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A PENALIZATION APPROACH IN SHAPE OPTIMIZATION

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ABSTRACT. We introduce a new approximation technique in general shape optimization problems, involving unknown/variable domains, that penalizes the given boundary condition in the cost functional. The essential technical ingredient is provided by the implicit parametrization theorem, using Hamiltonian systems, recently developed by the author and his collaborators. The method is of fixed domain type and reduces the original problem to a distributed control problem.

1. Introduction

Shape optimization problems arise in many applications in engineering. For instance, thickness optimization problems for plates and beams, geometric optimization problems for curved rods or shells (Neittaanmäki *et al.* 2006). Such examples take the form of optimal control problems, with the control acting in the coefficients, due to the formulation of the respective mechanical models. They are highly nonconvex and their investigation is challenging both from the theoretical and the computational points of view. General shape optimization problems involve unknown domains (Pironneau 1984; Sokolowski and Zolésio 1992; Bucur and Buttazzo 2005; Henrot and Pierre 2005; Delfour and Zolésio 2011). A typical example has the form (problem (P)):

$$\text{Min} \int_{\Omega} j(x, y(x))dx, \tag{1}$$

$$-\Delta y = f \quad \text{in } \Omega, \tag{2}$$

$$y = 0 \quad \text{in } \partial\Omega, \tag{3}$$

where $\Omega \subset D$ is the minimization parameter, an unknown open set in some prescribed family of open sets, D is a given bounded $C^{1,1}$ domain, $f \in L^2(D)$ is given and $j : D \times R \to R$ is a Caratheodory mapping. More constraints (for instance, on the state y) and more specific assumptions will be imposed as necessity appears. Other differential operators (for instance, of order four), other boundary conditions or cost functionals defined on some given subset

$E \subset \Omega \subset D$ (for any admissible open set Ω) may be considered as well in the problem (1) - (3).

One may compare (1) - (3) with optimal control problems (Lions 1971; Clarke 1990; Barbu and Precupanu 2012), the main difference and difficulty being the unknown and variable character of the geometry governing the minimization problem. This is a challenge both at the theoretical and numerical levels. The methodology to solve such problems is very rich and we mention here just the level set method (Osher and Sethian 1988), homogenization methods (Allaire 2002), the SIMP approach (Bendsøe and Sigmund 2003). Due to the variable character of the domain Ω, it is necessary to update, in each iteration of usual algorithms, the FEM grid and the mass matrix, which is very time consuming. That's why in recent papers (Neittaanmäki *et al.* 2009; Neittaanmäki and Tiba 2012) we have stressed the use of fixed domain methods, that are also relevant in the theory of variational inequalities, in free boundary problems (Philip and Tiba 2013; Halanay *et al.* 2016; Murea and Tiba 2016, 2018).

In this article, we introduce a new approach of fixed domain type, using a penalization idea, allowing both topological and boundary variations. Our penalization is in the cost functional, as usual in optimization theory (but not yet used in shape optimization, as far as we know). The state equation is perturbed just by introducing certain control parameters. This is possible due to recent developments related to the parametrization of implicitly defined manifolds, that provide an advantageous description of the unknown geometry (Tiba 2013; Nicolai and Tiba 2015; Tiba 2018).

We shall briefly analyze the Hamiltonian representation of implicitly defined manifolds in Section 2. In Section 3, we formulate the optimization problem and a fixed domain approach based on penalization. This is done for second order partial differential operators and in dimension two, which is a basic case in optimal design. We prove existence and approximation properties under appropriate regularity and coercivity assumptions. Our construction may allow the development of new numerical methods of fixed domain type for general shape optimization problems (1) - (3).

2. Implicit parametrizations

In this section, we give a brief presentation of parametric representations for implicitly defined manifolds, via Hamiltonian systems. We follow Tiba (2013) and Nicolai and Tiba (2015), where the case of dimension two and three is discussed, but we also investigate some new properties. Recently, a general finite dimensional treatment was introduced by Tiba (2018). The two dimensional case, that we discuss here (due to the subsequent applications in shape optimization) is also very shortly mentioned by Thorpe (1979, p. 63).

