

## TRANSVERSALS AND BLOCKING SETS IN $H^{(3)}$ -DESIGNS

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**ABSTRACT.** If  $H^{(h)}$  is a subhypergraph of order  $n$  of  $K_v^{(h)}$ , the complete and  $h$ -uniform hypergraph of order  $v$ , an  $H^{(h)}$ -decomposition of  $K_h^{(v)}$ , also called an  $H^{(h)}$ -design of order  $v$ , is a pair  $\Sigma = (X, \mathcal{B})$ , where  $\mathcal{B}$  is a partition of the edge-set of  $K_v^{(h)}$  into classes generating hypergraphs all isomorphic to  $H^{(h)}$ . The classes of the partition  $\mathcal{B}$  are said to be the *blocks* of  $\Sigma$ . Using hypergraph terminology, if  $\Sigma = (X, \mathcal{B})$  is an  $H^{(h)}$ -design, a *transversal*  $T$  of  $\Sigma$  is a subset of  $X$  intersecting every block of  $\Sigma$ . The *transversal number* of  $\Sigma$  is the minimum number  $\tau(\Sigma) = \tau$  for which there exists a transversal of  $\Sigma$  having cardinality  $\tau$ . A *blocking set*  $B$  of  $\Sigma$  is a subset of  $X$  such that both  $B$  and  $C_X(B)$  are *transversals*. In this paper, the existence of transversals and blocking sets in  $H^{(3)}$ -designs are studied.

### 1. Introduction

Let  $K_v^{(h)} = (X, \mathcal{P}_h(X))$  be the complete and  $h$ -uniform hypergraph of order  $v$  defined in  $X$ . Let  $H^{(h)}$  be any subhypergraph of  $K_v^{(h)}$ . An  $H^{(h)}$ -decomposition of  $K_v^{(h)}$ , also called an  $H^{(h)}$ -design of order  $v$ , is a pair  $\Sigma = (X, \mathcal{B})$ , where  $\mathcal{B}$  is a partition of  $\mathcal{P}_h(X)$ , the edge-set of  $K_v^{(h)}$ , into classes generating hypergraphs all isomorphic to  $H^{(h)}$ . The classes of the partition  $\mathcal{B}$  are said to be the *blocks* of  $\Sigma$ . Using hypergraph terminology, if  $\Sigma = (X, \mathcal{B})$  is an  $H^{(h)}$ -design, a *transversal*  $T$  of  $\Sigma$  is a subset of  $X$  intersecting every block of  $\Sigma$ . The *transversal number* of  $\Sigma$  is the minimum number  $\tau(\Sigma) = \tau$  for which there exists a transversal of  $\Sigma$  having cardinality  $\tau$ . A *blocking-set*  $B$  of  $\Sigma$  is a subset of  $X$  such that both  $B$  and  $C_X(B)$  are *transversals* of  $\Sigma$ . If a system  $\Sigma$  has order  $v$ , then we say that a blocking-set  $B$  of  $\Sigma$  is an *equi-blocking-set* if  $|B| = \frac{v}{2}$  in the case  $v$  even, and  $|B| = \frac{v+1}{2}$  in the case odd  $v$ . It is known that a *Steiner system*  $S(h, k, v)$  is a  $K_k^{(h)}$ -decomposition of  $K_v^{(h)}$ . The problem to determine the existence of possible *blocking sets* in Steiner systems and  $G$ -designs, where  $G$  is a graph, has been studied by several researchers (see Phelps and Rosa 1980; Gionfriddo *et al.* 1994; Linek and Wantland 1998; Gionfriddo *et al.* 2004; Rosa 2004; Lindner *et al.* 2005; Chang *et al.* 2008, 2013; Gionfriddo 2013; Gionfriddo *et al.* 2013). Some of the outcomes are included in the following theorems.

**Theorem 1.1.** (Gionfriddo *et al.* 2014) *For every  $v \equiv 4$  or  $8 \pmod{12}$ , there exists an  $SQS(v)$  having a blocking set.*

**Theorem 1.2.** (Gionfriddo and Lo Faro 1993) *If  $k$  is even and  $B$  is a blocking set in a Steiner System  $S(k-1, k, v)$ , then  $|B| = v/2$ . If  $k$  is odd, there does not exist any Steiner System  $S(k-1, k, v)$ ,  $v > k$ , having blocking sets.*

**Theorem 1.3.** (Hoffman *et al.* 1990; Franek *et al.* 2002) *For every admissible  $v$ , there exists an  $S(2, 4, v)$  having a blocking set.*

**Theorem 1.4.** (Lo Faro 1990) *Let  $B$  be a blocking set of an  $S(2, 4, v)$ , then:*

- i)  $|B| = \frac{v \pm 1}{2}$ , if  $v$  is odd;
- ii)  $|B| = \frac{v}{2}$  or  $|B|$  is odd, if  $v$  is even.

