

A NEW CHARACTERIZATION OF ELLIPTIC QUADRICS IN $\text{PG}(3, q)$, q ODD

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ABSTRACT. We prove that a set \mathcal{O} of points of $\text{PG}(3, q)$, q odd, of line-type $(0, m, n)_1$, $n \neq q$, with a point on which there are at most $q + 1$ lines intersecting \mathcal{O} in exactly m points is either an elliptic quadric or $n = q + 1$ and \mathcal{O} is the complement of a line in $\text{PG}(3, q)$.

1. Introduction

Barlotti (1955) and Panella (1995), independently, proved that in $\text{PG}(3, q)$, q odd, a subset of $q^2 + 1$ points no three of which are collinear is an elliptic quadric, thus providing an extension to the three-dimensional case of Beniamino Segre Theorem (1955) characterizing non-degenerate conics of $\text{PG}(2, q)$, q odd. These papers have been the starting point for the study of (special) subsets of points of the projective spaces $\text{PG}(d, q)$, $d \geq 2$, with respect to their intersection with at least one family of subspaces of $\text{PG}(d, q)$. There is a wide literature devoted to these sets, especially for the case with only two intersections also in view of their connections with coding theory (cf e.g. Calderbank and Kantor 1986).

In this paper, we give a new characterization of elliptic quadrics of $\text{PG}(3, q)$, q odd, weakening some of the assumptions of Barlotti and Panella.

Before stating our result, we briefly recall the definition of line-type (Napolitano 2015) of a set of points of a projective space. Let $0 \leq m_1 < \dots < m_s$ be a series of s non-negative integers. A set of points \mathcal{K} of $\text{PG}(3, q)$ is of line-type $(m_1, \dots, m_s)_1$ if $|\ell \cap \mathcal{K}| \in \{m_1, \dots, m_s\}$ for any line ℓ of $\text{PG}(3, q)$ and for any given m_j there is at least one line of $\text{PG}(3, q)$ intersecting \mathcal{K} into exactly m_j points). The integers m_1, \dots, m_s are the *intersection numbers* of \mathcal{K} .

Thus an elliptic quadric of $\text{PG}(3, q)$ is a set of line-type $(0, 1, 2)_1$ and a hyperbolic quadric of $\text{PG}(3, q)$ is a set of line-type $(0, 1, 2, q + 1)_1$.

Let \mathcal{K} be a subset of points of $\text{PG}(3, q)$ of line type $(m_1, \dots, m_s)_1$. A line intersecting \mathcal{K} in m_j points is a m_j -line. An *external* line is a 0-line and a *tangent* line is a 1-line.

Theorem 1.1. (Barlotti 1955; Panella 1995) *A subset \mathcal{O} of $q^2 + 1$ points of $\text{PG}(3, q)$, q odd, of line-type $(0, 1, 2)_1$ is an elliptic quadric.*

A key step in the proof of Barlotti is the fact that at each point of the set \mathcal{O} there are exactly $q+1$ coplanar tangent lines, that is on each point of the set \mathcal{O} the number of tangent lines is small with respect to the number of 2-lines. Vice versa, a set of points of $\text{PG}(3, q)$ of line-type $(0, 1, 2)_1$ and with at least one point on exactly $(q+1)$ tangent lines has size q^2+1 . So the assumption on the size of sets of line-type $(0, 1, 2)_1$ made by Barlotti and Panella is equivalent to assume that there is a point of the set on $q+1$ tangent lines.

Starting from this observation, we have considered sets of line-type $(0, m, n)_1$ and with few m -lines through one of their point, obtaining the following result.

Theorem 1.2. *Let m, n be two integers with $0 < m < n$. A subset \mathcal{O} of points of $\text{PG}(3, q)$, q odd, of line type $(0, m, n)_1$, $n \neq q$, with at least one point p on which there are at most $q+1$ m -lines is either an elliptic quadric or $n = q+1$ and \mathcal{O} is the complement of a line in $\text{PG}(3, q)$.*

Note that, in addition to the complement of a line in $\text{PG}(3, q)$, there are sets of points of $\text{PG}(3, q)$, q odd, of line type $(0, m, q)_1$ with few m -lines at each one of their points, which are not elliptic quadrics. Let π be a plane of $\text{PG}(3, q)$ and p be a point of $\text{PG}(3, q)$ not in π . Then the complement of $\pi \cup \{p\}$ in $\text{PG}(3, q)$ is a set of line-type $(0, q-1, q)_1$ and each point belongs to exactly one $(q-1)$ -line.

We end this section by recalling the definition of maximal arc of $\text{PG}(2, q)$ and a result on their non-existence for odd q , which will play a special role in the proof of Theorem 1.2.

Let $0 < n \leq q+1$ be an integer. A *maximal arc* in $\text{PG}(2, q)$ is a set of points of $\text{PG}(2, q)$ of line-type $(0, n)_1$. A maximal arc is *non-trivial* if $1 < n < q$. Single points and the complement of a line in a projective plane are examples of trivial maximal arc. Hyperovals ($n = 2$) and dual hyperovals ($n = q/2$) are examples of non-trivial maximal arcs in Desarguesian projective planes.

Theorem 1.3. (Ball *et al.* 1997) *For odd q , non-trivial maximal arcs in $\text{PG}(2, q)$ do not exist.*

2. The Proof of Theorem 1.2

Throughout this section, \mathcal{O} is a set of points of $\text{PG}(3, q)$ of line-type $(0, m, n)_1$ with $n \neq q$

Let p be a point of \mathcal{O} and μ_p be the number of m -lines containing p . Counting the number of points of \mathcal{O} via the lines on p gives:

$$|\mathcal{O}| = 1 + \mu_p(m-1) + (q^2 + q + 1 - \mu_p)(n-1) = 1 + (q^2 + q + 1)(n-1) - \mu_p(n-m). \quad (2.1)$$

It follows that on every point of \mathcal{O} there is a constant number of m -lines, say μ . Of course $\mu \geq 1$. Under the assumption of Theorem 1.2, we also have that $\mu \leq q+1$.