Let $g : D \subset R^2 \to R$ be a C^1 mapping and $(x^0, y^0) \in D$ (bounded $C^{1,1}$ domain) be such that the classical independence condition is satisfied:

$$g(x^0, y^0) = 0,\ \nabla g(x^0, y^0) \neq 0. \tag{4}$$

We consider the Hamiltonian system:

$$\begin{aligned} x'(t) &= -\frac{\partial g}{\partial y}(x(t), y(t)),\ t \in I, \\ y'(t) &= \frac{\partial g}{\partial x}(x(t), y(t)),\ t \in I, \end{aligned} \tag{5}$$

$$(x(0),y(0)) = (x^0,y^0), \tag{6}$$

where $I \subset R$ is some interval around the origin. The local existence in (5), (6) follows by the Peano theorem since $g \in C^1(\overline{D})$.

Proposition 1. *We have*

$$g(x(t),y(t)) = 0,\ \forall\, t \in I. \tag{7}$$

Moreover, the solution of (5), (6) is unique.

Proof. Relation (7) is the conservation property of the Hamiltonian, due to

$$\frac{d}{dt}[g(x(t),y(t))] = \nabla g(x(t),y(t)) \cdot (x'(t),y'(t)) = 0$$

and (4), (5), on I. The uniqueness (see Bouchut and Desvillets 2001) is a consequence of (4), (7) and the implicit functions theorem applied around $(x^0,y^0) \in D$. Then, (5) can be reduced to one ordinary differential equation with non zero right-hand side (by the independence assumption (4)) that is uniquely solvable. The uniqueness follows although the vector field appearing in (5) is just continuous.

Under the subsequent conditions (8), (9), the solution is even global and cannot leave D, as shown in the next proposition.

Remark 1. *For the general uniqueness property, see Tiba (2018), where regularity is also studied. In higher dimension, iterated Hamiltonian systems are to be used.*

Remark 2. *Notice that (5), (6) make sense even in the critical case, when* $\nabla g(x^0,y^0) = 0$ *(but, then, the solution of (5), (6) is constant). This observation, using some approximation procedure and the Hausdorff - Pompeiu metric allows to introduce the notion of generalized solution of the implicit system* $g(x,y) = 0$ *and to study its properties in the critical case (Tiba 2013; Nicolai and Tiba 2015; Tiba 2018) .*

We introduce now some basic assumptions on the family $\mathscr{O}$ of all admissible shape functions. We assume $g \in C^2(\overline{D})$ that will be used here and in the next section, in the study of shape optimization problems. We use the name shape function to underline that our approach is essentially different from the level set method of Osher and Sethian (1988). It may be partially compared with the optimization method discussed by Santosa (1995). The main hypotheses are:

$$g(x,y) > 0 \quad \text{on } \partial D, \tag{8}$$

$$|\nabla g(x,y)| > 0 \quad \text{on } G = \{(x,y) \in D;\ g(x,y) = 0\}. \tag{9}$$

Notice that the admissible family $\mathscr{O}$ as defined by (8), (9) is very rich. It includes, for instance, multimodal functions of class $C^2(\overline{D})$, that may have even an infinity of local extremal points in D, etc.

Proposition 2. *Under hypothesis (8), (9), G is a finite union of disjoint closed curves of class C^2, without self intersections and not intersecting ∂D, parametrized by the solution of (5), (6), when some initial point (x^0,y^0) is chosen on each component.*

Proof. The fact that G is a union of curves of class C^2 without self intersections and not intersecting ∂D, is obtained by contradiction from the implicit functions theorem and conditions (8), (9). By Proposition 1, the system (5), (6) provides a parametrization of each component of G, when (x^0, y^0) is on that component (at least locally). Using the structure theorem for maximal solutions of (5), (6) (see Pontryagin 1968, Thm. B, p. 182; Barbu 1985, Thm. 9, p. 48), since D is bounded and the trajectory remains far from ∂D by (8), it yields that the maximal existence interval is infinite in both senses: $I = (-\infty, \infty)$, for any $g \in \mathscr{O}$. Moreover, due to (9), the trajectory itself continues indefinitely since all the points on it are not critical for g.

