nonempty interior which has the intersection property (sip).
We have of course $0 < \rho(K) \leq 1$ and by Theorem 1 we have $\rho(K) = 1$ only for a disc. If $\rho(K)$ *is tending to 1* the aspect of the set K is nearer to that of a disc. If $\rho(K)$ *is tending to 0* the aspect of the set K is nearer to that of a line segment conv $\{a, b\}$ where a and b are two distinct points in $\mathbb{R}^2$.
We think that $\rho(K)$ can be calculated for special convex sets as for instance for the convex hull of the ellipse and other sets. It would also be interesting to generalize Theorem 1 and Theorem 4 for the n-dimensional Euclidean space $\mathbb{R}^n$.

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