DOI: 10.1478/AAPP.96S1A1

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 96, No. S1, A1 (2018)

SPECULATIVE AND HEDGING INTERACTION MODEL IN OIL AND U.S. DOLLAR MARKETS: NASH EQUILIBRIA FOR ONE OR MORE BANKS

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ABSTRACT. We determine the (refined) Nash equilibria for a bounded rational Carfì-Musolino speculative and hedging model. This model shows two types of operators: a real economic subject (Air) and one or more investment banks (Bank). We consider the bank agents' behavior to equilibrate much more quickly than that of Air, as they react to the move of Air. In this sense, Air is an acting external agent, whereas the action of the banks is 'annealed' – *i.e.*, equilibrates before Air makes its next transaction. When Air makes no purchases of oil futures as a hedge, two Nash equilibria exist for the bank agents. However, a unique Nash equilibrium exists for the bank agents when Air makes a purchase. This is a result of Air's purchase breaking a symmetry of the potential. The existence of multiple equilibriums in this two-market model is in the spirit of the Sonnenschein-Mantel-Debreu theorem, and is associated with phase transitions in statistical mechanics.

1. Introduction

A bounded rational Carfì-Musolino speculative and hedging model involving market transactions among a real economic subject ("Air") and a finite number of investment players ("Banks") was studied by Carfì and Campbell (2015). There it was shown that the model is a potential game (Monderer and Shapley 1996) and that the Gibbs measure from statistical mechanics is the bounded rational equilibrium for Air and any finite number of Bank subjects reacting to the transaction of Air.

For potential games, the maximum of the potential is considered to be an appropriate refinement of the Nash equilibrium (Carbonell-Nicolau and McLean 2014). The potential turns out to have a symmetry when Air does not make any purchases of oil futures as a hedge. This symmetry results in two maximums for the potential – *i.e.*, two equilibriums. The Sonnenschein-Mantel-Debreu theorem (Sonnenschein 1972, 1973; Debreu 1974; Mantel 1974) implies that a unique equilibrium may not exist for multiple markets that interact with each other. This is indeed the case here. It is also seen that when Air makes a purchase, the

This paper is dedicated to the memory of Prof. Gaetano Giaquinta (1945–2016) symmetry of the potential is broken. This results in a unique maximum, and hence a unique equilibrium exists in this case.

1.1. Literature review. In this paper, we shall refer to a wide variety of literature. First of all, we shall consider some papers on the complete study of differentiable games and related mathematical backgrounds, introduced and applied to economic theories since 2008 by Carfì and coworkers (Carfì 2008, 2009; Carfì and Ricciardello 2009, 2010; Agreste *et al.* 2012; Baglieri *et al.* 2012; Carfì and Ricciardello 2012a,b; Carfì and Schilirò 2012a,b,c; Carfì and Ricciardello 2013).

Specific applications of the previous methodologies, also strictly related to the present model, have been illustrated by Carfì and Musolino (2012a,b, 2013a,b,c, 2014a,b, 2015a,b).

Other important applications of the complete examination methodology were introduced by Agreste *et al.* (2012), Baglieri *et al.* (2012), Carfì and Ricciardello (2012a,b), Carfì and Schilirò (2012a,b,c), Carfì and Ricciardello (2013), and Carfì and Campbell (2015).

For other physics and economic bases we shall refer to Sonnenschein (1972, 1973), Debreu (1974), Mantel (1974), Beightler and Wilde (1996), Monderer and Shapley (1996), Campbell (2005), and Carbonell-Nicolau and McLean (2014).

In this section, for convenience of the reader, we reconsider the game model analyzed by Carfì and Campbell (2015), without significant changes.

1.2. Preliminary survey of the model. As discussed by Carfì and Musolino (2014b) and Carfì and Campbell (2015), we consider a game with a finite number N of "Bank" (investment) players, and all of these players belong to the set Λ. At any moment in time, a Bank player $i \in \Lambda$ can select an action or strategy

$$(y_1^{(i)}, y_2^{(i)}) \in F$$

and the $y_1^{(i)}$ and $y_2^{(i)}$ are the **strategy variables**.

The strategy $y_1^{(i)}$ represents the proportion of its resources that Bank i spends on the oil spot market, and $y_2^{(i)}$ represents the proportion of its resources spent on the (US) dollar futures market from its total resources $M > 0$.

A **configuration** $\vec{y}$ of the system is any possible state of the system:

$$\vec{y} = \left((y_1^{(1)}, y_2^{(1)}), (y_1^{(2)}, y_2^{(2)}), \ldots, (y_1^{(N)}, y_2^{(N)}) \right), \tag{1}$$

where each

$$(y_1^{(i)}, y_2^{(i)}) \in F\,.$$

The set of all possible configurations of the game is

$$\Phi_\Lambda := \prod_{i \in \Lambda} F^{(i)}, \tag{2}$$

which is called (pure) **state space**. The

$$F^{(i)} := F$$

here is the diamond set

$$F := \left\{ \vec{y} \in (\mathbb{R}^2)^N : \|(y_1^{(i)}, y_2^{(i)})\|_1 = \left| y_1^{(i)} \right| + \left| y_2^{(i)} \right| \leq 1 \right\}. \tag{3}$$

Now we will define the payoff functions as done by Carfì and Musolino (2014b).

