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ASYMPTOTIC WAVES IN VISCOANELASTIC MEDIA WITH MEMORY BY THE DOUBLE-SCALE METHOD

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ABSTRACT. In this paper asymptotic dissipative waves in viscoanelastic media with shape and volumetric memory, previously studied by the author in a more classical way, are studied from the point of view of the double-scale method. Then, a physical interpretation is given of a new (fast) variable ξ, related to the family of hypersurfaces S, across which the solutions or/and some of their derivatives vary steeply, whereas their variation is slow along S and related to the old variables x^α (called slow variables). The double-scale method is sketched. The thermodynamical model governing in the three-dimensional case the motion of the rheological media under consideration is presented and many results regarding the propagation of a particular solution into a uniform unperturbed state and the approximation of the first order of the wave front of this solution are given in full detail and reviewed by the double-scale method. Other novel results are worked out. The thermodynamic models for viscoanelastic media with memory are applied in rheology and in several technological sectors.

1. Introduction

The present paper is one of a series in the area of asymptotic waves in rheological media, a subject which had been already studied by the author with classical techniques (Ciancio and Restuccia 1985a,b, 1987), following Boillat (1976) and Fusco (1979) (see also Lax 1953, 1954; Jeffrey 1963; Jeffrey and Taniuti 1964; Boillat 1965; Choquet-Bruhat 1969; Jeffrey 1976; Donato and Greco 1986). Georgescu (1995) studied asymptotic solutions from the point of view of the double-scale method. Georgescu and Restuccia (2006) sketched out the general approach of the double-scale method to nonlinear hyperbolic partial differential equations (PDEs) in order to study asymptotic waves. The double-scale method had been applied to the case of asymptotic waves in Maxwell media as well as to viscoanelastic media without shape and volumetric memory in the one dimension and in three dimensions, respectively (Restuccia and Georgescu 2008; Georgescu and Restuccia 2011, 2017). Here, we apply this method to the study of asymptotic waves which had been previously studied by Ciancio and Restuccia (1987) in a more classical way in viscoanelastic media with

[†]*This paper is dedicated to the memory of Prof. Adelina Georgescu (1942-2010), unforgettable friend, great teacher and eminent scientist.*

shape and bulk memory in three dimensions. The one-dimensional case had been treated by Georgescu and Restuccia (2010) with original results.

The multiple-scale method, and, in particular, the double-scale approach, is applied when, for instance, the characteristics of the motion vary steeply at some well-determined times or space coordinates, whereas at larger scales the characteristics are slow and describe another type of motion. Furthermore, the scales are defined by some small parameters. The mathematical aspects involved into the study of asymptotic solutions of PDEs belong to the singular perturbation theory (Kryloff and Bogoliuboff 1950; Bogoliubov and Mitropolsky 1961; Mitropol'skij 1966; Cole 1968; O'Malley 1968; Kevorkian 1972; Lagerstrom and Casten 1972; Smith 1975; Wolkind 1977; Eckhaus 1979; Veronis 1980; Georgescu 1995).

In this paper, following the double-scale method, the solution is looked for in the form of an asymptotic expansion with respect to an asymptotic sequence of powers of some small parameter, which is related to the thickness of internal layers of a hypersurface S, across which solutions or/and some of their derivatives vary steeply. Then, a physical interpretation is given of the variable ξ, as a new (fast) variable related to the family of hypersurfaces S across which the solutions or/and some of their derivatives vary steeply, whereas along S their variation is slow and related to the old variables x^α (the so-called slow variables). The asymptotic method involving ξ is known as the double-scale method (Georgescu and Restuccia 2006). In Section 2 we present this method, obtaining the first and second asymptotic approximation. In Sections 3, 4, 5 and 6 we introduce the equations governing the motion of viscoanelastic media with memory in three dimensions, in the framework of classical irreversible thermodynamics (TIP) with internal variables (Kluitenberg 1968; Kluitenberg and Ciancio 1978; Ciancio and Kluitenberg 1979). Many results concerning the propagation of a particular solution into a uniform unperturbed state and the asymptotic approximation of first order of the wave front of this solution are given in full detail and are reviewed by the double-scale method. In addition, other novel results are derived.

