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ON SETS OF CLASS $[1, q+1, 2q+1]_2$ IN $PG(3, q)$

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ABSTRACT. In this note we prove that a set of class $[1, q+1, 2q+1]_2$ in $PG(3, q)$ is either a line, or an ovoid, or a (q^2+q+1)-set of type $(1, q+1, 2q+1)_2$ or a $(q+1)^2$-set of type $(q+1, 2q+1)_2$, or a unique, up to projective equivalence, sporadic 19-set of type $(1, 4, 7)$ in $PG(3, 3)$.

1. Introduction and motivation

A fundamental question in finite geometry is to recognize geometric substructures from combinatorial properties. Let $PG(3, q)$ be the projective space of dimension three and order $q = p^h$ a prime power. Let K denote a k-set, i.e. a set of k points, of $PG(3, q)$. A plane (line) of $PG(3, q)$ meeting K in exactly j points is simply called a *j-plane* (*j-line*) of K. A 0-plane (0-line) is also said an *external plane* (*external line*). For each integer j such that $0 \leq j \leq q^2+q+1$ ($0 \leq j \leq q+1$), let us denote by $t_j = t_j(K)$ the number of j-planes (j-lines) of K. The numbers t_j are called the *characters* of K with respect to the planes (lines). Let m_1, m_2, ..., m_h be h integers such that $0 \leq m_1 < m_2 < ... < m_h \leq q^2+q+1$ ($0 \leq m_1 < m_2 < ... < m_h \leq q+1$). A set K is said to be of *class* $[m_1, m_2, ..., m_h]_2$ (of *class* $[m_1, m_2, ..., m_h]_1$) if $t_j \neq 0$ only if $j \in \{m_1, m_2, ..., m_h\}$. Moreover K is said to be of *type* $(m_1, m_2, ..., m_h)_2$ (of *type* $(m_1, m_2, ..., m_h)_1$) if $t_j \neq 0$ if and only if $j \in \{m_1, m_2, ..., m_h\}$. The integers m_1, m_2, ..., m_h are called *intersection numbers* with respect to the planes (lines). Of particular interest are sets with few intersection numbers (see Coykendall and Dover 2001). The most studied sets are those ones with only two intersection numbers and not much seems to be known in the general case of sets with more than two (see Thas 1973; Hirschfeld 1985; Tallini Scafati 1985). Recently Napolitano (2014) and Innamorati and Zuanni (2015, 2017) studied some sets of class $[l, m, n]_2$ in affine spaces and in projective spaces and they showed how to recognize specific configurations only by using the intersection numbers.

Now, let P be a point of a k-set K and τ a plane passing through P. If any line in τ through P is either a 1-line or a $(q+1)$-line of K, then Biondi and Melone (1986) say that the plane τ is *tangent* to K in the point P. Moreover, if for any line l in $PG(3, q)$ the number of tangent planes to K through l belongs to the set $\{0, 1, 2, q+1\}$, they say that K is of *Plücker class two*. Then Biondi and Melone proved the following

Result 1.1. *Let K be a set of class $[1, q+1, 2q+1]_2$ of $PG(3,q)$. If K is of Plücker class two, then*

- *K is a line;*
- *K is an ovoid;*
- *K is of type $(1, q+1, 2q+1)_2$; furthermore, if K contains at least two lines, then K is a cone projecting an oval in a plane π from a point V not in π;*
- *K is of type $(q+1, 2q+1)_2$;*
  furthermore, if K contains at least five lines and there is at least one tangent plane to K, then K is the pointset of $q+1$ pairwise skew lines, which either have one or two trasversals, or form a hyperbolic quadric.