The real economic subject ("Air") is a player in the game, and is assigned zero as its player number. Its strategy variable

$$x \in [0,1]$$

represents the proportion of its resources $M^{(0)}$ spent on purchasing oil futures as a hedge. The remaining proportion of Air's resources, $1-x$, is spent purchasing jet fuel on the spot market. The oil payoff function for Air is then what it spent: the amount of jet fuel it bought on the spot market,

$$(1-x)M^{(0)},$$

multiplied by its savings on the price of jet fuel (hedge prices minus the actual prices, which depends on the actions of Bank and contains a *negated* [1] component related to what Bank spent on US dollar futures).

In the case of only a single Bank (player 1), this reduces to (Carfì and Musolino 2014b)

$$f_O^{(0)}(x,\vec{y}) = M^{(0)}(1-x)(u(un-\nu)y_1^{(1)} + uky_2^{(1)}), \tag{4}$$

where u is the capitalization factor ($1+r$, for risk-free interest rate r) resulting from the transaction occuring in the previous time step. The parameter

$$n > 0$$

represents the effect of Bank's strategy $y_1^{(1)}$ on the oil spot market price at time 1, and

$$k > 0$$

is the negative influence of Bank's strategy $y_2^{(1)}$ on the price of oil futures. Both n and k depend on Bank's ability to influence the oil spot market and the behavior of other financial agents. A tax parameter ν, with

$$0 \leq \nu \leq nu,$$

can be set within the range of no taxation ($\nu = 0$) to full taxation ($\nu = un$).

For the single Bank player, the Bank payoff function for dollar futures is the product of the amount purchased, and returns per unit of dollar futures:

$$f_\$^{(1)}(x,\vec{y}) = y_2^{(1)}M(-u^2ny_1^{(1)} + u(k-\kappa)y_2^{(1)} - umx), \tag{5}$$

where

$$k - \kappa = 0$$

in Carfì and Musolino (2014b) as a result of a tax, and there Bank gains nothing from its actions on the oil spot market. Here, we can vary κ within

$$0 \leq \kappa \leq k$$

to represent the range from no taxation ($\kappa = 0$) to full taxation ($\kappa = k$). The parameter

$$m > 0$$

measures the influence of Air's strategy x on the price of oil futures and the ability of Air to influence the oil market and the behavior of other financial agents.

[1] It is pointed out by Carfì and Musolino (2014b), and references therein, that rises in oil prices are associated with the depreciation of the US dollar. Hence there is a leading negative sign in front of $y_1^{(1)}$ in Eq. 5.

Similarly, Bank's payoff function from the oil spot market is the product of the amount purchased and returns per unit:

$$f_O^{(1)}(x,\vec{y}) = y_1^{(1)} M((un-\nu)y_1^{(1)} + u\delta y_2^{(1)}), \tag{6}$$

where as above, the tax paramter ν is set so that

$$un - \nu = 0$$

in Carfì and Musolino (2014b) as a result of a tax (we can take $0 \leq \nu \leq un$ to represent the range from full taxation to no taxation), and

$$m > 0$$

measures the influence of Air's strategy x on the price of oil futures and the ability of Air to influence the oil market and the behavior of other financial agents.

The model can be generalized to a single large-scale economic subject (Air player zero) and many investors (Bank players each labeled i, $1 \leq i \leq N$). For simplicity, we assume the Bank players are *interaction homogeneous*[2] , so that they all have the same resources and are identically affected by each other and by Air, within markets.

The **gains of each Bank *i* from the dollar futures market** would then be affected by Air and *all* Bank players:

$$p_\$(x,\vec{y}) = \sum_{j=1}^{N}\left(-\frac{u^2 n}{N} y_1^{(j)} + \frac{u(k-\kappa)}{N} y_2^{(j)}\right) - umx, \tag{7}$$

where the "interaction" terms are

$$-\frac{u^2 n}{N} \qquad \text{and} \qquad \frac{u(k-\kappa)}{N},$$

and the "field" term umx.

As mentioned by Campbell (2005), the interaction terms are divided by N so that demand is based on the average production. Thus demand stays nonnegative for large N. For example, if each Bank used all resources for oil spot market purchases (all $y_1^{(j)} = 1$), then

$$p_\$(x, \overrightarrow{(1,0)}) = -u^2 n - umx$$

is well-behaved and non-trivial (*i.e.*, doesn't go to negative infinity or zero). In a similar manner, the **gain from the oil spot market for each Bank agent** would be

$$p_O(x,\vec{y}) = \sum_{j=1}^{N}\left(\frac{un-\nu}{N} y_1^{(j)} - \frac{u\delta}{N} y_2^{(j)}\right). \tag{8}$$

From these gains, Bank i ($1 \leq i \leq N$) has oil spot market payoff function

$$f_O^{(i)} = y_1^{(i)} M p_O, \tag{9}$$

and dollar futures market payoff

$$f_\$^{(i)} = y_2^{(i)} M p_\$. \tag{10}$$

[2] Bank agents are *heterogeneous* agents, since they can play different strategies.