2. The double-scale method applied to nonlinear PDEs

Let E^{3+1} be an Euclidean space, let $P \in E^{3+1}$ be a current point, let $\mathbf{U} = \mathbf{U}(P)$ be the unknown vector function solution of a system of PDEs written in the following matrix form

$$\mathbf{A}^\alpha(\mathbf{U})\mathbf{U}_\alpha + \omega^{-1}\left[\mathbf{H}^k \frac{\partial^2 \mathbf{U}}{\partial t \partial x^k} + \mathbf{H}^{ik}\frac{\partial^2 \mathbf{U}}{\partial x^i \partial x^k}\right] = \mathbf{B}(\mathbf{U}), \tag{1}$$

$$(\alpha = 0,1,2,3) \quad \text{and} \quad (i,k = 1,2,3),$$

where $x^0 = t$ (time), x^1, x^2, x^3 are the space coordinates, $\mathbf{U}$ is the vector of the unknown functions (which depend on x^α), $\mathbf{U}_\alpha = \frac{\partial \mathbf{U}}{\partial x^\alpha}$, $\mathbf{A}^\alpha$, $\mathbf{H}^k$, $\mathbf{H}^{ik}$ are appropriate matrices, and

$$\mathbf{A}^\alpha(\mathbf{U})\mathbf{U}_\alpha = \mathbf{B}(\mathbf{U}), \tag{2}$$

is the associated system of hyperbolic partial differential equations (PDEs).

Ciancio and Restuccia (1987) showed that the motion of viscoanelastic media with memory, in the isothermal case, is described by a system of nonlinear partial differential equations (PDEs) having the matrix form (1), where $\mathbf{A}^\alpha$, $\mathbf{H}^k$, $\mathbf{H}^{ik}$ are appropriate matrices 13×13. The system of PDEs (1) include terms containing second order derivative multiplied by a very small parameter $\omega^{-1} \to 0$, with $\omega \gg 1$ very large), that have a balancing effect on the non-linear steepening of waves.

The non-linear dissipative waves were worked out by Ciancio and Restuccia (1987) following a procedure illustrated by Boillat (1976) and generalized by Fusco (1979). Following Jeffrey (1976), the solution hypersurfaces for systems of PDEs are referred to as waves because they may be interpreted as representing propagating wavefronts. When physical problems are associated with such interpretation the solution on the side of the wavefront towards which propagation takes place may then be regarded as being the *undisturbed solution* ahead of the wavefront, whilst the solution on the other side may be regarded as a propagating *disturbance wave* which is entering a region occupied by the undisturbed solution. We deal with those smooth waves $\mathbf{U}(x^\alpha)$, called asymptotic waves, which evolve as *progressive waves*. Now, introducing the point of view of the double-scale method, we see that there exists a family of moving hypersurfaces *S (t)*, parametrized by the time t, (defined by the equation $\varphi(x^\alpha) = 0$) moving in the Euclidean space E^{3+1} (consisting of points of coordinates x^α, $\alpha = 0,1,2,3$, or, equivalently, of the time $t = x^0$ and the space coordinates x^i, $i = 1,2,3$):

$$\varphi(t,x^i) = \bar{\xi} = const, \tag{3}$$

such that $\mathbf{U}$ or/and some of their derivatives vary steeply across S, while along S their variation is slow (Georgescu and Restuccia 2006). This means that around S there exist (asymptotic) *internal layers*, such that the order of magnitude (*i.e.*, the scale) of some derivatives of the solution inside these layers and far away from them differ very much. Therefore, it is natural to introduce a new variable ξ, related to the hypersurfaces S:

$$\xi = \omega\bar{\xi} = \omega\varphi(t,x^i), \tag{4}$$

where $\xi = \frac{\varphi(t,x^i)}{\omega^{-1}}$ is asymptotically fixed, *i.e.*, $\xi = Ord(1)$ as $\omega^{-1} \to 0$, and $\omega \gg 1$ is a very large parameter, to assume that the solution depends on the old variables x^α as well as the new variable ξ, *i.e.*, $\mathbf{U} = \mathbf{U}(x^\alpha, \xi)$. The coefficient ω^{-1} is the small parameter associated with the order of magnitude of the interior layers. Consequently, the derivatives with respect to x^α must be replaced by