Assume that the trajectory is not periodic. We denote by Λ the corresponding ω-limit set, contained in G and by $(\hat{x}, \hat{y}) \in G$ some point in Λ. It is not a critical point for g due to (9). The trajectory passing through $(\hat{x}, \hat{y})$ remains in Λ (Hirsch *et al.* 2013, Ch. 9.2). Denote by $(\tilde{x}, \tilde{y})$ another point on this second trajectory. Notice that the tangential (to the second trajectory) derivatives of g are null in both points due to Proposition 1. We also consider the normals to this second trajectory, in both points. The first trajectory has to intersect at least one of these normals an infinity of times. Otherwise, a direct argument shows that not all the points on the second trajectory can be approximated by points on the first trajectory, i.e. the second trajectory is not contained in Λ - a contradiction. One can consider the infinite sequence of intersections points with one of the normals and infer that the normal (to the second trajectory) derivative of g in at least one of the points is also null. Then, this is a critical point of g on G - a contradiction to (9).

Consequently, the solution of (5), (6) has to be periodic, and each considered trajectory is a closed curve in D. It is defined on some interval denoted by I_g, with length equal to the period (that may be different for each component of G). Similar arguments can be found in (Pontryagin 1968, §28) or in (Hirsch *et al.* 2013, Ch.10, the Poincare-Bendixson theorem).

Finally, the number of components of G has to be finite. They are all disjoint closed curves in the bounded domain $D \subset R^2$. If their number is infinite, on a subsequence, the closure of their interior subdomains converges in the Hausdorff-Pompeiu metric to some compact (the case of infinitely many embedded subdomains can be excluded by an argument as in the above paragraph). We can pick up critical points of g (by Weierstrass existence theorem for extremal points on compacts and the fact that g is constant on the boundary of these subdomains) from this subsequence of open sets, such that the distance to G goes to 0. In the limit we get some point $(\bar{x}, \bar{y}) \in G$ that is critical for g - a contradiction to (9).

Consider now a second Hamiltonian function $h \in \mathscr{O}$, i.e. $h \in C^2(\overline{D})$ and satisfying (8), (9). We investigate the perturbation $g + \lambda h$, $\lambda \in R$ "small", and we denote by $G_\lambda \subset D$ the compact set

$$G_\lambda = \{(x,y) \in D;\ (g + \lambda h)(x,y) = 0\}. \tag{10}$$

We also denote by $V_\varepsilon \subset D$ an ε - neighborhood of G (where $d[(x,y), G]$ is the distance from a point to G):

$$V_\varepsilon = \{(x,y) \in D;\ d[(x,y), G] < \varepsilon\}. \tag{11}$$

Proposition 3. *If $\varepsilon > 0$ is small enough, there is $\lambda(\varepsilon) > 0$ such that for $\lambda \in R$, $|\lambda| < \lambda(\varepsilon)$, we have $G_\lambda \subset V_\varepsilon$ and G_λ is a finite union of C^2 curves.*

Proof. On G_λ, we have

$$|g(x,y)| = |\lambda||h(x,y)| \leq |\lambda| C, \tag{12}$$

where C is independent of λ and depends just on $h \in \mathscr{O}$. On $D \setminus V_\varepsilon$, there is some constant $c_\varepsilon > 0$, depending on g, such that

$$|g(x,y)| \geq c_\varepsilon \tag{13}$$

due to the Weierstrass theorem on the existence of extremal points for continuous mappings on compacts.

