

## EVOLUTION OF TZITZEICA HYPERSURFACES

CONSTANTIN UDRIȘTE <sup>a\*</sup>, IONEL ȚEVY <sup>a</sup> AND ALI SAPEEH RASHEED <sup>a</sup>

**ABSTRACT.** Our aim is to study the evolutions of Tzitzeica hypersurfaces which appear in understanding the dynamics of some geometric programming problems and reliability optimal allocation problems. Section 2 analyses the convexity of a Tzitzeica hypersurface. Sections 3 – 6 refer to standard Tzitzeica hypersurfaces and their evolutions by convenient geometrical flows: (i) evolution along the normal vector field, (ii) infinitesimal normal transformation of a Tzitzeica hypersurface, (iii) evolution along a centro-affine vector field, and (iv) evolution along an affine vector field. Sections 7 – 8 include results on the Tzitzeica law in economics and the evolution of Tzitzeica surfaces described by PDEs: (v) Tzitzeica hypersurfaces as invariants w.r.t. excess demand flow; (vi) parametric Tzitzeica surfaces based on PDEs and their evolutions.

### 1. Introduction and motivation

The nature of hypersurface changes and how the hypersurface evolves with time, in the differential geometry sense, is not yet entirely known. Generally, the mechanism of hypersurface evolution in a differential process is investigated through changes in hypersurface profiles with process time.

Hypersurface processing tools and techniques are widespread in computer graphics, animation, medical imaging, computer aided modeling, and computer vision (Morigi 2010). In recent years, there has been an increasing interest in hypersurface deformation (motion) governed by geometric PDEs. Our paper introduces a new approach to modelling geometrical flows of contours and hypersurfaces having in view future topics in optimization problems. Differential evolution has shown to be very efficient when solving global optimization problems with simple bounds. In this paper, we propose a modified constrained differential evolution based on different Tzitzeica constraints handling techniques, namely, feasibility and dominance rules, especially for geometric programming and reliability optimal allocation (Udriște *et al.* 2017).

In Sections 2 – 8 we are interested both in the original forms and in the ending forms, beneficial to some optimization problems.

## 2. Convexity of Tzitzeica hypersurfaces

The Tzitzeica surfaces theory is well-known (Tzitzéica 1907; Udriște *et al.* 2004; Agnew *et al.* 2010). The constant level sets  $\prod_{i=1}^n x_i = c$  attached to the function

$$f: \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = \prod_{i=1}^n x_i, x = (x_1, \dots, x_n),$$

are Tzitzeica hypersurfaces in  $\mathbb{R}^n$ . Each of these hypersurfaces reduces to: union of  $2^{n-1}$  disjoint and closed sets if  $c \neq 0$ , union of coordinate hyperplanes if  $c = 0$ .

**Proposition 2.1.** (Udriște *et al.* 2016) *The Tzitzeica hypersurface  $\prod_{i=1}^n x_i = c$  is convex.*

*Proof.* Since

$$\frac{\partial^2 f}{\partial x_i^2} = 0, \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{c}{x_i x_j},$$

we find

$$d^2 f = 2f \sum_{i < j} \frac{dx_i}{x_i} \frac{dx_j}{x_j} = c \left( \left( \sum_{i=1}^n \frac{dx_i}{x_i} \right)^2 - \sum_{i=1}^n \frac{dx_i^2}{x_i^2} \right).$$

The restriction of this Hessian to the tangent plane  $c \sum_{i=1}^n \frac{dx_i}{x_i} = 0$ , *i.e.*,

$$d^2 f = -c \left( \sum_{i=1}^n \frac{dx_i^2}{x_i^2} \right)$$

is negative definite. Consequently, the second fundamental form is positive definite.  $\square$

## 3. Normal evolution of Tzitzeica hypersurfaces

The monographs by Evans (2010), Morigi (2010), and Antontsev and Shmarev (2015) offer the reader a treatment of the theory of evolution PDEs. The monographs by Udriște (1994), Udriște *et al.* (2004), and Calin and Udriște (2014) contain related topics. All of these suggested the following ways of deformation.