Adding these yields the **payoff function for Bank** *i* below:

$$\begin{aligned} f^{(i)}(x,\vec{y}) &= f_O^{(i)} + f_\$^{(i)} = \\ &= y_1^{(i)} M \sum_{j=1}^{N} \left(\frac{un-\nu}{N} y_1^{(j)} - \frac{u\delta}{N} y_2^{(j)} \right) + \\ &\quad + y_2^{(i)} M \sum_{j=1}^{N} \left(-\frac{u^2 n}{N} y_1^{(j)} + \frac{u(k-\kappa)}{N} y_2^{(j)} \right) - uMmxy_2^{(i)} \end{aligned} \tag{11}$$

For brevity, we relabel the four interaction terms and field term as

$$E := M(un-\nu) \geq 0, \tag{12}$$
$$D := Mu\delta \geq 0, \tag{13}$$
$$K := Mu^2 n \geq 0, \tag{14}$$
$$J := Mu(k-\kappa) \geq 0, \tag{15}$$
$$h_x := -uMmx \leq 0. \tag{16}$$

1.3. Potential Games. A potential game (Monderer and Shapley 1996) with potential $V(\vec{q})$ and payoff functions $f_i(\vec{q})$, with

$$\vec{q} = (q_1, ..., q_N),$$

for each agent $i \in \Lambda$ satisfies, by definition,

$$\frac{\partial}{\partial q_i} f_i(\vec{q}) = \frac{\partial}{\partial q_i} V(\vec{q}). \tag{17}$$

The salient point is that, for each *i*, the gradient of the potential with respect to the variables of agent *i* is the same as the gradient of the *i*th agent's payoff (with respect the *i*th agent's variables). In a dynamical interpretation, agents would follow the gradient of their payoff function for "myopic decisions" (agents look at the best *local* choice), and for potentials with an interior maximum, this would lead to the Nash equilibrium (Monderer and Shapley 1996).

1.4. Potential analysis of the model. In the model presented here, each bank agent has two variables:

$$(y_1^{(i)}, y_2^{(i)}).$$

The conditions in Eq. 17 above for a potential require the "externality symmetry" condition

$$D = K, \qquad i.e., \qquad u\delta = u^2 n$$

which is to say the negative correlation of the US dollar and oil markets must have the same effect on each other (accounting for *u*) for there to be a potential. If this is the case, then the potential for the payoff functions in Eq. 11 is:

$$\begin{aligned} V(h_x,\vec{y}) &= \sum_{i,j=1}^{N} \frac{E}{2N} y_1^{(i)} y_1^{(j)} + \sum_{i,j=1}^{N} \frac{J}{2N} y_2^{(i)} y_2^{(j)} - \frac{K}{N} \sum_{i,j=1}^{N} y_1^{(i)} y_2^{(j)} + \\ & -\frac{K}{N} \sum_{i=1}^{N} y_1^{(i)} y_2^{(i)} + \frac{E}{2N} \sum_{i=1}^{N} \left[y_1^{(i)}\right]^2 + \frac{J}{2N} \sum_{i=1}^{N} \left[y_2^{(i)}\right]^2 + \\ & +h_x \sum_{i=1}^{N} y_2^{(i)} \end{aligned} \tag{18}$$

For computations, it will be easier to change variables from $\vec{y}$ to $\vec{v} \in \Omega_\Lambda$, where for the agents i $(1 \leq i \leq N)$ in the set Λ,

$$v_1^{(i)} = \left(y_2^{(i)} + y_1^{(i)}\right)/2, \quad v_2^{(i)} = \left(y_2^{(i)} - y_1^{(i)}\right)/2, \tag{19}$$

$v_\alpha^{(i)} \in [-1/2, 1/2]$ for $\alpha = 1, 2$, the rotated domain

$$\tilde{F}^{(i)} := [-1/2, 1/2] \times [-1/2, 1/2] \tag{20}$$

is a square, and the rotated configuration space is

$$\Omega_\Lambda := \prod_{i \in \Lambda} \tilde{F}^{(i)} = [-1/2, 1/2]^{2N}. \tag{21}$$

The potential is then

$$\begin{aligned} V(h_x,\vec{v}) &= \sum_{i,j=1}^{N} \frac{I_-}{N} v_1^{(i)} v_1^{(j)} + \sum_{i,j=1}^{N} \frac{I_+}{N} v_2^{(i)} v_2^{(j)} - \frac{2I}{N} \sum_{i,j=1}^{N} v_1^{(i)} v_2^{(j)} + \\ & -\frac{2I}{N} \sum_{i=1}^{N} v_1^{(i)} v_2^{(i)} + \frac{I_-}{N} \sum_{i=1}^{N} \left[v_1^{(i)}\right]^2 + \frac{I_+}{N} \sum_{i=1}^{N} \left[v_2^{(i)}\right]^2 + \\ & +h_x \sum_{i=1}^{N} \left(v_1^{(i)} + v_2^{(i)}\right) \end{aligned} \tag{22}$$

where

$$I := (J-E)/2, \tag{23}$$
$$I_+ := (J+E)/2 + K, \tag{24}$$
$$I_- := (J+E)/2 - K. \tag{25}$$

1.5. Parameter Analysis. The general possibilities for the parameters result from the possibilities for J, E, and K in the definitions 23, 24, 25, and the parameter

$$\begin{aligned} \Delta := \frac{I_- I_+ - I^2}{I_-} &= \frac{[(J+E)/2-K][(J+E)/2+K] - (J-E)^2/4}{(J+E)/2-K} = \\ &= \frac{(\sqrt{JE}+K)(\sqrt{JE}-K)}{(J+E)/2-K}. \end{aligned} \tag{26}$$