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} + \omega\frac{\partial}{\partial \xi}\frac{\partial \varphi}{\partial x^\alpha} \qquad (\alpha = 0,1,2,3). \tag{5}$$

Taking into account that $\mathbf{U}$ is sufficiently smooth, hence it has sufficiently many bounded derivatives. It follows that, except for the terms containing ω, all other terms are asymptotically fixed and the computation can proceed formally. In this way, if $x^\alpha = x^\alpha(s)$ are the parametric equations of a curve C in E^{3+1}, taking into account Eq. (5) we have

$$\frac{d\mathbf{U}}{ds} = \left(\frac{\partial \mathbf{U}}{\partial x^\alpha} + \frac{\partial \mathbf{U}}{\partial \xi}\frac{\partial \xi}{\partial x^\alpha}\right)\frac{dx^\alpha}{ds} = \frac{\partial \mathbf{U}}{\partial x^\alpha}\frac{dx^\alpha}{ds} + \omega\frac{\partial \mathbf{U}}{\partial \xi}\frac{d\varphi}{ds},$$

with $\mathbf{U} = \mathbf{U}(x^\alpha(s), \xi(x^\alpha(s)))$, where the dummy index convention is understood. This relation shows that, indeed, along C, $\mathbf{U}$ does not vary too much if C belongs to the hypersurface $S(t)$ (in this case $\frac{d\varphi}{ds} = 0$) but it has a large variation if C is not situated on $S(t)$. For these reasons ξ is referred to as the fast variable.

Then, we look for the solution of the equations as an asymptotic series of powers of a small parameter, ε, namely with respect to the asymptotic sequence $\{1, \varepsilon^{a+1}, \varepsilon^{a+2}, ...\}$ or

$\left\{1,\varepsilon^{\frac{1}{p}},\varepsilon^{\frac{2}{p}},...\right\}$, as $\varepsilon \to 0$. In particular we consider $p=1$ and $\varepsilon=\omega^{-1}$, such that $\mathbf{U}(x^\alpha,\xi)$ is written as an asymptotic power series of the small parameter ω^{-1} around the initial unperturbed state $\mathbf{U}^0(x^\alpha,\xi)$, *i.e.*, with respect to the asymptotic sequence $1,\omega^{-1},\omega^{-2},...,$ as $\omega^{-1}\to 0$, where $\mathbf{U}^i$ $(i=1,2,...)$ are functions of x^α and ξ:

$$\mathbf{U}(x^\alpha,\xi)\sim\mathbf{U}^0(x^\alpha,\xi)+\omega^{-1}\mathbf{U}^1(x^\alpha,\xi)+O(\omega^{-2}),\, as\ \omega^{-1}\to 0, \tag{6}$$

where $\mathbf{U}^0(x^\alpha,\xi)$ is a known solution of (2)

$$\mathbf{A}^\alpha(\mathbf{U}^0)\mathbf{U}_\alpha(\mathbf{U}^0)=\mathbf{B}(\mathbf{U}^0), \tag{7}$$

and it is taken constant and as the initial, unperturbed state, where no small parameters occur. We recall that $\xi=\omega\varphi(x^\alpha)$, $\omega\gg 1$ is a real parameter and $\varphi(x^\alpha)$ is the unknown wavefront which is to be determined. Then, the derivative $\mathbf{U}_\alpha=\frac{\partial\mathbf{U}}{\partial x^\alpha}$ has the following form:

$$\frac{\partial\mathbf{U}}{\partial x^\alpha}\sim\omega^{-1}\left(\frac{\partial\mathbf{U}^1}{\partial x^\alpha}+\omega\frac{\partial\mathbf{U}^1}{\partial\xi}\frac{\partial\varphi}{\partial x^\alpha}\right)+\omega^{-1}\frac{\partial\mathbf{U}^2}{\partial\xi}\frac{\partial\varphi}{\partial x^\alpha}+O(\omega^{-2}),\ \text{as}\,\omega^{-1}\to 0\,,$$

and the following asymptotic expansions are deduced for $\mathbf{A}^\alpha$, $\mathbf{H}^k$, $\mathbf{H}^{ik}$ and $\mathbf{B}$:

$$\mathbf{A}^\alpha(\mathbf{U})\sim\mathbf{A}^\alpha(\mathbf{U}^0)+\frac{1}{\omega}\nabla\mathbf{A}^\alpha(\mathbf{U}^0)\mathbf{U}^1+O\left(\frac{1}{\omega^2}\right),\, as\ \omega^{-1}\to 0, \tag{8}$$

$$\mathbf{H}^k(\mathbf{U})\sim\mathbf{H}^k(\mathbf{U}^0)+\frac{1}{\omega}\nabla\mathbf{H}^k(\mathbf{U}^0)\mathbf{U}^1+O\left(\frac{1}{\omega^2}\right),\, as\ \omega^{-1}\to 0\ (k=1,2,3), \tag{9}$$

$$\mathbf{H}^{ik}(\mathbf{U})\sim\mathbf{H}^{ik}(\mathbf{U}^0)+\frac{1}{\omega}\nabla\mathbf{H}^{ik}(\mathbf{U}^0)\mathbf{U}^1+O\left(\frac{1}{\omega^2}\right),\, as\ \omega^{-1}\to 0\ (i,k=1,2,3), \tag{10}$$

$$\mathbf{B}(\mathbf{U})\sim\mathbf{B}(\mathbf{U}^0)+\frac{1}{\omega}\nabla\mathbf{B}(\mathbf{U}^0)\mathbf{U}^1+O\left(\frac{1}{\omega^2}\right),\, as\ \omega^{-1}\to 0, \tag{11}$$

where $\nabla=\frac{\partial}{\partial\mathbf{U}}$.

Then, we introduce the asymptotic expansions (8) - (11) into the equations of the system (1) in order to obtain the sets of equations of various order of asymptotic approximation. Each such set has as a solution one approximation $\mathbf{U}^r(r=1,2,...)$ of $\mathbf{U}$. For the first and second asymptotic approximation we have:

$$(\mathbf{A}^\alpha)_0\Phi_\alpha\frac{\partial\mathbf{U}^1}{\partial\xi}=\mathbf{0}\quad(\alpha=0,1,2,3), \tag{12}$$

$$\begin{gathered}(\mathbf{A}^\alpha)_0\left(\Phi_\alpha\frac{\partial\mathbf{U}^2}{\partial\xi}\right)+(\nabla\mathbf{A}^\alpha)_0\mathbf{U}^1\left(\Phi_\alpha\frac{\partial\mathbf{U}^2}{\partial\xi}\right)=\\-(\mathbf{A}^\alpha)_0\left(\frac{\partial\mathbf{U}^1}{\partial x^\alpha}\right)-(\mathbf{H}^k)_0\Phi_0\Phi_k\frac{\partial^2\mathbf{U}^1}{\partial\xi^2}-(\mathbf{H}^{ik})_0\Phi_i\Phi_k\frac{\partial^2\mathbf{U}^1}{\partial\xi^2}+(\nabla\mathbf{B})_0\mathbf{U}^1,\end{gathered} \tag{13}$$

where $\Phi_\alpha=\frac{\partial\varphi}{\partial x^\alpha}$ $(\Phi_k=\frac{\partial\varphi}{\partial x^k},\ k=1,2,3)$ and the symbol $(...)_0$ indicates that the quantities are calculated in $\mathbf{U}^0$. Eq. (12) is linear in $\mathbf{U}^1$, while (13) is affine in $\mathbf{U}^2$.

We conclude this section by recalling that the wavefront φ is an unknown function. In order to determine it, we recall its equation is $\varphi(t,x^1,x^2,x^3)=0$. This implies that along the wavefront we have $d\varphi/dt = 0$, implying $\partial\varphi/\partial t + \mathbf{v}\cdot grad\varphi = 0$, or equivalently, $(\partial\varphi/\partial t)|grad\varphi|^{-1} + \mathbf{v}\cdot grad\varphi|grad\varphi|^{-1} = 0$. Obviously,

$$\frac{grad\varphi}{|grad\varphi|} = \mathbf{n}, \tag{14}$$

such that the previous equality reads

$$\frac{\frac{\partial\varphi}{\partial t}}{|grad\varphi|} + \mathbf{v}\cdot\mathbf{n} = 0. \tag{15}$$

Introducing the notation

$$\lambda = -\frac{\frac{\partial\varphi}{\partial t}}{|grad\varphi|}, \tag{16}$$

we have

$$\lambda(\mathbf{U},\mathbf{n}) = \mathbf{v}\cdot\mathbf{n}, \tag{17}$$

where λ is the velocity normal to the progressive wave.