In relation to deformation of Tzitzeica surfaces, we have to say from the beginning what is well-known (Tzitzéica 1907, 1909): *the Tzitzeica property is preserved only by affine transformations and centro-affine ones.*

**3.1. Steepest increase lines.** Let  $f(x) = \prod_{i=1}^n x_i$  be the Tzitzeica function. The *gradient vector field*  $\nabla f = \left( \frac{f}{x_i} \right)$  has two properties: (i) shows at every point  $(x_1, \dots, x_n)$  the direction and the sense of steepest increase of  $f$ ; (ii) is orthogonal to the constant level sets of  $f$  (Tzitzeica hypersurfaces). These are the field lines of  $\nabla f$ , *i.e.*, the solutions of the ODE system  $\frac{dx_i}{dt} = \frac{c}{x_i}$ ,  $i = 1, \dots, n$ . Since  $x_1 dx_1 = \dots = x_n dx_n = c dt$ , the steepest increase orbits are intersections of hyperbolic hyper-cylinders  $x_2^2 - x_1^2 = c_2, \dots, x_n^2 - x_1^2 = c_n$  (see first integrals).

The field  $\nabla f$  is solenoidal, and consequently its flow conserves the volume.

**3.2. Evolution along the normal vector field.** Let us consider the Tzitzeica hypersurface  $f(x) = x_1 \cdots x_n = c$  and the vector field  $N = \nabla f$  with the components  $\left(\frac{c}{x_1}, \dots, \frac{c}{x_n}\right)$ . Along the given Tzitzeica hypersurface, the field  $N$  is a normal vector field.

**Theorem 3.1.** *A normal deformation of a Tzitzeica hypersurface  $x_1 \cdots x_n = c$  can be written as*

$$(x_1^2 - 2ct)(x_2^2 - 2ct) \cdots (x_n^2 - 2ct) = c^2.$$

*Proof.* The evolution of the surface  $f(x) = c$  along the direction of the vector field  $N$ , i.e., with respect to a parameter  $t \in [0, \varepsilon]$ , is made taking into consideration the extension (variation) function  $\mathcal{F}(x_1, \dots, x_n, t)$ , solution of the Cauchy problem

$$\frac{\partial \mathcal{F}}{\partial t} = -(N, \nabla \mathcal{F}), \quad \mathcal{F}(x_1, \dots, x_n, 0) = f(x_1, \dots, x_n) - c.$$

Explicitly, the PDE is

$$\frac{\partial \mathcal{F}}{\partial t} + \frac{c}{x_1} \frac{\partial \mathcal{F}}{\partial x_1} + \dots + \frac{c}{x_n} \frac{\partial \mathcal{F}}{\partial x_n} = 0,$$

with the characteristic system

$$\frac{dt}{1} = x_1 \frac{dx_1}{c} = \dots = x_n \frac{dx_n}{c}.$$

We use the first integrals (general solution of the characteristic system)

$$x_1^2 = 2ct + c_1, x_2^2 - x_1^2 = c_2, \dots, x_n^2 - x_1^2 = c_n.$$

Let  $\Phi: \mathbb{R}^n \rightarrow R$  be an arbitrary  $C^1$  function. Then

$$\mathcal{F}(x_1, \dots, x_n, t) = \Phi(x_1^2 - 2ct, x_2^2 - x_1^2, \dots, x_n^2 - x_1^2),$$

together with the initial data

$$\mathcal{F}(x_1, \dots, x_n, 0) = f(x_1, \dots, x_n) = x_1 \cdots x_n - c.$$

The condition

$$\Phi(x_1^2, x_2^2 - x_1^2, \dots, x_n^2 - x_1^2) = x_1 \cdots x_n - c$$

implies

$$\Phi(y_1, y_2, \dots, y_n) = \sqrt{y_1(y_2 + y_1) \cdots (y_n + y_1)} - c.$$

We find the solution

$$\mathcal{F}(x_1, \dots, x_n, t) = \sqrt{(x_1^2 - 2ct)(x_2^2 - 2ct) \cdots (x_n^2 - 2ct)} - c$$

and the result follows.  $\square$

The normal deformation of a Tzitzeica hypersurface is not a Tzitzeica hypersurface since the deformation is realised by the diffeomorphism  $y_i = x_i^2 - 2ct$ ,  $i = 1, \dots, n$ , which is not a centro-affine one.

**Remark 3.1.** (i) *The evolution Hamilton-Jacobi PDE*

$$\frac{\partial \mathcal{F}}{\partial t} = \pm (\nabla_x \mathcal{F}, \nabla_x \mathcal{F})$$

describes the movement of the surface  $\mathcal{F}(x_1, \dots, x_n; t) = 0$  along its normal vector field  $\pm \nabla_x \mathcal{F}$ .