Lemma 2.1. $\mu \geq 3$.

PROOF. Assume to the contrary that $\mu \leq 2$. Let ℓ be an n -line, and p be a point of $\ell \cap \mathcal{O}$. Since on ℓ there are at least three different planes and there are at most two m -lines on p , it follows that there is at least one plane, say π , through ℓ with no m -line passing

through p . Thus, π intersects \mathcal{O} in $1 + (q + 1)(n - 1)$ points and so π contains no m -line. If $n = q + 1$, the plane π is contained in \mathcal{O} and so each line of $\text{PG}(3, q)$ has at least one point in common with \mathcal{O} contradicting the assumption that \mathcal{O} is of line-type $(0, m, n)_1$. Thus, $n \leq q$ and so there is at least one point of π not in \mathcal{O} . Let p be a point of $\pi \setminus \mathcal{O}$. If all the lines through p are n -lines, counting the size of $\pi \cap \mathcal{O}$ via the lines on p gives $|\pi \cap \mathcal{O}| = (q + 1)n > 1 + (q + 1)(n - 1)$, a contradiction. Hence π contains external lines to \mathcal{O} . It follows that $\pi \cap \mathcal{O}$ is a set of line type $(0, n)_1$, with $n \geq 2$, in a projective plane of odd order. So, by Theorem 1.3 and since $n \neq q$ it follows that $n = 1$, a contradiction. \square

Lemma 2.2. *The m -lines on any point p of \mathcal{O} are coplanar.*

PROOF. By Lemma 2.1 on every point of \mathcal{O} there are at least three different m -lines. Let p be a point of \mathcal{O} . Assume by the way of contradiction that there are three non-coplanar m -lines t_1, t_2 and t_3 on p . Let α be the plane containing t_1 and t_2 . The plane α contains an n -line through p since there are at most q m -lines through p contained in α . Let ℓ be such an n -line. At least one plane, say π , through ℓ contains no m -line through p . So, as in the proof of Lemma 2.1, we get a contradiction. \square

Lemma 2.3. *$\mu = q + 1$ and either $m = 1$ or $m = q$, $n = q + 1$ and \mathcal{O} is the complement of a line in $\text{PG}(3, q)$.*

PROOF. Assume that $\mu \leq q$. Let p be a point of \mathcal{O} . Let π be the plane through p containing all the m -lines on p . Since $\mu \leq q$, π contains at least one n line, say ℓ , passing through p . Thus, ℓ is contained in at least one plane, say α , with no m -line on p . The same argument used in the proof of the previous lemma shows that either α is contained in \mathcal{O} and so there is no external line to \mathcal{O} or $n = 1$, which is not possible. Therefore, $\mu = q + 1$. Hence, for any point p the plane containing the $q + 1$ m -lines intersects \mathcal{O} in a set of line-type $(0, m)_1$ and so, since $m < n \leq q + 1$, by Theorem 1.3 it follows that either $m = 1$ or $m = q$ and $n = q + 1$. \square

Since \mathcal{O} is of line-type $(0, m, n)_1$, \mathcal{O} has at least one external line L_0 . If $m = q$ and $n = q + 1$, then by Equation (2.1) with $\mu = q + 1$ we get $|\mathcal{O}| = q^3 + q^2$. It follows that \mathcal{O} is the complement of L_0 in $\text{PG}(3, q)$.

Finally, let us consider the case $m = 1$. From Lemmas 2.2 and 2.3 it follows that \mathcal{O} is a semiovoid and so by a theorem of Thas (1974) it follows that \mathcal{O} is an elliptic quadric. However, here we also give a direct proof to not recall further definitions and to make the reading of the article as much as possible self-contained. Since $m = 1$ and $\mu = q + 1$, by Equation (2.1) it follows that $|\mathcal{O}| = q^2(n - 1) + 1$.

If π is a plane intersecting \mathcal{O} in at least two points, the line ℓ connecting these points is an n -line and on each point of $\ell \cap \mathcal{O}$ there is one tangent line and q n -lines, so $|\pi \cap \mathcal{O}| = 1 + q(n - 1)$.

Let E be an external line to \mathcal{O} , let x be the number of tangent planes on E and let y be the number of planes intersecting \mathcal{O} in at least two points and containing E . Then $x + y \leq q + 1$.

We have:

$$q^2(n - 1) + 1 = |\mathcal{O}| = x + y(q(n - 1) + 1)$$

that is

$$q \geq x + y - 1 = q(n - 1)(q - y) \quad (2.2)$$

and so

$$1 \geq (n - 1)(q - y). \quad (2.3)$$

Since $n \geq 2$, from (2.3) it follows that $y \geq q - 1$.

If $y = q + 1$ then $x = 0$ and the equality in (2.2) becomes $q = -q(n - 1)$ which is not possible. If $y = q$, then (2.2) gives $x + y - 1 = 0$ and so either $x = 1$ and $y = 0$ or $x = 0$ and $y = 1$. Both possibilities contradicts the fact that $y = q \geq 2$. Hence $y = q - 1$, and $n = 2$ by (2.3). From the inequality in (2.2) it follows that $x = 2$. Hence \mathcal{O} is a set of size $q^2 + 1$ and of line-type $(0, 1, 2)_1$ and by Theorem 1.1 it follows that \mathcal{O} is an elliptic quadric. This, completes the proof of Theorem 1.2.

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