The following assumptions generalize those made about the model by Carfì and Musolino (2014b):

$$\textbf{Assumption} \qquad I_- < 0. \tag{27}$$

$$\textbf{Assumption} \qquad \left|\frac{I}{I_-}\right| < 1. \tag{28}$$

The first assumption (Eq. 27) is the same as in Carfì and Musolino (2014b), and the latter assumption (Eq. 28) generalizes the $I = 0$ assumption in Carfì and Musolino (2014b) by keeping I small relative to I_-. The assumption $I_- < 0$ implies $K > 0$, and because the arithmetic mean is greater than or equal to the geometric mean we have

$$\textbf{Consequence} \qquad I_+ > 0, \Delta > 0, \tag{29}$$

since

$$\sqrt{JE} - K \leq (J+E)/2 - K = I_- < 0,$$
$$J \geq 0, \qquad E \geq 0, \qquad K > 0,$$

and thus

$$\sqrt{JE} + K > 0.$$

2. Nash Equilibrium

2.1. Existing results. We will now find the maximum(s) of the potential in Eq. 18. This is the appropriate refinement of the Nash equilibrium that we consider here (see Carbonell-Nicolau and McLean 2014). The potential is quadratic in the $v_\alpha^{(i)}$, with

$$1 \leq i \leq N, \qquad \alpha = 1, 2$$

and by continuity it will have a maximum on the domain Ω_Λ, which may occur in the interior or on the boundary depending on the parameters I, I_+, and I_-. To this end, the second-degree part of the potential is an

$$2N \times 2N$$

quadratic form Q equal to half of the second-derivative matrix D^2V $(1 \leq i, j \leq N; 1 \leq \alpha, \bar{\alpha} \leq 2)$,

$$Q = \frac{1}{2}\left[\frac{\partial^2 V}{\partial v_\alpha^{(i)} \partial v_{\bar{\alpha}}^{(j)}}\right] = \frac{1}{N}\left[\begin{array}{c|c} I_- Q_0 & I Q_0 \\ \hline I Q_0 & I_+ Q_0 \end{array}\right] \tag{30}$$

with

$$Q_0 = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{bmatrix}$$

where

- the upper-left quadrant of the matrix contains the $\alpha = \bar{\alpha} = 1$ terms,
- the upper-right contains the $\alpha = 1, \bar{\alpha} = 2$ terms,
- the lower-left has $\alpha = 2, \bar{\alpha} = 1$ terms,

- the lower-right has $\alpha = \bar{\alpha} = 2$ terms.

We will use the LDL^* decomposition (L is an invertible, lower-triangular matrix with ones on the diagonal, L^* is the transpose of L, D is a diagonal matrix) of the symmetric quadratic form Q of the potential V in Eq. 30 to facilitate finding the maximum of the potential V. The matrix L is determined from Q using elementary row operations (EROs) as outlined by Beightler and Wilde (1996). The first step is to reduce Q to upper triangular form using EROs, and then to determine D and L, which is done in Appendix A of Carfì and Campbell (2015).

Now that the quadratic form corresponding to the potential V has been diagonalized as

$$Q = LDL^*,$$

we need to find the maximum of $V(\vec{v})$ on its domain

$$\Omega_\Lambda = [-1/2, 1/2]^{2N}.$$

The potential can be written as inner products

$$\begin{aligned} V(\vec{v}) &= \langle \vec{v}, Q\vec{v} \rangle + h_x \left\langle \vec{1}, \vec{v} \right\rangle = \\ &= \langle \vec{v}, LDL^*\vec{v} \rangle + h_x \left\langle \vec{1}, (L^*)^{-1} L^* \vec{v} \right\rangle = \\ &= \langle L^*\vec{v}, DL^*\vec{v} \rangle + h_x \left\langle L^{-1}\vec{1}, L^*\vec{v} \right\rangle, \end{aligned} \tag{31}$$

where

$$\vec{1}_M := [1\ 1\ \cdots\ 1]^*, \qquad \vec{1} := \vec{1}_{2N}, \tag{32}$$

are the column vectors with M rows and $2N$ rows, respectively, having all entries equal to one. Using

$$\vec{\mathrm{v}} := L^*\vec{v}, \tag{33}$$

the matrix D can be written as a direct sum of its $N \times N$ negative definite and positive definite submatrices

$$D = D_- \oplus D_+, \tag{34}$$

and the vector $\vec{\mathrm{v}}$ can be decomposed into a direct sum over the subspaces corresponding to the N-dimensional negative definite and positive definite submatrices of D as

$$\vec{\mathrm{v}} = \vec{\mathrm{v}}_- \oplus \vec{\mathrm{v}}_+. \tag{35}$$

The same can be done for

$$\vec{M} := L^{-1}\vec{1} \tag{36}$$

to get

$$\vec{M} = \vec{M}_- \oplus \vec{M}_+. \tag{37}$$

This direct sum decomposition will split the inner products as

$$\begin{aligned} V(\vec{v}) &= \langle (D_- \oplus D_+)(\vec{v}_- \oplus \vec{v}_+), \vec{v}_- \oplus \vec{v}_+ \rangle + h_x \left\langle \vec{M}_- \oplus \vec{M}_+, \vec{v}_- \oplus \vec{v}_+ \right\rangle = \\ &= \langle D_- \vec{v}_-, \vec{v}_- \rangle + h_x \left\langle \vec{M}_-, \vec{v}_- \right\rangle + \langle D_+ \vec{v}_+, \vec{v}_+ \rangle + h_x \left\langle \vec{M}_+, \vec{v}_+ \right\rangle = \\ &= -\left\| \, |D_-|^{1/2} \vec{v}_- - \frac{h_x}{2} |D_-|^{-1/2} \vec{M}_- \right\|^2 + \\ &\quad + \left\| D_+^{1/2} \vec{v}_+ + \frac{h_x}{2} D_+^{-1/2} \vec{M}_+ \right\|^2 + \\ &\quad + \frac{h_x^2}{4} \left\| \, |D_-|^{-1/2} \vec{M}_- \right\|^2 - \frac{h_x^2}{4} \left\| D_+^{-1/2} \vec{M}_+ \right\|^2 , \end{aligned} \tag{38}$$

where the last line is a result of completing the square,

$$|D_-| := -D_- \geq 0 \tag{39}$$

as a matrix, and the square root of a diagonal matrix with non-negative entries d_{ii} is the matrix having diagonal entries $\sqrt{d_{ii}}$.