(ii) Let  $F(x, t)$ . The ODE  $\frac{dx}{dt}(t) = \nabla_x F(x(t), t)$ , gives the PDE

$$\frac{d}{dt} F(x(t), t) = \|\nabla_x F\|^2 + \frac{\partial F}{\partial t}.$$

Consequently, the hypersurfaces in the evolution (i) are first integrals of ODE (ii).

#### 4. Infinitesimal normal transformation of Tzitzeica hypersurface

Our aim is to deform a Tzitzeica hypersurface  $x_1 \cdots x_n = c$  by infinitesimal transformation associated to its normal vector field  $N = \left( \frac{c}{x_1}, \dots, \frac{c}{x_n} \right)$ . Between the deformed hypersurfaces there exist one which is Tzitzeica?

**Theorem 4.1.** *The infinitesimal transformation of the normal flow deforms the Tzitzeica hypersurface  $x_1 \cdots x_n = c$  into the hypersurface*

$$\prod_{i=1}^n \frac{y_i \pm \sqrt{y_i^2 - 4ct}}{2} = C.$$

*Proof.* The infinitesimal transformation produced by the normal vector field  $N$  is

$$y_1 = x_1 + t \frac{c}{x_1}, \dots, y_n = x_n + t \frac{c}{x_n}, \quad x_1 \cdots x_n = c, \quad t \in [0, \varepsilon].$$

One determines  $x_1, \dots, x_n$  (one reverses the infinitesimal transformation),

$$x_i = \frac{y_i \pm \sqrt{y_i^2 - 4ct}}{2}, \quad i = 1, \dots, n$$

(positive roots) and we introduce in the first equation. □

These new hypersurfaces are obtained from a Tzitzeica hypersurface using a diffeomorphism different from a centro-affine one. Consequently they are not Tzitzeica hypersurfaces.

#### 5. Evolution along a centro-affine vector field

Let us consider the Tzitzeica hypersurface  $f(x) = x_1 \cdots x_n = c$  and the centro-affine vector field  $Ax$ ,  $\det A \neq 0$ . The evolution along this vector field is described by the PDE

$$\frac{\partial \mathcal{F}}{\partial t} = -(Ax, \nabla \mathcal{F}), \quad \mathcal{F}(x_1, \dots, x_n, 0) = f(x_1, \dots, x_n) - c.$$

The characteristic system  $\frac{dt}{1} = \frac{dx_1}{A_1 x} = \dots = \frac{dx_n}{A_n x}$ , i.e.,  $\frac{dx}{dt} = Ax$ , has the general solution  $x = e^{At}C$ . Then

$$\begin{aligned} \mathcal{F}(x_1, \dots, x_n, t) &= \Phi(e^{-At}x), \\ \mathcal{F}(x_1, \dots, x_n, 0) &= f(x_1, \dots, x_n) - c = x_1 \cdots x_n - c. \end{aligned}$$

It follows the equation of deformed hypersurface

$$\mathcal{F}(x_1, \dots, x_n, t) = (e^{-At})_1 x \cdots (e^{-At})_n x - c = 0.$$

The centro-affine diffeomorphism  $y = e^{-At} x$  shows that our deformation is a Tzitzeica hypersurface.

**Problem** Let  $B(t), B(0) = I$  be a given family of centro-affine diffeomorphisms. Does there exist one centro-affine vector field  $Ax$  that, deforming the Tzitzeica hypersurfaces, creates the given family  $B(t)$ ?

**Proposition 5.1.** *The centro-affine diffeomorphisms corresponding to centro-affine deforming field are those of exponential type only.*

**Proof** Setting  $e^{-At} = B(t)$ , we find  $A = -B^{-1}(t)\dot{B}(t)$ . Since the right hand member is independent on  $t$ , we find the matrix ODE:  $B^{-1}(t)\dot{B}(t)^2 - \ddot{B}(t) = 0$ , with the solution  $B(t) = e^{Ct}$ . It follows that  $A = -C$ . Consequently, the Proposition is true.