3. Stress-strain relations for viscoanelastic media with memory

Assuming that several microscopic phenomena due to effects of lattice defects give rise to inelastic strains, Kluitenberg (1968) formulated a general thermodynamic theory for mechanical phenomena in continuous media, in the framework of classical irreversible thermodynamics with internal variables.

The total strain is given by elastic and inelastic deformations $\varepsilon_{\alpha\beta} = \varepsilon^{(el)}_{\alpha\beta} + \varepsilon^{(in)}_{\alpha\beta}$ (Kluitenberg 1968). The inelastic strain is due to lattice defects and related to microscopic phenomena: $\varepsilon^{(in)}_{\alpha\beta} = \sum_{k=1}^{n}\varepsilon^{(k)}_{\alpha\beta}$, where $\varepsilon^{(k)}_{\alpha\beta}$ is the k-th contribution to the inelastic strain of the k-th microscopic phenomenon. $\varepsilon^{(k)}_{\alpha\beta}$ is called partial inelastic strain tensor. Kluitenberg (1968) introduced $n-1$ partial inelastic strain tensors $\varepsilon^{(k)}_{\alpha\beta}$ as internal variables in the thermodynamic state vector. Furthermore, ε_{ik} is assumed to be small, i. e. $\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial}{\partial x^k}u_i + \frac{\partial}{\partial x^i}u_k\right)$, where u_i is the i-th component of the displacement field $\mathbf{u}$ and x^i is the i-th component of the position vector $\mathbf{x}$ in Eulerian coordinates in a Cartesian reference frame. From the theory it follows that several types of macroscopic stress fields may occur in a medium: a stress field $\tau^{(eq)}_{\alpha\beta}$ of thermoelastic nature, a stress field $\tau^{(vi)}_{\alpha\beta}$ analogous to the viscous stress in ordinary fluids and n stress fields $\tau^{(k)}_{\alpha\beta}(k = 1,2,...,n)$, in arbitrary number, connected with microscopic stress fields surrounding imperfections in the medium. The stress field $\tau^{(eq)}_{\alpha\beta} + \tau^{(vi)}_{\alpha\beta}$ is the mechanical stress field which occurs in the equations of motion and in the first law of thermodynamics. Temperature effects were fully taken into account. Explicit stress-strain relations, with constant phenomenological coefficients, were deduced, linearizing equations of state. For distortional phenomena in isotropic media the following linear stress-strain relation among the deviators of the mechanical stress tensor, the first n derivatives with respect to time of this tensor, the tensor of total strain (the sum of the elastic and inelastic strains) and the first $n+1$ derivatives with respect to time of this last tensor was deduced, where n is the number of phenomena that give rise to inelastic deformations

$$R^{(\tau)}_{(d)0}\tilde{\tau}_{ik}+\sum_{m=1}^{n-1}R^{(\tau)}_{(d)m}\frac{d^m}{dt^m}\tilde{\tau}_{ik}+\frac{d^n}{dt^n}\tilde{\tau}_{ik}=R^{(\varepsilon)}_{(d)0}\tilde{\varepsilon}_{ik}+\sum_{m=1}^{n+1}R^{(\varepsilon)}_{(d)m}\frac{d^m}{dt^m}\tilde{\varepsilon}_{ik}\ (i,k=1,2,3).$$