2.2. Nash Analysis: first case. We first find the Nash equilibrium for the case when $h_x \neq 0$; when Air purchases some amount of oil futures. First, using the upper-diagonal form for L^* in Eq. (44) of Carfì and Campbell (2015), we decompose the matrix L^* as

$$L^* = \left[\begin{array}{c|c} L_1^* & \left(\frac{I}{I_-} \right) L_1^* \\ \hline \mathbf{0} & L_1^* \end{array} \right] \tag{40}$$

where L_1^* is the $N \times N$ submatrix of L^* appearing in the upper-left quadrant of L^* (L^* can be found explicitly in Eq. (44) of Carfì and Campbell (2015)). Then using the notation of Eq. 38, we can use a direct sum to represent the vector

$$\vec{v} \in \Phi_\Lambda$$

as

$$\vec{v} = \vec{v}_t \oplus \vec{v}_b, \tag{41}$$

where

$$\vec{v}_t, \vec{v}_b \in [-1/2, 1/2]^N.$$

Furthermore,

$$L^* \vec{v} = \left(L_1^* \vec{v}_t + \frac{I}{I_-} L_1^* \vec{v}_b \right) \oplus L_1^* \vec{v}_b = \vec{v}_- \oplus \vec{v}_+, \tag{42}$$

$$DL^* \vec{v} = D_- \left(L_1^* \vec{v}_t + \frac{I}{I_-} L_1^* \vec{v}_b \right) \oplus D_+ L_1^* \vec{v}_b = D_- \vec{v}_- \oplus D_+ \vec{v}_+. \tag{43}$$

From Eq. 38, we define the **restricted potential**

$$\begin{aligned} V^{\vec{v}_{t_0}}(\vec{v}_b) \;:=\; & \left\langle D_-\left(L_1^*\vec{v}_{t_0}+\frac{I}{I_-}L_1^*\vec{v}_b\right), L_1^*\vec{v}_{t_0}+\frac{I}{I_-}L_1^*\vec{v}_b\right\rangle + \langle D_+L_1^*\vec{v}_b, L_1^*\vec{v}_b\rangle + \\ & + \left\langle h_x \mathrm{Proj}_- L^{-1}\vec{1}, L_1^*\vec{v}_{t_0}+\frac{I}{I_-}L_1^*\vec{v}_b\right\rangle + \left\langle h_x \mathrm{Proj}_+ L^{-1}\vec{1}, L_1^*\vec{v}_b\right\rangle. \end{aligned} \tag{44}$$

The expression in Eq. 44 can be simplified by decomposing L^{-1} as

$$L^{-1} = \left[\begin{array}{c|c} L_1^{-1} & \mathbf{0} \\ \hline \left(\frac{I}{I_-}\right)L_1^{-1} & L_1^{-1} \end{array}\right], \tag{45}$$

where L_1^{-1} is the $N \times N$ submatrix of L^{-1} appearing in the upper-left quadrant of L^{-1} (the transpose of L^{-1} is seen explicitly in equation (55) of Carfì and Campbell (2015)). Some calculation shows

$$\mathrm{Proj}_- L^{-1}\vec{1}_{2N} \;=\; L_1^{-1}\vec{1}_N, \tag{46}$$

$$\mathrm{Proj}_+ L^{-1}\vec{1}_{2N} \;=\; \left(1-\frac{I}{I_-}\right)L_1^{-1}\vec{1}_N. \tag{47}$$

We verify convexity of $V^{\vec{v}_{t_0}}$ in

$$\vec{v}_b \in [-1/2, 1/2]^N$$

by substituting Eqs. 46, 47, expanding, and rearranging Eq. 44:

$$\begin{aligned} V^{\vec{v}_{t_0}}(\vec{v}_b) \;:=\; & \left\langle \left(D_+ + \frac{I^2}{I_-^2}D_-\right)L_1^*\vec{v}_b, L_1^*\vec{v}_b\right\rangle + \langle D_-L_1^*\vec{v}_{t_0}, L_1^*\vec{v}_{t_0}\rangle + \\ & +2\left\langle D_-L_1^*\vec{v}_{t_0}, \frac{I}{I_-}L_1^*\vec{v}_b\right\rangle + \left\langle h_x L_1^{-1}\vec{1}_N, L_1^*\vec{v}_{t_0}+\frac{I}{I_-}L_1^*\vec{v}_b\right\rangle + \\ & + \left\langle h_x\left(1-\frac{I}{I_-}\right)L_1^{-1}\vec{1}_N, L_1^*\vec{v}_b\right\rangle, \end{aligned} \tag{48}$$

which is convex if

$$D_+ + \frac{I^2}{I_-^2}D_- \geq 0. \tag{49}$$

To see this, define

$$D_0 := \mathrm{diag}\left[2/1, \ldots, (k+1)/k, \ldots, (N+1)/N\right]. \tag{50}$$