### 6. Evolution along an affine vector field

Let us consider the Tzitzeica hypersurface  $f(x) = x_1 \cdots x_n = c$  and the affine vector field  $Ax + h$ ,  $\det A \neq 0$ . The evolution along this vector field is described by the PDE

$$\frac{\partial \mathcal{F}}{\partial t} = -(AR + h, \nabla \mathcal{F}), \quad \mathcal{F}(x_1, \dots, x_n, 0) = f(x_1, \dots, x_n) - c.$$

The characteristic system

$$\frac{dt}{1} = \frac{dx_1}{A_1x + h_1} = \dots = \frac{dx_n}{A_nx + h_n},$$

i.e.,  $\frac{dx}{dt} = Ax + h$ , has the general solution

$$x = e^{At} \{C + (I - e^{-At})h\}.$$

Then

$$\begin{aligned} \mathcal{F}(x_1, \dots, x_n, t) &= \Phi(e^{-At}x - (I - e^{-At})h), \\ \mathcal{F}(x_1, \dots, x_n, 0) &= f(x_1, \dots, x_n) - c = x_1 \cdots x_n - c. \end{aligned}$$

It follows:

**Theorem 6.1.** *The equation of deformed hypersurface is*

$$\mathcal{F}(x_1, \dots, x_n, t) = \prod_{i=1}^n [(e^{-At}x)_i - ((I - e^{-At})h)_i] - c = 0.$$

The affine diffeomorphism  $y = e^{-At}x - (I - e^{-At})h$  shows that our deformation is a Tzitzeica hypersurface, with respect to a new center point  $(e^{-At} - I)h$ .

Let us highlight an affine diffeomorphism that preserve the quality of Tzitzeica hypersurface, like in duality between series and parallel systems in reliability (Abed *et al.* 2017). Let  $x_1 \cdots x_n = c$  be a Tzitzeica hypersurface, with the center  $(0, \dots, 0)$ . The affine diffeomorphism (symmetry and translation)  $x_1 = 1 - y_1, \dots, x_n = 1 - y_n$  change the given Tzitzeica hypersurface into a Tzitzeica hypersurface  $(1 - y_1) \cdots (1 - y_n) = c$ , with the center  $(1, \dots, 1)$ . Here, the associated matrix has the determinant  $\pm 1$  (equi-affine diffeomorphism).

In our context, a well-known result (Tzitzéica 1907, 1909) can be written in the general form:

**Theorem 6.2.** *Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ . An affine diffeomorphism  $x = Ay + b$  changes the Tzitzéica hypersurface  $x_1 \cdots x_n = c$ , with the center  $(0, \dots, 0)$ , into a Tzitzéica hypersurface  $(Ay_1 + b_1) \cdots (Ay_n + b_n) = c$ , with the center  $A^{-1}b$ .*

**7. Tzitzéica hypersurfaces as invariants w.r.t. excess demand flow**

We extend the theory of single-time dynamics produced by the excess demand vector field (see Udriște *et al.* 2004; Calapso and Udriște 2005), from 3 to  $n = \text{odd}$  dimensions.

Let  $p = (p_1, \dots, p_n)$  be the vector of prices. The general excess demand vector field  $E$  has the components

$$\begin{aligned} E_1(p) &= -\frac{p_2}{p_1+p_2} + \frac{p_3}{p_1+p_3} - \dots + (-1)^{n-1} \frac{p_n}{p_1+p_n} \\ E_2(p) &= -\frac{p_3}{p_2+p_3} + \frac{p_4}{p_2+p_4} - \dots + (-1)^{n-1} \frac{p_1}{p_2+p_1} \\ \dots & \dots \dots \dots \dots \dots \\ E_n(p) &= -\frac{p_1}{p_n+p_1} + \frac{p_2}{p_n+p_2} - \dots + (-1)^{n-1} \frac{p_{n-1}}{p_n+p_{n-1}}. \end{aligned}$$

Using these data, we can write the excess demand flow  $\frac{dp}{dt}(t) = E(p(t))$ .

The positive homogeneity of degree zero of each component  $E_i(p)$  means that if all prices are multiplied by the same positive constant, the excess demands do not vary (is invariant with respect to homothetic transformations). The Walras Law  $\sum_{i=1}^n p_i E_i = 0$  means that  $f_1(p) = \frac{1}{2} (p_1^2 + p_2^2 + \dots + p_n^2)$  is a first integral. The associated constant level sets of  $f_1(p)$  are spheres (compact hypersurfaces, invariant under the excess demand flow). Consequently, the vector field  $E$  is complete.

Let us introduce and explain what is a Tzitzéica Law.