The quantities $R^{(\tau)}_{(d)m}$ $(m=0,1,...,n-1)$, $R^{(\varepsilon)}_{(d)m}$ $(m=0,1,...,n+1)$ are algebraic functions of the coefficients occurring in the phenomenological equations and in the equations of state. We also have the following stress-strain relation:

$$R^{(\tau)}_{(v)0}\tau'+\sum_{m=1}^{n-1}R^{(\tau)}_{(v)m}\frac{d^m\tau'}{dt^m}+\frac{d^n\tau'}{dt^n}=R^{(\varepsilon)}_{(v)0}\varepsilon+\sum_{m=1}^{n+1}R^{(\varepsilon)}_{(v)m}\frac{d^m\varepsilon}{dt^m}+R^{(T)}_{(v)0}(T-T^0)+\sum_{m=1}^{n}R^{(T)}_{(v)m}\frac{d^m}{dt^m}T, \tag{18}$$

where τ and ε are the scalar parts of the stress tensor and strain tensor, respectively, $\tau'=\tau-\tau^0$, and τ^0 and T^0 are the scalar part of the stress tensor and the temperature of the medium in a state of thermodynamic equilibrium. This equilibrium state plays the role of a reference state. The quantities $R^{(\tau)}_{(v)m}$ $(m=0,1,...,n-1)$, $R^{(\varepsilon)}_{(v)m}$ $(m=0,1,...,n+1)$ and $R^{(T)}_{(v)m}$ $(m=0,1,...,n)$ are algebraic functions of the coefficients occurring in the phenomenological equations and in the equations of state. Stress-strain relations for Maxwell, Kelvin (Voigt), Jeffreys, Poyting-Thomson, Prandtl-Reuss and other rheological media were deduced from the mentioned above general relations and in the case in which only one microscopic phenomenon (described by one internal variable) was assumed giving rise to inelastic strain, in the isothermal case, the stress strain relations describing the behaviour of isotropic viscoanelastic media, of order one $(n=1)$, with memory were derived in the following form (see Kluitenberg and Ciancio 1978; Ciancio and Kluitenberg 1979):

$$R^{(\tau)}_{(d)0}\tilde{P}_{ik}+\frac{d}{dt}\tilde{P}_{ik}+R^{(\varepsilon)}_{(d)0}\tilde{\varepsilon}_{ik}+R^{(\varepsilon)}_{(d)1}\frac{d}{dt}\tilde{\varepsilon}_{ik}+R^{(\varepsilon)}_{(d)2}\frac{d^2}{dt^2}\tilde{\varepsilon}_{ik}=0, \tag{19}$$

$$R^{(\tau)}_{(v)0}P'+\frac{d}{dt}P'+R^{(\varepsilon)}_{(v)0}\varepsilon+R^{(\varepsilon)}_{(v)1}\frac{d}{dt}\varepsilon+R^{(\varepsilon)}_{(v)2}\frac{d^2}{dt^2}\varepsilon=0. \tag{20}$$

Introducing the deviator $\tilde{P}_{ik}$ and the scalar part P of the mechanical pressure tensor P_{ik}, and defining P_{ik} in terms of the symmetric Cauchy tensor $P_{ik}=-\tau_{ik}$ $(i,k=1,2,3)$, $P'=P-P^0=-(\tau-\tau^0)$, one has

$$\tilde{P}_{ik}=P_{ik}-\frac{1}{3}P_{ss}\delta_{ik},\quad P=\frac{1}{3}P_{ss},\quad P_{ss}=trP,$$
$$P_{ik}=\tilde{P}_{ik}+P\delta_{ik},\quad \tilde{P}_{ss}=0.$$

Analogous definitions are valid for $\tilde{\varepsilon}_{ik}$ and ε. In Eqs. (19) and (20) the coefficients satisfy the following relations

$$R^{(\tau)}_{(d)0}=a^{(1,1)}\eta_s^{(1,1)}\geq 0, \tag{21}$$

$$R^{(\varepsilon)}_{(d)0}=a^{(0,0)}(a^{(1,1)}-a^{(0,0)})\eta_s^{(1,1)}\geq 0, \tag{22}$$

$$R^{(\varepsilon)}_{(d)1}=\left\{a^{(0,0)}\left(1+2\eta_s^{(0,1)}\right)+a^{(1,1)}\left[\eta_s^{(0,0)}\eta_s^{(1,1)}+\left(\eta_s^{(0,1)}\right)^2\right]\right\}\geq 0, \tag{23}$$