Durante *et al.* (2010) proved the following

Result 1.2. *Let K be a set of class $[1, q+1, 2q+1]_2$ of $PG(3,q)$. If K contains at least two lines, then*

- *K is a cone projecting an oval in a plane π from a point V not in π; hence if q is odd, then K is a quadratic cone;*
- *K is a $(q+1)^2$-set of type $(q+1, 2q+1)_2$.*

In the groove traced by the previous results, we prove the following

Theorem 1.3. *If K is a set of class $[1, q+1, 2q+1]_2$ of $PG(3,q)$, then*

- *K is a line;*
- *K is an ovoid;*
- *K is a (q^2+q+1)-set of type $(1, q+1, 2q+1)_2$;*
- *K is a $(q+1)^2$-set of type $(q+1, 2q+1)_2$;*
- *$q = 3$ and K is a 19-set of type $(1,4,7)_2$; furthermore, up to projective equivalence, there is a unique 19-set of type $(1,4,7)_2$ in $PG(3,3)$.*

2. The proof of the theorem

Let K be a k-set of class $[1, q+1, 2q+1]_2$ in $PG(3,q)$. By counting in double way the number of planes, the number of pairs (P, π) where $P \in K$ and π is a plane through P, and the number of pairs $((P,Q), \pi)$ where $\{P,Q\} \subset K$ and π is a plane through P and Q, we get the following equations on the integers $t_i = t_i(K)$

$$\begin{cases} t_1 + t_{q+1} + t_{2q+1} = q^3 + q^2 + q + 1 \\ t_1 + (q+1)t_{q+1} + (2q+1)t_{2q+1} = k(q^2+q+1) \\ q(q+1)t_{q+1} + 2q(2q+1)t_{2q+1} = k(k-1)(q+1) \end{cases}$$

or, equivalently,

$$\begin{cases} 2q^2 t_1 = 2q^5 + 5q^4 + (6-3k)q^3 + (6-4k)q^2 + (k-4)(k-1)q + (k-1)^2 \\ q^2 t_{q+1} = -2q^4 + (2k-3)q^3 + 3(k-1)q^2 - (k-3)(k-1)q - (k-1)^2 \\ 2q^2 t_{2q+1} = (q+1)[q^3 - (k-1)q^2 - (k-1)q + (k-1)^2] \end{cases}$$

It is immediate to see that $k \equiv 1 \ (mod\ q)$. If $k=1$, then K is a point, so it is not a set of class $[1,q+1,2q+1]_2$. Putting $k=aq+1$ with $a \geq 1$ we get

$$\begin{cases} 2t_1 = (q+2-a)[(q+1)(2q-1)+2-a(q+1)] \\ t_{q+1} = 3q+(2q-a)[(a-1)q+(a-2)] \\ 2t_{2q+1} = (q+1)(a-1)(a-q) \end{cases}$$

Being $t_{2q+1} \geq 0$, we have that $a=1$ or $a \geq q$. If $a=1$, then $k=q+1$, $t_1 = q^2(q+1)$, $t_{q+1} = q+1$, and $t_{2q+1}=0$. So K is a line (see Thas 1973).

From now on, $a \geq q$. Being $t_1 \geq 0$, we get $a \leq q+2$ or $a \geq 2q-1+2/(q+1)$, i.e. $a \geq 2q$ since $0 < 2/(q+1) < 1$. If $a \geq 2q+1$, then $t_{q+1} \leq -(q-1)(2q+1) < 0$, a contradiction. Finally, we have that $q \leq a \leq q+2$ or $a=2q$.

If $a=q$, then $k=q^2+1$, $t_1=q^2+1$, $t_{q+1}=q^3+q$, and $t_{2q+1}=0$; so K is a (q^2+1)-set of type $(1,q+1)_2$, i.e. K is an ovoid, see (see Thas 1973).

If $a=q+1$, then $k=q^2+q+1$, $t_1=q(q-1)/2$, $t_{q+1}=q^3+q+1$, and $t_{2q+1} = q(q+1)/2$; so K is a (q^2+q+1)-set of type $(1,q+1,2q+1)_2$.

If $a=q+2$, then $k=(q+1)^2$, $t_1=0$, $t_{q+1}=q^3-q$, and $t_{2q+1}=(q+1)^2$; so K is a $(q+1)^2$-set of type $(q+1,2q+1)_2$.