**Theorem 1.5.** (Gionfriddo *et al.* 1991) *For every admissible  $v$ , there exists a  $(K_4 - e)$ -design of order  $v$  admitting blocking sets.*

In this paper, the existence of transversals and blocking sets in  $H^{(3)}$ -designs is studied. Observe that, given a 3-uniform hypergraph  $H^{(3)} = (V, \mathcal{E})$ , a triple  $\{x, y, z\} \in \mathcal{E}$  is usually called an *edge* of  $H^{(3)}$ . In what follows, we will say also that such a triple  $\{x, y, z\} \in \mathcal{E}$  is a 3-edge of  $H^{(3)}$  and that a pair  $\{x, y\} \subset E \in \mathcal{E}$  is an edge, or also a 2-edge, of  $H^{(3)}$ . Denote by  $r_1$  the *minimum degree* of the vertices of  $H^{(3)}$ , and by  $r_2$  the *minimum degree* of the 2-edges of  $H^{(3)}$ .

In what follows we will denote by:

- $[x, (y, z), t]$  the 3-uniform hyperpath  $P^{(3)}(2, 4)$ , having vertex set  $\{x, y, z, t\}$  and edge set  $\{\{x, y, z\}, \{y, z, t\}\}$ ;
- $[x, y, (z), t, w]$  the 3-uniform hyperpath  $P^{(3)}(1, 5)$ , having vertex set  $\{x, y, z, t, w\}$  and edge set  $\{\{x, y, z\}, \{z, t, w\}\}$ ;
- $[x, (y), z]$  the path  $P_3$ , having vertex set  $\{x, y, z\}$  and edge set  $\{\{x, y\}, \{y, z\}\}$ .

We remember that:

**Theorem 1.6.** (Bermond *et al.* 1977; Gionfriddo *et al.* 2014) *There exists a  $P^{(3)}(2, 4)$  or  $P^{(3)}(1, 5)$ -design of order  $v$  if and only if  $v \not\equiv 3 \pmod{4}$ ,  $v \geq 4$  or 5, respectively.*

**Theorem 1.7.** (Heinrich 1992) *There exists a  $P_3$ -design of order  $v$  if and only if  $v \equiv 0, 1 \pmod{4}$ .*

## 2. Transversals in $H^{(3)}$ -designs

The following theorem gives a necessary condition for the existence of transversals and blocking sets in  $H^{(3)}$ -designs.

**Theorem 2.1.** *Let  $\Sigma = (X, \mathcal{B})$  be an  $H^{(3)}$ -design of order  $v$ ,  $r_1$  the minimum degree of the vertices of  $H^{(3)}$ ,  $r_2$  the minimum degree of the 2-edges of  $H^{(3)}$ . If  $T$  a transversal of  $\Sigma$  and  $|T| = p \geq 3$ , then*

$$(1) \quad \binom{p}{3} + \binom{p}{2} \left[ \frac{v-2}{r_2} - (p-2) \right] + p \cdot \left[ \frac{(v-1)(v-2)}{2r_1} - \binom{p-1}{2} \right] \geq |\mathcal{B}|.$$

*In the case:  $r_1 = r_2 = 1$ , (1) becomes:*

$$(2) \quad \binom{p}{3} + \binom{p}{2} (v-p) + p \cdot \frac{(v-1)(v-2) - (p-1)(p-2)}{2} \geq |\mathcal{B}|.$$

**Proof.** Let  $\Sigma = (X, \mathcal{B})$  be an  $H^{(3)}$ -design of order  $v$ . Let  $P$  be a subset of  $X$  such that:  $|P| = p \geq 3$ . Let  $r_1, r_2$  be respectively the minimum degree of vertices and of 2-edges in  $H^{(3)}$ . Let  $M(P)$  be the maximum number of blocks of  $\Sigma$  which can intersect  $P$ . Observe that:

1) Considering that every triple of distinct elements of  $X$  belongs to exactly one block, there are at most  $\binom{p}{3}$  blocks containing the triples of  $P$ .