$$R_{(d)2}^{(\varepsilon)} = \eta_s^{(0,0)} \geq 0, \quad R_{(v)0}^{(\tau)} = b^{(1,1)}\eta_{(v)}^{(1,1)} \geq 0, \tag{24}$$

$$R_{(v)0}^{(\varepsilon)} = b^{(0,0)}(b^{(1,1)} - b^{(0,0)})\eta_{(v)}^{(1,1)} \geq 0, \quad R_{(v)2}^{(\varepsilon)} = \eta_v^{(0,0)} \geq 0, \tag{25}$$

$$R_{(v)1}^{(\varepsilon)} = \left\{ b^{(0,0)}\left(1 + 2\eta_v^{(0,1)}\right) + b^{(1,1)}\left[\eta_v^{(0,0)}\eta_v^{(1,1)} + \left(\eta_v^{(0,1)}\right)^2\right]\right\} \geq 0, \tag{26}$$

$$\begin{gathered} R_{(d)1}^{(\varepsilon)} - R_{(d)0}^{(\tau)}R_{(d)2}^{(\varepsilon)} \geq 0\,,\, R_{(d)1}^{(\varepsilon)}R_{(d)0}^{(\tau)} - R_{(d)0}^{(\varepsilon)} \geq 0\,, \\ R_{(v)1}^{(\varepsilon)} - R_{(v)0}^{(\tau)}R_{(v)2}^{(\varepsilon)} \geq 0\,,\, R_{(v)1}^{(\varepsilon)}R_{(v)0}^{(\tau)} - R_{(v)0}^{(\varepsilon)} \geq 0. \end{gathered}$$

In Eqs. (21) - (26) $a^{(0,0)}$, $a^{(1,1)}, b^{(0,0)}$ and $b^{(1,1)}$ are scalar constants which occur in the equations of state, while the coefficients $\eta_s^{(0,0)}$, $\eta_s^{(0,1)}$, $\eta_s^{(1,1)}$, $\eta_v^{(0,0)}$, $\eta_v^{(0,1)}$, $\eta_v^{(1,1)}$ are called *phenomenological coefficients* and represent fluidities. Indicating by v_i the ith-component of the velocity field $\mathbf{v}$, we have

$$v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + v_j \frac{\partial u_i}{\partial x^j}, \tag{27}$$

where the definition of material derivative was taken into consideration. The balance equations for the mass density and momentum in the case of viscoanelastic media with shape an bulk memory read

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \qquad (i = 1,2,3), \tag{28}$$

$$\rho\left(\frac{\partial}{\partial t}v_i + v_k\frac{\partial}{\partial x^k}v_i\right) + \frac{\partial}{\partial x^k}\tilde{P}_{ik} + \frac{\partial}{\partial x_i}P = 0. \tag{29}$$

Equations (19) and (20) read

$$\begin{aligned} & R_{(d)0}^{(\tau)}\tilde{P}_{ik} + \frac{\partial}{\partial t}\tilde{P}_{ik} + v_p\frac{\partial}{\partial x^p}\tilde{P}_{ik} + R_{(d)0}^{(\varepsilon)}\left[\frac{1}{2}\left(\frac{\partial}{\partial x^k}u_i + \frac{\partial}{\partial x^i}u_k\right) - \frac{1}{3}\frac{\partial}{\partial x^p}u_p\delta_{ik}\right] \\ & + R_{(d)1}^{(\varepsilon)}\left[\frac{1}{2}\left(\frac{\partial}{\partial x^k}v_i + \frac{\partial}{\partial x^i}v_k\right) - \frac{1}{3}\frac{\partial}{\partial x^p}v_p\delta_{ik}\right] \\ & + R_{(d)2}^{(\varepsilon)}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial t\partial x^k}v_i + v_p\frac{\partial^2}{\partial x^p\partial x^k}v_i + \frac{\partial^2}{\partial t\partial x^i}v_k + v_p\frac{\partial^2}{\partial x^p\partial x^i}v_k\right)\right] \\ & - R_{(d)2}^{(\varepsilon)}\left[\frac{1}{3}\left(\frac{\partial^2}{\partial t\partial x^p}v_p + v_q\frac{\partial^2}{\partial x^q\partial x^p}v_p\right)\delta_{ik}\right] = 0 \quad (p,q = 1,2,3), \end{aligned} \tag{30}$$