**Theorem 7.1.** *The Tzitzéica function  $f(p) = p_1 \cdots p_n$  is a first integral for the excess demand flow.*

*Proof.* We can verify the relation

$$\sum_{CP} p_1 p_2 \cdots p_{n-1} E_n = 0 \text{ (summation by cyclic permutation).}$$

Indeed, for  $n = \text{odd}$  number, we can write

$$\begin{aligned} E_1(p) &= p_1 \left( \frac{1}{p_1 + p_2} - \frac{1}{p_1 + p_3} + \dots \right) \\ E_2(p) &= p_2 \left( \frac{1}{p_2 + p_3} - \frac{1}{p_2 + p_4} + \dots \right) \\ & \dots \dots \dots \dots \dots \\ E_n(p) &= p_n \left( \frac{1}{p_n + p_1} - \frac{1}{p_n + p_2} + \dots \right), \end{aligned}$$

whence the previous relation follows. Consequently, a second first integral of the flow is  $f_2(p) = p_1 \cdots p_n$ , *i.e.*, the excess demand flow conserves the standard Tzitzéica hypersurfaces. □

Practically this means that the "volume of the prices" is constant.

**Remark 7.1.** For  $n=$ odd number, we can write

$$\begin{pmatrix} E_1(p) \\ \dots \\ E_n(p) \end{pmatrix} = A(p) \begin{pmatrix} p_1 \\ \dots \\ p_n \end{pmatrix},$$

where  $A$  is a skew-symmetric  $n \times n$  matrix.

The previous flow and the Euclidean metric determine the least squares Lagrangian

$$L = \frac{1}{2} \delta^{ij} \left( \frac{dp_i}{dt} - E_i \right) \left( \frac{dp_j}{dt} - E_j \right).$$

Its Euler-Lagrange ODEs are

$$\frac{d^2 p_i}{dt^2} = \delta^{jk} \left( \frac{\partial E_i}{\partial p_j} - \frac{\partial E_j}{\partial p_i} \right) \frac{dp_k}{dt} + \frac{\partial f}{\partial p_i},$$

where  $f = \frac{1}{2} \delta^{ij} E_i E_j$  is the *energy density*. The last ODEs describe the generated geometric dynamics (geodesic motion in a gyroscopic field of forces), dynamics that include the excess demand flow.

### 8. General evolution of Tzitzeica surfaces

A parametric Tzitzeica surface is characterized by the completely integrable PDE system

$$r_{uu} = \frac{h_u}{h} r_u + \frac{1}{h} r_v, r_{uv} = hr, r_{vv} = \frac{1}{h} r_u + \frac{h_v}{h} r_v, (\ln h)_{uv} = h - \frac{1}{h^2}.$$

**Theorem 8.1.** *The evolution surface  $r(u, v, t), t \in [0, \varepsilon)$  is a Tzitzeica surface if and only if  $\frac{\partial r}{\partial t}(u, v, t) = A(t)r(u, v, t)$ , where  $A(t)$  is a non-degenerate matrix.*

*Proof.* Let  $(u, v) \rightarrow r(u, v, t), t \in [0, \varepsilon)$  be an evolution of the Tzitzeica surface  $r(u, v) = r(u, v, 0)$ .

If we consider  $\frac{\partial r}{\partial t}(u, v, t) = A(t)r(u, v, t)$ , with  $A(t)$  a non-singular matrix, then using the fundamental matrix  $\Omega(t)$ , we find the solution  $r(u, v, t) = \Omega(t)r(u, v, 0) = \Omega(t)r(u, v)$ , which will satisfy the Tzitzeica system, and hence the evolution is a Tzitzeica surface.

Conversely, let us take the evolution surface  $r(u, v, t)$  as Tzitzeica for any  $t \in [0, \varepsilon)$ . By partial derivative, we find

$$\frac{\partial}{\partial t} r_{uu} = \frac{h_u}{h} \frac{\partial}{\partial t} r_u + \frac{1}{h} \frac{\partial}{\partial t} r_v, \frac{\partial}{\partial t} r_{uv} = h \frac{\partial}{\partial t} r, \frac{\partial}{\partial t} r_{vv} = \frac{1}{h} \frac{\partial}{\partial t} r_u + \frac{h_v}{h} \frac{\partial}{\partial t} r_v.$$

Hence the surface  $\frac{\partial}{\partial t} r(u, v, t)$  is again a Tzitzeica surface. According to the well-known theory, there exists a time dependent centro-affine transformation, which applies the surfaces one to another, i.e.,  $\frac{\partial r}{\partial t}(u, v, t) = A(t)r(u, v, t)$ . □