Combining (12), (13), one can find $\lambda(\varepsilon) > 0$, such that $G_\lambda \subset V_\varepsilon$. Moreover, again by the Weierstrass theorem and $g \in C^2(\overline{D})$ satisfying (8), (9), there is some constant $\widetilde{c}_\varepsilon > 0$, depending on g, such that, on V_ε we have

$$|\nabla g(x,y)| \geq \widetilde{c}_\varepsilon, \tag{14}$$

due to (9), if ε is small enough. Taking $\lambda(\varepsilon)$ even smaller (if necessary), we get that

$$|\nabla (g + \lambda h)(x,y)| > 0 \quad \text{on } G_\lambda, \tag{15}$$

due to (14) and to $G_\lambda \subset V_\varepsilon$, if $|\lambda| < \lambda(\varepsilon)$. Then, by the previous Proposition 2, it yields that G_λ is a finite union of C^2 curves, for λ small enough.

Remark 3. *The inclusion $G_\lambda \subset V_\varepsilon$ shows that $G_\lambda \to G$, for $\lambda \to 0$, in the Hausdorff - Pompeiu metric (Neittaanmäki* et al. *2006). Moreover, the family $\mathscr{O}$ is a cone and small perturbations are admissible, $g + \lambda h \in \mathscr{O}$ for $\lambda > 0$ small, $h \in \mathscr{O}$.*

3. Shape optimization and penalization

We discuss the problem (P) given by (1) - (3). The family of admissible open sets $\Omega \subset D \subset R^2$, denoted by $\mathscr{F}$, is assumed to be generated starting from the family $\mathscr{O}$ of admissible shape functions $g \in C^2(\overline{D})$, via the definition

$$\Omega_g = \{(x,y) \in D;\ g(x,y) < 0\}. \tag{16}$$

By the results from the previous section, under hypotheses (8), (9), for any $g \in \mathscr{O}$, we have that $\Omega_g \subset D$ is an open set with a finite number of connected components, not intersecting ∂D. This point of view was introduced by Neittaanmäki *et al.* (2009), and was further discussed by Neittaanmäki *et al.* (2006), Philip and Tiba (2013), and Tiba (2018). The family $\mathscr{F}$ of admissible open sets is rich and, for cost functionals (1) defined on some given subset $E \subset D$, the natural constraint

$$E \subset \Omega,\ \forall\, \Omega \in \mathscr{F}, \tag{17}$$

can be expressed as

$$g < 0 \text{ in } E,\ \forall\, g \in \mathscr{O}. \tag{18}$$

In this case, the definition (16) can be modified such that Ω_g is given just by the subdomain that contains E since the other components from (16) have no contribution to the performance index.

The mappings $g \in \mathscr{O}$ will be interpreted as a control parameter and we introduce a supplementary control unknown $u \in L^2(D)$ and consider the perturbed state system defined in D:

$$-\Delta y = f + H(g)u \quad \text{in } D, \tag{19}$$

$$y = 0 \quad \text{on } \partial D, \tag{20}$$

where $H : R \to R$ is the Heaviside function. Then, $H(g)$ is the characteristic function of $D \setminus \Omega_g$.

We introduce the following state constrained optimal control problem, defined in the fixed domain D:

$$\underset{g,u}{\text{Min}} \int_D (1 - H(g))\, j(x, y(x))dx \tag{21}$$

subject to (19), (20) and (22)

$$\int_{\partial\Omega_g} |y(\sigma)|^2 d\sigma = 0, \tag{22}$$

for any $g \in \mathscr{O}$ and $u \in L^2(D)$.

Proposition 4. *For any $g \in \mathscr{O}$, there is $u_g \in L^2(D)$ (not unique) such that the solution of (19), (20) coincides in Ω_g with the solution of (2), (3) and satisfies (22). The costs (1) and (21) coincide.*

Proof. Let $y \in H^2(\Omega_g) \cap H_0^1(\Omega_g)$ denote the solution of (2), (3) in Ω_g. Since $\partial\Omega_g$ is C^2 and ∂D is $C^{1,1}$, one can use the trace theorem in $D \setminus \Omega_g$ and find (not unique) $\widetilde{y} \in H^2(D \setminus \Omega_g)$ such that $\widetilde{y} = y$, $\partial\widetilde{y}/\partial n = \partial y/\partial n$ on $\partial\Omega_g$, $\widetilde{y} = 0$ on ∂D (n is the outside normal to $\partial\Omega_g$). We denote by $u_g = -\Delta\widetilde{y} - f$ in $D \setminus \Omega_g$ and it can be extended arbitrarily to $L^2(D)$. The concatenation of y and $\widetilde{y}$, denoted by $\bar{y}$, satisfies in D relations (19), (20) with u_g in the right-hand side:

$$\begin{gathered}\int_D \nabla\bar{y}\nabla\phi dx = \int_{D\setminus\Omega_g} \nabla\widetilde{y}\nabla\phi dx + \int_{\Omega_g} \nabla y\nabla\phi dx = -\int_{D\setminus\Omega_g} \Delta\widetilde{y}\phi dx - \\ \int_{\partial\Omega_g} \partial\widetilde{y}/\partial n\phi d\sigma - \int_{\Omega_g} \Delta y\phi dx + \int_{\partial\Omega_g} \partial y/\partial n\phi d\sigma = \int_D [f + H(g)u_g]\phi dx\end{gathered} \tag{23}$$

for any $\phi \in H_0^1(D)$. Relation (22) is obvious by $y \in H^2(\Omega_g) \cap H_0^1(\Omega_g)$ and the above construction.

Corollary 1. *The shape optimization problem (1) - (3) on the family $\mathscr{F}$ is equivalent with the constrained optimal control problem (19) - (22), defined in D.*

Proof. By Proposition 4, for any admissible open set Ω_g, $g \in \mathscr{O}$ one can find admissible control pairs $[g, u_g]$ with the same cost. Conversely, for any $[g, u] \in \mathscr{O} \times L^2(D)$, admissible for (19) - (22), the corresponding Ω_g and $y|_{\Omega_g}$ is admissible for (1) - (3) with the same cost.

Remark 4. *The state constraint (22) has an implicit character and the unknown geometry $\partial\Omega_g$ is still fully present in it, that shows the difficulty of the problem.*

In this section, we denote by $z_g(t) = (z_g^1(t), z_g^2(t))$, $t \in I_g$, the unique solution of the Hamiltonian system (5), (6) associated to some initial condition on $\partial\Omega_g$, where $I_g = [0, T_g]$ is the period corresponding to $z_g(\cdot)$ (according to the Proposition 2). The penalized optimal control problem is:

$$\underset{g,u}{\mathrm{Min}}\left\{\int_D (1-H(g))j(x,y(x))dx+\frac{1}{\varepsilon}\int_{I_g}|y(z_g(t))|^2\sqrt{\left|z_g'(t)\right|^2}dt\right\}, \tag{24}$$

subject to (19), (20), $g \in \mathscr{O}$, $u \in L^2(D)$ and for $\varepsilon > 0$ given. Notice that $y \in H^2(D)$ by (19), (20) and elliptic regularity. Consequently, $y \in C(\bar{D})$ if $D \subset R^2$, by the Sobolev embedding theorem. Then, (24) makes sense. The last term in (24) is a penalization of (22) and replaces it. In case $\partial\Omega_g$ has several connected components, then the last integral in (24) has to be replaced by a finite sum of integrals, due to Proposition 2.

Lemma 1. *Let $j(x,y)$ be a Caratheodory function on $D \times R$, bounded from below by a constant. Denote by $[y_n^\varepsilon, g_n^\varepsilon, u_n^\varepsilon]$ a minimizing sequence in the penalized problem (19), (20), (24). Then, on a subsequence denoted by $m(n)$, the (not necessarily admissible) pairs $[\Omega_{g_{m(n)}^\varepsilon}, y_{m(n)}^\varepsilon]$ give a minimizing cost in (1), $y_{m(n)}^\varepsilon$ satisfies (2) in $\Omega_{g_{m(n)}^\varepsilon}$ and (3) is fulfilled with a perturbation of order $\varepsilon^{\frac{1}{2}}$ on $\partial\Omega_{g_{m(n)}^\varepsilon}$.*