Now, let us consider the case $a=2q$. First, let us note that if $q=2$, then $q+2=2q$. So $q \geq 3$. Being $t_1 = 1+q(q-3)/2 \geq 1$, let π be a 1-plane and let P the point of K on π. If $t_1 \geq 2$, then let π' be another 1-plane and let r be the line $\pi \cap \pi'$. Let us denote by u_i the number of i-planes through r with $i \in \{1,q+1,2q+1\}$. Counting the planes through r, and the points of K by the planes through r we obtain

$$u_1 + u_{q+1} + u_{2q+1} = q+1 \tag{1}$$

$$(1-x)u_1 + (q+1-x)u_{q+1} + (2q+1-x)u_{2q+1} = 2q^2+1-x \tag{2}$$

with $x=0$ if $P \notin r$ or $x=1$ if $P \in r$. Being $u_1 \geq 2$, by (1) we have that $u_{2q+1} \leq q-1$. By the following combination (2)$-(q+1-x)$(1) we get $u_{2q+1} = q+x+(u_1-2) \geq q$, a contradiction. So $t_1=1$ which implies $q=3$. Being $a=2q=6$, we have $k=19$, $t_4=9$, and $t_7=30$. Hence, K is a 19-set of type $(1,4,7)_2$.

3. The unique 19-set of type $(1,4,7)_2$ in $PG(3,3)$

In this section K will ever be a 19-set of type $(1,4,7)_2$ in $PG(3,3)$. Let π be the unique 1-plane of K and let P be the unique point of K on π.

By $v_i(P)$ we denote the number of i-planes of K passing through P with $i \in \{1,4,7\}$. Of course, it is $v_1(P)=1$.

Lemma 3.1. *We have that $v_4(P)=0$ and $v_7(P)=12$.*

Proof. Counting the pairs (Q,α) where Q is a point of $K \setminus \{P\}$ and α is a plane through Q and P we obtain $3v_4(P)+6v_7(P) = (q+1)(k-1) = 72$. Being $v_4(P)+v_7(P) = 13 - v_1(P) = 12$, we get $v_4(P)=0$ and $v_7(P)=12$. □

If r_h is an h-line of K, by $\tau_i(r_h)$ we denote the number of i-planes passing through r_h with $h \in \{0,1,2,3,4\}$ and $i \in \{1,4,7\}$.

Remark 3.2. *If r_h lyes on π, then $h \in \{0,1\}$ and $\tau_1(r_h)=1$.*

Lemma 3.3. *If r_h lyes on π, then $\tau_7(r_h) = 2+h$ and $\tau_4(r_h) = 1-h$.*

Proof. Counting the points of $K \setminus \pi$ by the planes through r_h different from π, we obtain $(4-h)\tau_4(r_h)+(7-h)\tau_7(r_h) = 18$. Being $\tau_4(r_h)+\tau_7(r_h) = 3$, we get $\tau_7(r_h) = 2+h$ and $\tau_4(r_h) = 1-h$. □

Remark 3.4. *If r_h does not lye on π, then $\tau_1(r_h) = 0$.*

Lemma 3.5. *If r_h does not lye on π, then $\tau_7(r_h) = 1+h$ and $\tau_4(r_h) = 3-h$. Furthermore, $h \leq 3$ and so K does not contain lines.*

Proof. Counting the points of K by the planes passing through r_h we obtain $(4-h)\tau_4(r_h)+(7-h)\tau_7(r_h) = 19-h$. Being $\tau_4(r_h)+\tau_7(r_h) = 4$, we get $\tau_7(r_h) = 1+h$ and $\tau_4(r_h) = 3-h$. Finally, $\tau_4(r_h) \geq 0$ implies $h \leq 3$. □

Corollary 3.6. *A 4-plane of K does not contain a 3-line of K.*

Lemma 3.7. *If we denote by $w_i(P)$ the number of i-lines of K passing through P with $i \in \{1,2,3\}$, then $w_1(P) = 4$, $w_2(P) = 0$, and $w_3(P) = 9$.*

Proof. The four lines through P on the plane π are 1-lines of K. So if we denote by x the number of 1-secant lines of K passing through P and not lying on π, then $w_1(P) = 4+x$. Counting the number of points $Q \in K \setminus \{P\}$ by the lines through P, we obtain $w_2(P)+2w_3(P) = 18$. Being $w_2(P)+w_3(P) = 13-w_1(P) = 9-x$, we get $w_3(P) = 9+x$ and $w_2(P) = -2x$. Finally, $w_2(P) \geq 0$ implies $x = 0$. So $w_1(P) = 4$, $w_2(P) = 0$, and $w_3(P) = 9$. □