**Remark 8.1.** *If the system of PDEs is linear in  $r(u, v)$ , then an evolution  $\frac{\partial r}{\partial t}(u, v, t) = A(t)r(u, v, t)$  produces surfaces of the same type.*

## References

- Abed, S. A., Udriște, C., and Țevy, I. (2017). “Optimal reliability allocation for redundancy series-parallel systems”. *European Journal of Pure and Applied Mathematics* **10**(4), 877–889. URL: <https://www.ejpam.com/index.php/ejpam/article/view/3014>.
- Agnew, A., Bobe, A., Boskoff, W., and Suceava, B. (2010). “Tzitzéica curves and surfaces”. *The Mathematica Journal* **12**. DOI: [10.3888/tmj.12-3](https://doi.org/10.3888/tmj.12-3).
- Antontsev, S. and Shmarev, S. (2015). *Evolution PDEs with Nonstandard Growth Conditions*. Atlantis Press. DOI: [10.2991/978-94-6239-112-3](https://doi.org/10.2991/978-94-6239-112-3).
- Calapso, M. T. and Udriște, C. (2005). “Tzitzéica and Walras laws in geometric economics”. *Analele Universității București, Matematică* **54**, 213–218.
- Calin, O. and Udriște, C. (2014). *Geometric Modeling in Probability and Statistics*. Springer International Publishing. DOI: [10.1007/978-3-319-07779-6](https://doi.org/10.1007/978-3-319-07779-6).
- Evans, L. C. (2010). *Partial differential equations*. 2nd ed. Graduate studies in mathematics v. 19. Providence, R.I: American Mathematical Society. URL: <https://bookstore.ams.org/gsm-19-r>.
- Morigi, S. (2010). “Geometric surface evolution with tangential contribution”. *Journal of Computational and Applied Mathematics* **233**(5), 1277–1287. DOI: [10.1016/j.cam.2007.04.052](https://doi.org/10.1016/j.cam.2007.04.052).
- Tzitzéica, M. G. (1907). “Sur une nouvelle classe de surfaces”. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* **144**, 1257–1259. URL: <http://gallica.bnf.fr/ark:/12148/bpt6k3098j/>.
- Tzitzéica, M. G. (1909). “Sur une nouvelle classe de surfaces (2ème Partie)”. *Rendiconti del Circolo Matematico di Palermo (1884-1940)* **28**(1), 210–216. DOI: [10.1007/BF03018218](https://doi.org/10.1007/BF03018218).
- Udriște, C. (1994). *Convex Functions and Optimization Methods on Riemannian Manifolds*. Springer Netherlands. DOI: [10.1007/978-94-015-8390-9](https://doi.org/10.1007/978-94-015-8390-9).
- Udriște, C. (2005). “Tzitzéica theory – opportunity for reflection in Mathematics”. *Balkan Journal of Geometry and Its Applications* **10**(1), 110–120. URL: <http://www.mathem.pub.ro/bjga/v10n1/B10-1-UDR.pdf>.
- Udriște, C., Abed, S. A., and Rasheed, A. S. (2016). “Optimal reliability allocation”. *American Review of Mathematics and Statistics* **4**(2), 82–91. URL: <http://armsnet.info/vol-4-no-2-december-2016-abstract-9-arms>.
- Udriște, C., Abed, S. A., and Țevy, I. (2017). “Geometric programming approaches of reliability allocation”. *U.P.B. Scientific Bulletin, Series A* **79**(3), 3–10. URL: [https://www.scientificbulletin.upb.ro/rev\\_docs\\_arhiva/full47\\_435415.pdf](https://www.scientificbulletin.upb.ro/rev_docs_arhiva/full47_435415.pdf).
- Udriște, C., Ferrara, M., and Opreș, D. (2004). *Economic Geometric Dynamics*. Monographs and textbooks. Geometry Balkan Press. URL: <http://www.mathem.pub.ro/dept/cu-bk.pdf>.

<sup>a</sup> University Politehnica of Bucharest  
Department of Mathematics & Informatics  
Splaiul Independentei 313, Bucharest 060042, Romania

To whom correspondence should be addressed | email: [udriste@mathem.pub.ro](mailto:udriste@mathem.pub.ro)

Manuscript received 12 December 2017; communicated 20 June 2018; published online 22 June 2018



© 2018 by the author(s); licensee *Accademia Peloritana dei Pericolanti* (Messina, Italy). This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution 4.0 International License](https://creativecommons.org/licenses/by/4.0/) (<https://creativecommons.org/licenses/by/4.0/>).