2) Since every pair of distinct vertices of  $X$  forms  $v - 2$  triples with all the other vertices of  $\Sigma$  and its minimum degree in  $H^{(3)}$  is  $r_2$ , it follows that for every pair  $x, y \in X$  there are at most other  $v - 2 - r_2(p - 2)$  triples containing them.

3) Since every vertex of  $X$  forms  $v - 1$  pairs with all the other vertices of  $\Sigma$  and its minimum degree in  $H^{(3)}$  is  $r_1$ , it follows that for every vertex  $x \in X$  there are at most other  $\binom{v-1}{2} - r_1 \binom{p-1}{2}$  triples containing it.

Hence, it follows that:

$$\binom{p}{3} + \binom{p}{2} \left[ \frac{v-2}{r_2} - (p-2) \right] + p \cdot \left[ \frac{(v-1)(v-2)}{2r_1} - \binom{p-1}{2} \right] \geq M(P).$$

At this point, we can conclude that if  $T$  is a transversal of  $\Sigma$ , then  $M(P) = |\mathcal{B}|$ . Hence:

$$\binom{p}{3} + \binom{p}{2} \left[ \frac{v-2}{r_2} - (p-2) \right] + p \cdot \left[ \frac{(v-1)(v-2)}{2r_1} - \binom{p-1}{2} \right] \geq |\mathcal{B}|,$$

which proves (1). The assertion (2) follows easily. □

### 3. Equi-blocking-sets in $P^{(3)}(2, 4)$ -designs

In this section we determine the spectrum for  $P^{(3)}(2, 4)$ -designs having equi-blocking-sets.

**Theorem 3.1.** *For every  $v$  even,  $v \geq 4$ , there exist  $P^{(3)}(2, 4)$ -designs having equi-blocking-sets.*

**Proof.** For  $v = 4$ , on  $X = \{x_0, x_1, y_0, y_1\}$  consider the set of blocks  $\mathcal{B} = \{[x_0, (y_0, y_1), x_1], [y_0, (x_0, x_1), y_1]\}$ ; then  $(X, \mathcal{B})$  is a  $P^{(3)}(2, 4)$ -design admitting  $\{x_0, x_1\}$  as an equi-blocking-set.

**Construction**  $v = 2h \rightarrow v + 2$ .

Let  $(X, \mathcal{B})$  be a  $P^{(3)}(2, 4)$ -design of order  $v = 2h$ ,  $h \geq 2$ , defined on  $X = \{x_0, x_1, \dots, x_{h-1}, y_0, y_1, \dots, y_{h-1}\}$  and having an equi-blocking-set  $B = \{x_0, x_1, \dots, x_{h-1}\}$ . Let  $\infty_1, \infty_2 \notin X$ ,  $X' = X \cup \{\infty_1, \infty_2\}$ . Consider the sets of blocks:

$$\Gamma = \{[x_i, (\infty_1, \infty_2), y_i] : i = 0, 1, \dots, h - 1\},$$

$$\Gamma' = \{[\infty_1, (z, t), \infty_2] : z, t \in X\}.$$

If  $\mathcal{B}' = \mathcal{B} \cup \Gamma \cup \Gamma'$ , then  $(X', \mathcal{B}')$  is a  $P^{(3)}(2, 4)$ -design of order  $v + 2$  where  $B' = B \cup \{\infty_1\}$  is an equi-blocking-set.  $\square$

**Theorem 3.2.** *For every  $v = 4h + 1$ ,  $h \in N$ ,  $v \geq 5$ , there exist  $P^{(3)}(2, 4)$ -designs having equi-blocking-sets.*

**Proof.** For  $v = 5$ , consider the following blocks on  $X = \{1, 2, 3, 4, 5\}$ :

$$[3, (1, 2), 4], [4, (2, 3), 5], [5, (3, 4), 1], [1, (4, 5), 2], [2, (5, 1), 3].$$

They generate a  $P^{(3)}(2, 4)$ -design of order  $v = 5$  having the equi-blocking-set  $B = \{1, 2\}$ .