Theorem 3.8. *Up to projective equivalence, there is a unique 19-set K of type $(1,4,7)_2$ in $PG(3,3)$.*

Proof. Let us suppose that there exists a 19-set K of type $(1,4,7)_2$ in $PG(3,3)$. Here we use homogeneous coordinates (x_0,x,y,z) for the points of $PG(3,3)$. U.t.p.e. (up to projective equivalence), let $\pi : x_0 = 0$ be the 1-plane of K and $P = Z_\infty(0,0,0,1)$ be the point of K on π. The line $r_0 : x_0 = z = 0$ is a line on π not through P. Hence, by (**3.3**) $\tau_4(r_0) = 1$ and $\tau_7(r_0) = 2$. U.t.p.e., let $\alpha : z = 0$ be the 4-plane through r_0, $\beta : z = 1$ and $\gamma : z = 2$ be the two 7-planes through r_0. By (**3.6**) the four points of $\alpha \cap K$ are the vertices of a square. U.t.p.e., $K \cap \alpha = \{(1,0,0,0),(1,1,0,0),(1,0,1,0),(1,1,1,0)\}$. By (**3.7**) the nine lines $r_{ij} : x-ix_0 = y-jx_0 = 0$ with $i,j \in \{0,1,2\}$ are 3-lines. So the ten points $(1,0,2,z)$, $(1,1,2,z)$, $(1,2,2,z)$, $(1,2,1,z)$, $(1,2,0,z)$ with $z \in \{1,2\}$ are in K. So both on the plane β and on the plane γ there are other two points of K. Since the line $r_{11} : x-x_0 = y-x_0 = 0$ is a 3-line of K, one and only one of the points $(1,1,1,1)$ or $(1,1,1,2)$ is in K. U.t.p.e., let us suppose that $(1,1,1,1) \in K$ and $(1,1,1,2) \notin K$. If we consider the plane $\delta : x+y+z = 0$, then the five points $(1,0,0,0)$, $(1,2,0,1)$, $(1,1,1,1)$, $(1,0,2,1)$, and $(1,2,2,2)$ are in $K \cap \delta$. So δ is a 7-plane of K. The other two points on $\delta \cap \gamma$ are $(1,1,0,2)$ and $(1,0,1,2)$ necessarily. Since the lines $r_{10} : x-x_0 = y = 0$ and $r_{01} : x = y-x_0 = 0$ are 3-lines of K, then $(1,1,0,1)$ and $(1,0,1,1)$ are not in K. Thus, the 7^{th} of $K \cap \beta$ is $(1,0,0,1)$ necessarily.

Finally, if there exists a 19-set K of type $(1,4,7)_2$ in $PG(3,3)$, then, up to projective equivalence, the set K must contain the following 19 points:

$(0,0,0,1)$, $(1,0,0,0)$, $(1,1,0,0)$, $(1,0,1,0)$, $(1,1,1,0)$, $(1,0,2,1)$, $(1,1,2,1)$, $(1,2,2,1)$, $(1,2,1,1)$, $(1,2,0,1)$, $(1,0,2,2)$, $(1,1,2,2)$, $(1,2,2,2)$, $(1,2,1,2)$, $(1,2,0,2)$, $(1,1,1,1)$, $(1,1,0,2)$, $(1,0,1,2)$, $(1,0,0,1)$.
Viceversa, by computer, we checked that the set containing those 19 points is of type $(1,4,7)_2$. □

3.1. A geometric description of the 19-set K. Let Σ be the pointset of $PG(3,3)$. Let π be the pointset of the plane $x_0 = 0$. Let Q be the pointset of the hyperbolic quadric $xy + (x+y+z+x_0)x_0 = 0$. By computer we checked that $K = (\Sigma \setminus (Q \cup \pi)) \cup \{(0,0,0,1)\}$.

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