From the decomposition for D in (45) of Appendix A of Carfì and Campbell (2015), it is seen that

$$D_- = \frac{I_-}{N}D_0, \tag{51}$$

$$D_+ = \frac{\Delta}{N}D_0, \tag{52}$$

and that

$$D_+ + \frac{I^2}{I_-^2} D_- = \frac{1}{N}\left(\Delta + \frac{I^2}{I_-}\right) D_0 = \\ = \frac{I_+}{N} D_0 > 0 \quad (53)$$

since $I_+ > 0$ from Eq. 24 which, along with $D_0 > 0$, implies that $V^{\vec{v}_{t_0}}$ is convex in $\vec{v}_b$. Consequently, its maximum must occur at one or more extreme points of the domain for any such fixed $\vec{v}_{t_0}$. It is then evident that any point at which a maximum of the potential V occurs must be of the form

$$\vec{v}^{\wedge} = \vec{v}_t \oplus \vec{v}_e, \quad (54)$$

where

$$\vec{v}_t \in [-1/2, 1/2]^N$$

and $\vec{v}_e$ is an extreme point which we write as

$$\vec{v}_e = \frac{1}{2}\left[(-1)^{p_1} (-1)^{p_1} \cdots (-1)^{p_N}\right]^*, \quad (55)$$

with $p_i = 0$ or $p_i = 1$ for $1 \leq i \leq N$.

The norm-form of the potential in Eq. 38 can be written

$$V(\vec{v}_{t_0} \oplus \vec{v}_e) = -\left\| |D_-|^{1/2}\left(L_1^* \vec{v}_{t_0} + \frac{I}{I_-} L_1^* \vec{v}_e\right) - \frac{h_x}{2} |D_-|^{-1/2} L_1^{-1} \vec{1}_N \right\|^2 + \\ + \left\| D_+^{1/2} L_1^* \vec{v}_e + \frac{h_x}{2} D_+^{-1/2}\left(1 - \frac{I}{I_-}\right) L^{-1} \vec{1}_N \right\|^2 + \\ + \frac{h_x^2}{4} \left\| |D_-|^{-1/2} L_1^{-1} \vec{1}_N \right\|^2 + \\ - \frac{h_x^2}{4} \left\| D_+^{-1/2}\left(1 - \frac{I}{I_-}\right) L^{-1} \vec{1}_N \right\|^2. \quad (56)$$

Since the maximum of the restricted potential in Eq. 44 occurs at an extreme point for each $\vec{v}_{t_0}$, finding the maximum of the potential in Eq. 56 amounts to (1) minimizing the first negative norm term of V in $\vec{v}_{t_0}$ (Eq. 56) for an arbitrary extreme

$$\vec{v}_b = \vec{v}_e$$

and then (2) finding the extreme point $\vec{v}_e$ that maximizes V as shown below. The negative norm term in Eq. 56 is minimized when the term

$$|D_-|^{1/2} L_1^* \vec{v}_{t_0}$$

in the first norm is antiparallel to the other terms, which amounts to

$$\vec{v}_{t_0} = -\omega\left(\frac{I}{I_-}\vec{v}_e - \frac{h_x}{2}|D_-|^{-1/2} L_1^{-1} \vec{1}_N\right), \qquad 0 \leq \omega \leq 1 \quad (57)$$

and ω is the greatest number so that $\vec{v}_t$ is still in its domain

$$[-1/2, 1/2]^N.$$

Multiplying out the matrices in Eq. 57 results in

$$\vec{v}_\omega := \vec{v}_{t_0} = -\omega\left(\frac{I}{I_-}\vec{v}_e + \frac{h_x}{2I_-}\frac{N}{N+1}\vec{1}_N\right). \tag{58}$$

Using Eqs. 41, 42, 43, 51, and 52, we can substitite the expression in Eq. 58 into Eq. 31 along with the extreme point $\vec{v}_b = \vec{v}_e$ to obtain

$$\begin{aligned} V(\vec{v}_\omega \oplus \vec{v}_e) &= \left\langle \frac{\Delta I_- + (1-\omega)^2 I^2}{NI_-} Q_0 \vec{v}_e\,, \vec{v}_e \right\rangle + \\ &+ \left\langle \vec{v}_e\,, h_x\left[\frac{I}{I_-}\{(\omega-1)^2-1\}+1\right]\vec{1}_N \right\rangle + \\ &+ \frac{{h_x}^2}{4I_-}\frac{N^2}{N+1}(\omega^2 - 2\omega) \end{aligned} \tag{59}$$

where

$$Q_0 := L_1 D_0 L_1^*$$

is the $N \times N$ matrix

$$Q_0 = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{bmatrix} \tag{60}$$

and we used the result

$$Q_0 \vec{1}_N = (N+1)\vec{1}_N.$$

The scalar in the first inner product of Eq. 59 is positive. This is evident since

$$I_- < 0, \qquad I_+ > 0,$$

and

$$(\omega-1)^2 - 1 \leq 0$$

imply

$$\Delta I_- + (1-\omega)^2 I^2 = I_- I_+ + [(1-\omega)^2 - 1]I^2 < 0. \tag{61}$$