Proof. Let $[y_m, g_m]$ be a minimizing sequence in the original problem (1) - (3) with $\Omega = \Omega_g \in \mathscr{F}$ and $g \in \mathscr{O}$. By the same argument as in Proposition 4, there is $u_m \in L^2(D)$, such that $[y_m, g_m, u_m]$ is admissible for the constrained control problem (19) - (22), with the same cost. It is also admissible for the penalized control problem (19), (20), (24) with the same cost and we have:

$$\begin{aligned}\int_D (1-H(g_{m(n)}^\varepsilon))j(x,y_{m(n)}^\varepsilon(x)) + \frac{1}{\varepsilon}\int_{I_{g_{m(n)}^\varepsilon}} \left|y_{m(n)}^\varepsilon(z_{g_{m(n)}^\varepsilon}(t))\right|^2 \sqrt{\left|z_{g_{m(n)}^\varepsilon}'(t)\right|^2}dt \le \\ \le \int_D (1-H(g_m))j(x,y_m(x))dx \underset{m\to\infty}{\longrightarrow} \inf(P) \le C,\end{aligned} \tag{25}$$

where C is some constant independent of ε, n, m and $m(n)$ is big enough. Notice that $y_n^\varepsilon \in C(D)$ by the Sobolev theorem and (25) makes sense. By the positive sign of the penalization term, we infer

$$\begin{aligned}\int_{\Omega_{g_{m(n)}^\varepsilon}} j(x,y_{m(n)}^\varepsilon(x))dx = \int_D (1-H(g_{m(n)}^\varepsilon))j(x,y_{m(n)}^\varepsilon(x))dx \le \\ \le \int_D (1-H(g_m))j(x,y_m(x))dx \to \inf(P).\end{aligned} \tag{26}$$

Moreover, again by (25) and the hypothesis j bounded from below, we also get

$$\left|y_{m(n)}^\varepsilon\right|^2_{L^2(\partial\Omega_{g_{m(n)}^\varepsilon})} \le C_1\varepsilon, \tag{27}$$

with C_1 independent of $\varepsilon > 0, m, n$. Relations (26), (27) end the proof.

Let $\varepsilon_k \to 0$ be some sequence of positive quantities. Taking into account Lemma 1, with $\varepsilon = \varepsilon_k$, we denote shortly $y^k = y_{m(n)_k}^{\varepsilon_k}$, $g^k = g_{m(n)_k}^{\varepsilon_k}$, $\Omega^k = \Omega_{g_{m(n)_k}^{\varepsilon_k}}$, with $m(n)_k$ conveniently big. Then, Ω^k is a bounded sequence of open sets and we have $\Omega^k \to \Omega^*$ in the Hausdorff - Pompeiu complementary sense, on a subsequence denoted again by ε_k. We suppose that

$\bar{\Omega}_k \to \bar{\Omega}^*$ in the Hausdorff-Pompeiu metric is also valid (see Neittaanmäki *et al.* 2006, Proposition A 3.10).

Proposition 5. *Assume that $j(x,\cdot)$ is coercive in $D \times R$:*

$$j(x,y) \geq \alpha|y|^2 - \beta,\ \alpha > 0;\ \beta \in R. \tag{28}$$

Then, there is an extension $\hat{y}_k$ of $y^k|_{\Omega^k}$, bounded in $L^2(D)$. If y^* is its weak limit on a subsequence, in $L^2(D)$, then $y^*|_{\Omega^*}$ satisfies (2) in the distributions sense. If Ω^* is of class C and $y^* \in H^1(D)$, (3) is also satisfied.

Proof. By (25), (28) and using the short notations from above, we get $\left\{|y^k|_{L^2(\Omega^k)}\right\}$ bounded by a constant independent of k. Notice that the boundedness of y^k in $L^2(D)$ is not guaranteed.