**Construction**  $v = 4h \rightarrow v + 1$ .

Let  $(X, \mathcal{B})$  be a  $P^{(3)}(2, 4)$ -design of order  $v = 4h$ ,  $h \in N$ , defined on

$$X = \{x_0, x_1, \dots, x_{2h-1}, y_0, y_1, \dots, y_{2h-1}\},$$

and admitting the blocking set  $B = \{x_0, x_1, \dots, x_{2h-1}\}$  of cardinality  $2h$  (see Theorem 3.1). Let  $\infty \notin X$ ,  $X' = X \cup \{\infty\}$ , and  $(X, \mathcal{C})$  be a  $P_3$ -design of order  $4h$ . Define the following family of blocks:

$$\Gamma = \{[x, (\infty, y), z] : [x, (y), z] \in \mathcal{C}\}.$$

If  $\mathcal{B}' = \mathcal{B} \cup \Gamma$ , then  $(X', \mathcal{B}')$  is a  $P^{(3)}(2, 4)$ -design of order  $v = 4h + 1$ , where  $B' = B \cup \{\infty\}$  is a blocking-set of cardinality  $2h + 1$  and, of course,  $|C_{X'}B'| = 2h - 1 = \frac{v-1}{2}$ .  $\square$

Collecting together Theorems 3.1 and 3.2, it follows the conclusive Theorem:

**Theorem 3.3.** *For every admissible  $v$ , there exist  $P^{(3)}(2, 4)$ -designs having equi-blocking-sets.*

#### 4. Equi-blocking-sets in $P^{(3)}(1, 5)$ -designs

In this section we determine the spectrum for  $P^{(3)}(1, 5)$ -designs having equi-blocking-sets.

**Theorem 4.1.** *There exist  $P^{(3)}(1, 5)$ -designs of order  $v = 5, 6$ , having equi-blocking-sets.*

**Proof.** For  $v = 5$ , on  $X_1 = Z_5 = \{0, 1, 2, 3, 4\}$  define the following family of  $P^{(3)}(1, 5)$ :

$$\mathcal{B}_1 = \{[i, 2 + i, (1 + i), 3 + i, 4 + i] : i \in Z_5\}$$

Then  $(X_1, \mathcal{B}_1)$  is a  $P^{(3)}(1, 5)$ -design of order  $v = 5$ , where  $B_1 = \{0, 1\}$  is an equi-blocking-set.

For  $v = 6$ , on  $X_2 = Z_5 \cup \{\infty\}$  define the following family of  $P^{(3)}(1, 5)$ :

$$\mathcal{B}_2 = \{[i, 2 + i, (1 + i), 3 + i, 4 + i] : i \in Z_5\} \cup \{[i, 1 + i, (\infty), 2 + i, 4 + i], i \in Z_5\}$$

Then  $(X_2, \mathcal{B}_2)$  is a  $P^{(3)}(1, 5)$ -design of order  $v = 6$ , where  $B_2 = \{0, 1, \infty\}$  is an equi-blocking-set. □

**Theorem 4.2.** *There exist  $P^{(3)}(1, 5)$ -designs of order  $v$  even,  $v \geq 6$ , having equi-blocking-sets.*

**Proof.** The existence for  $v = 6$  is proved in Theorem 4.1.

**Construction**  $v = 2h \rightarrow v + 2; h \geq 3$

Let  $(X, \mathcal{B})$  be a  $P^{(3)}(1, 5)$ -design of order  $v = 2h, h \geq 3, h \in N$ , defined on  $X = \{x_0, x_1, \dots, x_{h-1}, y_0, y_1, \dots, y_{h-1}\}$  and having the equi-blocking-set  $B = \{x_0, x_1, \dots, x_{h-1}\}$ . Let  $\infty_1, \infty_2 \notin X$  and consider the following set of blocks on  $X' = X \cup \{\infty_1, \infty_2\}$ :

$$\Gamma = \{[x_i, x_j, (\infty_1), y_i, y_j], [x_i, x_j, (\infty_2), y_i, y_j] : i, j = 0, 1, \dots, h - 1, i \neq j\}.$$

Now distinguish the following cases:

*Case 1: even  $h$ .* Let

$$\Gamma_1 = \{[x_i, \infty_2, (\infty_1), x_{1+i}, y_{1+i}], [y_i, \infty_1, (\infty_2), x_{1+i}, y_{1+i}] : i = 0, 1, \dots, h - 1\},$$

$$\Gamma_2 = \{[x_j, y_{h-1+j}, (\infty_1), x_{\frac{h}{2}+j}, y_{\frac{h}{2}-1+j}], [x_j, y_{h-1+j}, (\infty_2), x_{\frac{h}{2}+j}, y_{\frac{h}{2}-1+j}] : j = 0, 1, \dots, \frac{h}{2} - 1\},$$

and for every  $i = 0, 1, \dots, h - 1$

$$\Gamma_{3,i} = \{[\infty_2, y_{1+i+2k}, (x_i), y_{2+i+2k}, \infty_1], [\infty_1, y_{1+i+2k}, (x_i), y_{2+i+2k}, \infty_2] : k = 0, 1, \dots, \frac{h}{2} - 2\}.$$

If  $\mathcal{B}' = \mathcal{B} \cup \Gamma \cup \Gamma_1 \cup \Gamma_2 \cup (\bigcup_{i=1}^{h-1} \Gamma_{3,i})$ , then  $(X', \mathcal{B}')$  is a  $P^{(3)}(1, 5)$ -design of order  $v + 2$ , where  $B' = B \cup \{\infty\}$  is an equi-blocking-set.

*Case 2: odd  $h$ .* Let

$$\Gamma'_1 = \{[x_i, \infty_2, (\infty_1), x_{1+i}, y_{1+i}], [y_i, \infty_1, (\infty_2), x_{1+i}, y_{1+i}] : i = 0, 1, \dots, h - 1\},$$

and for every  $i = 0, 1, \dots, h - 1$

$$\Gamma'_{2,i} = \{[\infty_2, y_{1+i+2k}, (x_i), y_{2+i+2k}, \infty_1], [\infty_1, y_{1+i+2k}, (x_i), y_{2+i+2k}, \infty_2] : k = 0, 1, \dots, \frac{h-1}{2}\}.$$

If  $\mathcal{B}' = \mathcal{B} \cup \Gamma \cup \Gamma'_1 \cup (\bigcup_{i=1}^{h-1} \Gamma'_{2,i})$ , then  $(X', \mathcal{B}')$  is a  $P^{(3)}(1, 5)$ -design of order  $v + 2$ , where  $B' = B \cup \{\infty\}$  is an equi-blocking-set. □

**Theorem 4.3.** *There exist  $P^{(3)}(1, 5)$ -designs of order  $v = 4h + 1, v \geq 5$ , having equi-blocking-sets.*

**Proof.** The existence for  $v = 5$  is proved in Theorem 4.1.

**Construction**  $v = 4h \rightarrow v + 1; h \geq 2$

Let  $(X, \mathcal{B})$  be a  $P^{(3)}(1, 5)$ -design of order  $v = 4h$ ,  $h \geq 2$ , defined on  $X = \{x_0, x_1, \dots, x_{2h-1}, y_0, y_1, \dots, y_{2h-1}\}$ , having equi-blocking-set  $B = \{x_0, x_1, \dots, x_{2h-1}\}$ . Let  $\mathcal{F} = \{F_1, F_2, \dots, F_{2h}\}$  be a 1-factorization of the complete bipartite graph with disjoint vertex sets  $B$  and  $X \setminus B$ , and let  $\infty \notin X$ . Consider the following sets of blocks:

$$\Gamma = \{[x_i, x_j, (\infty), y_i, y_j] : i, j = 0, 1, \dots, 2h - 1, i \neq j\}.$$

and for every  $F_j = \{\{x_i^j, y_i^j\} : i = 0, 1, \dots, 2h - 1\}$ ,  $j = 1, 2, \dots, 2h$ ,

$$\Gamma_j = \{[x_{2t}^j, y_{2t}^j, (\infty), x_{1+2t}^j, y_{1+2t}^j] : t = 0, 1, \dots, \frac{h}{2} - 1\}.$$

If  $X' = X \cup \{\infty\}$  and  $\mathcal{B}' = \mathcal{B} \cup \Gamma \cup (\bigcup_{j=1}^{2h} \Gamma_j)$ , then  $(X', \mathcal{B}')$  is a  $P^{(3)}(1, 5)$ -design of order  $v + 1$  having  $B' = B \cup \{\infty\}$  as an equi-blocking-set.  $\square$

Collecting together Theorems 4.1, 4.2 and 4.3, we have the following conclusive result:

**Theorem 4.4.** *For every admissible  $v$ , there exist  $P^{(3)}(1, 5)$ -designs having equi-blocking-sets.*

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