Using this, the fact that Q_0 has all positive entries, and the representation for $\vec{v}_e$ in Eq. 55, the first inner product in Eq. 59 can be written

$$\sum_{i,j=1}^{N} g_{ij}\frac{(-1)^{p_i}}{2}\frac{(-1)^{p_j}}{2}, \tag{62}$$

where

$$g_{ij} > 0$$

and

$$p_i \in \{0,1\}$$

for all $1 \leq i, j \leq N$. It is then clear that the maximum of the first inner product in Eq. 59 will occur when all of the p_i are equal to each other, which happens in Eq. 55 for the domain points

$$\vec{v}_e^{\pm} = \pm\frac{1}{2}\vec{1}_N. \tag{63}$$

To determine which of the two points in Eq. 63 gives the maximum for the potential, we need to look at the second inner product of the potential in Eq. 59. The assumption in Eq. 28 along with the fact that

$$-1 \leq (\omega - 1)^2 - 1 \leq 0$$

implies that the vector in the second inner product of Eq. 59 has direction completely determined by h_x. This can be seen in the case

$$I/I_{-} < 0$$

by observing that this implies

$$\frac{I}{I_{-}}[(\omega - 1)^2 - 1] + 1 > 0. \tag{64}$$

Likewise, when

$$I/I_{-} > 0,$$

we see

$$\frac{I}{I_{-}}[(\omega - 1)^2 - 1] + 1 > (1)[-1] + 1 = 0, \tag{65}$$

since we assume

$$I/I_{-} < 1$$

from Eq. 28 in this case. In both cases, Eq. 28 implies that the coefficient of the vector $\vec{1}_N$ in Eq. 59 satisfies

$$\operatorname{sign}\left(h_x\left[\frac{I}{I_{-}}\{(\omega - 1)^2 - 1\} + 1\right]\right) = \operatorname{sign}(h_x) = -1 \tag{66}$$

from the condition for h_x in Eq. 12, with

$$\operatorname{sign}(x) := x/|x| \qquad \text{for} \quad x \neq 0.$$

From this it is seen that the second inner product in Eq. 59 is maximized by the extreme point

$$\vec{v}_e^{-} = -\frac{1}{2}\vec{1}_N, \tag{67}$$

which also maximizes the first inner product of Eq. 59. Finally, we need to determine the most maximum value for ω so that $\vec{v}_\omega$ is in the domain

$$[-1/2, 1/2]^N.$$

Using Eq. 67 in Eq. 58 yields

$$\vec{v}_\omega = \frac{1}{2}\omega\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\vec{1}_N \quad \in \quad [-1/2, 1/2]^N. \tag{68}$$

This imposes the following conditions on $\omega \in [0, 1]$:

$$-\frac{1}{2} \leq \frac{1}{2}\omega\frac{(N+1)I - Nh_x}{(N+1)I_{-}} \leq \frac{1}{2}. \tag{69}$$

From Eq. 57, we want to choose the most maximum value of

$$\omega \in [0,1]$$

subject to Eq. 69. This results in

$$\omega = \min\left(1,\ \frac{(N+1)|I_{-}|}{|(N+1)I - Nh_x|}\right). \tag{70}$$

Substituting Eq. 70 into Eq. 68 , we see that the maximum of the potential V, over its domain Ω_Λ occurs at the Nash equilibrium $\vec{v}^{\wedge}$, where

$$(v^{\wedge})_1^{(k)} = \begin{cases} \frac{1}{2}\frac{(N+1)I - Nh_x}{(N+1)I_{-}} & \text{if} \left|\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right| \leq 1, \\ \frac{1}{2}\text{sign}\left(\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right) & \text{if} \left|\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right| > 1, \end{cases} \tag{71}$$

$$(v^{\wedge})_2^{(k)} = \frac{1}{2}\text{sign}(h_x) = -\frac{1}{2}, \tag{72}$$

$$\text{for } 1 \leq k \leq N,$$

are the respective entries for the first N and last N of coordinates of $\vec{v}^{\wedge}$ and

$$\text{sign}(x) = x/|x| \qquad \text{for} \quad x \neq 0.$$

We want the Nash equilibrium coordinates in Eqs. 71 and 72 in terms of the original coordinates $\vec{y}$ in 3.

Converting back to the original "diamond" coordinates in Eq. 19, we then see that the **original coordinates of the *single* (refined) Nash equilibrium $\vec{y}^{\wedge}$ when Air purchases oil futures** ($h_x < 0$) are

$$(y^{\wedge})_1^{(k)} = \begin{cases} \frac{1}{2}\frac{(N+1)I - Nh_x}{(N+1)I_{-}} + \frac{1}{2} & \text{if} \left|\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right| \leq 1, \\ \frac{1}{2}\text{sign}\left(\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right) + \frac{1}{2} & \text{if} \left|\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right| > 1, \end{cases} \tag{73}$$

$$(y^{\wedge})_2^{(k)} = \begin{cases} \frac{1}{2}\frac{(N+1)I - Nh_x}{(N+1)I_{-}} - \frac{1}{2} & \text{if} \left|\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right| \leq 1, \\ \frac{1}{2}\text{sign}\left(\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right) - \frac{1}{2} & \text{if} \left|\frac{(N+1)I - Nh_x}{(N+1)I_{-}}\right| > 1, \end{cases} \tag{74}$$

$$\text{for } 1 \leq k \leq N,$$

as the (original) coordinates of the Nash equilibrium, which is where the maximum of the potential occurs.