We denote by V^k a small strict neighbourhood of Ω^k, $V^k \cap \partial D$ void, and we truncate y^k in V^k by 0. We denote by $\widehat{y}^k$ a smoothing of this truncated function that remains null in Ω^k if the smoothing parameter is small enough. It may be assumed to satisfy (20) by applying the same truncation procedure in a small neighbourhood of ∂D. We take $\hat{y}_k = y^k - \widehat{y}^k$ and $\hat{y}_k$ satisfies (19), (20) with some $\hat{u}_k = -\Delta\hat{y}_k - f$ in $D \setminus \Omega^k$ that may be unbounded in $L^2(D \setminus \Omega^k)$ (but this is not important in the sequel). If the smoothing parameter is small enough, then $\hat{y}_k$ is bounded in $L^2(D)$ by the convergence properties of smoothing in $L^2(D \setminus \Omega^k)$.

Denote by y^*, the weak limit in $L^2(D)$ of $\{\widehat{y}_k\}$, on a subsequence. Let $\mathscr{K} \subset \Omega^*$ be any compact subdomain and $\varphi \in C_0^\infty(D)$ be such that supp $\varphi = \mathscr{K}$. By the Γ - property (Neittaanmäki *et al.* 2006, Proposition A3.8) of the Hausdorff - Pompeiu complementary convergence, we have that supp $\varphi \subset \Omega^{\mathrm{k}}$, for k big enough. Then, we can use φ as a test function in (19) and obtain

$$\begin{aligned} -\int_{\Omega^*} y^*\Delta\varphi dx = -\int_{\Omega^k} y^*\Delta\varphi dx = -\lim_{k\to\infty}\int_D \widehat{y}_k\Delta\varphi dx = \\ = -\lim\int_D \Delta\widehat{y}_k\varphi dx = \int_D f\varphi dx = \int_{\Omega^*} f\varphi dx \end{aligned}$$

since supp $\varphi \cap (\mathrm{D} \setminus \Omega^{\mathrm{k}}) = \emptyset$, for "big" k. This shows that y^* satisfies (2) in Ω^*, in the distributions sense.

Consider now any compact $A \subset D \setminus \Omega^*$. We again get that $A \subset D \setminus \Omega^k$, for k big enough. Therefore, by the above construction, $|\widehat{y}^k|_{L^2(A)} \to 0$ for $k \to \infty$, that is $y^* = 0$ a.e on A and on $D \setminus \Omega^*$. Since Ω^* is assumed to be of class C and $y^* \in H^1(D)$, the Hedberg-Keldys stability property (Neittaanmäki *et al.* 2006, Cor. 2.3.11) shows that (3) is also satisfied and the proof is finished.

In the next result, we bring more clarifications, when j depends on ∇y as well.

Corollary 2. *Under the above conditions, assume that j satisfies the stronger coercivity assumption on $D \times R \times R^2$:*

$$j(x,y,v) \geq \alpha_1|v|^2 + \beta_1|y|^2 - \gamma,\ \alpha_1 > 0,\ \beta_1 > 0,\ \gamma \in R, \tag{29}$$

and $j(x,y,\cdot)$ is convex. Then, $[y^*,\Omega^*]$ is an optimal pair for the shape optimization problem (P).

Proof. For any $\mathscr{K} \subset \Omega^*$ compact subdomain, we have $\mathscr{K} \subset \Omega^k$ for k big enough, by the Γ - property. Relations (25), (29) show that $\left\{ |y^k|_{\mathscr{K}}|_{H^1(\mathscr{K})} \right\}$ is bounded. By the general lower semicontinuity theorem A3.15 (Neittaanmäki *et al.* 2006) and inequality (25), we infer

$$\int_{\Omega^*} j(x, y^*(x), \nabla y^*(x))dx \leq \inf(P).$$

Since the pair $[y^*, \Omega^*]$ is admissible for (P) by Proposition 5, the proof is finished.

Remark 5. *In applications, it is enough to apply gradient methods to the penalized problem (19), (20), (24). To ensure the necessary differentiability properties a further regularization should be performed (see Philip and Tiba 2013, for a related shape optimization problem). Proposition 3 will play an essential role in this respect. Under appropriate compactness assumptions on $\mathscr{O}$ (see Chenais 1975; Neittaanmäki* et al. *2006), one can prove the regularity conditions on Ω^*, y^*. Such developments are quite complex and will be discussed in a subsequent paper.*

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