2.3. Nash Analysis: zero-field case. Now we analyze the degenerate **"zero-field" case** $h_x = 0$**; when Air makes no purchase of oil futures**. The analysis is the same as the above

$h_x < 0$ case up to Eq. 57. When $h_x = 0$, Eq. 57 becomes

$$\vec{v}_{t_0} = -\omega \frac{I}{I_-} \vec{v}_e \qquad 0 \leq \omega \leq 1, \tag{75}$$

where ω is the greatest number so that $\vec{v}_t$ is still in its domain

$$[-1/2, 1/2]^N$$

as in Eq. 70:

$$\omega = \min\left(1,\ \frac{|I_-|}{|I|}\right). \tag{76}$$

Furthermore, the potential in Eq. 59 only consists of the first inner product, and the result in Eq. 63 still holds: the two points that maximize the potential are

$$\vec{v}_e^{\pm} = \pm \frac{1}{2} \vec{1}_N. \tag{77}$$

Putting together Eqs. 75, 76, and 77, we see the potential has the *two* maximum points

$$\vec{v}_{\pm}^{\wedge} = -\omega \frac{I}{I_-} \vec{v}_e \oplus \vec{v}_e$$

given by

$$(v_{\pm}^{\wedge})_1^{(k)} = \begin{cases} \pm \frac{1}{2}\left(\frac{-I}{I_-}\right) & \text{if} \left|\frac{I}{I_-}\right| \leq 1, \\ \pm \frac{1}{2} \operatorname{sign}\left(\frac{-I}{I_-}\right) & \text{if} \left|\frac{I}{I_-}\right| > 1, \end{cases} \tag{78}$$

$$(v_{\pm}^{\wedge})_2^{(k)} = \pm \frac{1}{2}, \tag{79}$$

for $1 \leq k \leq N$.

The original coordinates of the of the *two* (refined) Nash equilibriums $\vec{y}_{\pm}^{\wedge}$ when Air purchases no oil futures (*i.e.*, zero-field case $h_x = 0$) are then

$$(y_{\pm}^{\wedge})_1^{(k)} = \begin{cases} \pm \frac{1}{2}\left(\frac{-I}{I_-} - 1\right) & \text{if} \left|\frac{I}{I_-}\right| \leq 1, \\ \pm \frac{1}{2}\left(\operatorname{sign}\left(\frac{-I}{I_-}\right) - 1\right) & \text{if} \left|\frac{I}{I_-}\right| > 1, \end{cases} \tag{80}$$

$$(y_{\pm}^{\wedge})_2^{(k)} = \begin{cases} \pm \frac{1}{2}\left(\frac{-I}{I_-} + 1\right) & \text{if} \left|\frac{I}{I_-}\right| \leq 1, \\ \pm \frac{1}{2}\left(\operatorname{sign}\left(\frac{-I}{I_-}\right) + 1\right) & \text{if} \left|\frac{I}{I_-}\right| > 1, \end{cases} \tag{81}$$

for $1 \leq k \leq N$.

In general, the type of maximums of the potential depend on Q, the quadratic part of the potential, which is affected by the various cases of the parameters. There are three major cases involving the parameters that characterize the quadratic form Q. Table 1 summarizes this, and we point out that the sub-cases are not exhaustive.

TABLE 1. Quadratic Form Parameter Cases

Case	Sub-Case	Quadratic Form
I. $K \geq \frac{J+E}{2} \geq \sqrt{JE}$	(a) $K = \frac{J+E}{2}; I_- = 0, \Delta$ undefined	**degenerate**
	(b) $K > \frac{J+E}{2}; I_- < 0, \Delta > 0$	**neg-def $\oplus$ pos-def**
II. $\frac{J+E}{2} \geq K \geq \sqrt{JE}$	$I_- > 0, \Delta < 0,$	**pos-def $\oplus$ neg-def**
III. $\frac{J+E}{2} \geq \sqrt{JE} \geq K$	(a) $\frac{J+E}{2} \geq \sqrt{JE} > K; I_- > 0, \Delta > 0$	**positive definite**
	(b) $\frac{J+E}{2} > \sqrt{JE} = K; I_- > 0, \Delta = 0$	**degenerate**

In this paper, we only consider a generalization of the model set up by Carfì and Musolino (2014b), which is case I(b) in Table 1, along with the condition $|I/I_-| < 1$ in Eq. 28.

3. Conclusions

We have found the maximum(s) of the potential for this model, which is the (refined) Nash equilibrium(s). These equilibriums have some of the same characteristics as those for the Nash equilibrium discussed by Carfì and Musolino (2014b).

We have seen that when the real economic subject "Air" *does not* purchase oil futures, the field term h_x is equal to zero, and there are two Nash equilibriums due to a symmetry in the quadratic potential. This maximum of the potential occurs at two extreme points on the domain. When Air purchases oil futures, the field term h_x different from zero, affects all "Bank" investment players in such a way that there is one Nash equilibrium. Air's purchase adds a linear term to the potential which breaks the zero-field symmetry, and consequently the absolute maximum of the potential only occurs at a single point.

The existence of multiple equilibriums in this interdependent two-market system is not surprising, in light of the Sonnenschein-Mantel-Debreu theorem. Another point of view is that the occurrence of symmetry-breaking under certain conditions, which is associated with more than one Nash equilibrium, is also consequence of a phase transition in certain bounded rational economic models. These phase transitions characterize very different behaviors of agents above and below a "critical temperature", or degree of bounded rationality (Campbell 2005). The symmetry breaking in this model for N Banks, along with the persistence of two equilibria, as N tends to infinity, is motivation to investigate the existence of a phase transition for a statistically large number of Banks. This will be done in a subsequent work.

In Memoriam

The authors wish to dedicate —with humility and great respect— this little work to our friend and esteemed colleague Prof. Gaetano Giaquinta, a person who has truly enriched, substantially, the world of science, of research, of thought and its dissemination. A true scientist, standing in the eld until the last moment of his life. A tireless researcher, a scholar of those who aim at the discovery of the profound truths of nature and of the human soul, beyond any fashion and any academic need: a man of thought and of great consistency and honesty, as few remain today.

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Manuscript received 21 March 2017; published online 13 August 2018

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