where the relation $\frac{d\varepsilon_{ik}}{dt} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$ was used,

$$\begin{aligned} & R_{(v)0}^{(\tau)}P' + \frac{\partial}{\partial t}P' + v_p\frac{\partial}{\partial x^p}P' + \frac{1}{3}R_{(v)0}^{(\varepsilon)}\frac{\partial}{\partial x^p}u_p \\ & + \frac{1}{3}R_{(v)1}^{(\varepsilon)}\frac{\partial}{\partial x^p}v_p + \frac{1}{3}R_{(v)2}^{(\varepsilon)}\left(\frac{\partial^2}{\partial t\partial x^p}v_p + v_q\frac{\partial^2}{\partial x^q\partial x^p}v_p\right) = 0 \end{aligned} \tag{31}$$

It is easy to see how the above mentioned system of equations takes the matrix form (1), where $A^0(\mathbf{U}) = I$ is the identity matrix and A^i, $\mathbf{H}^k$, $\mathbf{H}^{ik}$, $(i,k = 1,2,3)$ are appropriate 13×13 square matrices which follow directly from the system (see the Appendix). In particular, indicating by $a = R^{(\varepsilon)}_{(d)1}, b = R^{(\varepsilon)}_{(v)1}, \alpha = R^{(\varepsilon)}_{(d)0}$ and $\beta = R^{(\varepsilon)}_{(v)0}$, $A^i(i,k = 1,2,3)$ have the following form

$$A^1 = \begin{pmatrix}
v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v_1 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 & \frac{1}{\rho} \\
0 & 0 & 0 & 0 & 0 & v_1 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v_1 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\
\frac{2}{3}\alpha & 0 & 0 & 0 & \frac{2}{3}a & 0 & 0 & v_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}a & 0 & 0 & v_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}a & 0 & 0 & v_1 & 0 & 0 & 0 \\
-\frac{1}{3}\alpha & 0 & 0 & 0 & -\frac{1}{3}a & 0 & 0 & 0 & 0 & 0 & v_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1 & 0 \\
\frac{1}{3}\beta & 0 & 0 & 0 & \frac{1}{3}b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1
\end{pmatrix},$$

$$A^2 = \begin{pmatrix}
v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & \frac{1}{\rho} \\
0 & 0 & 0 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 \\
0 & -\frac{1}{3}\alpha & 0 & 0 & 0 & -\frac{1}{3}a & 0 & v_2 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}a & 0 & 0 & 0 & v_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 \\
0 & \frac{2}{3}\alpha & 0 & 0 & 0 & \frac{2}{3}a & 0 & 0 & 0 & 0 & v_2 & 0 & 0 \\
0 & 0 & \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}a & 0 & 0 & 0 & 0 & v_2 & 0 \\
0 & \frac{1}{3}\beta & 0 & 0 & 0 & \frac{1}{3}b & 0 & 0 & 0 & 0 & 0 & 0 & v_2
\end{pmatrix},$$

$$A^3 = \begin{pmatrix}
v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v_3 & -\frac{1}{\rho} & 0 & 0 & -\frac{1}{\rho} & 0 & 0 \\
0 & 0 & -\frac{1}{3}\alpha & 0 & 0 & 0 & -\frac{1}{3}a & v_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 & 0 \\
\frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}a & 0 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{3}\alpha & 0 & 0 & 0 & -\frac{1}{3}a & 0 & 0 & 0 & v_3 & 0 & 0 \\
0 & \frac{1}{2}\alpha & 0 & 0 & 0 & \frac{1}{2}a & 0 & 0 & 0 & 0 & 0 & v_3 & 0 \\
0 & 0 & \frac{1}{3}\beta & 0 & 0 & 0 & \frac{1}{3}b & 0 & 0 & 0 & 0 & 0 & v_3
\end{pmatrix}.$$

Furthermore,

$$\mathbf{U} = (u_1, u_2, u_3, \rho, v_1, v_2, v_3, \tilde{P}_{11}, \tilde{P}_{12}, \tilde{P}_{13}, \tilde{P}_{22}, \tilde{P}_{23}, P')^T, \tag{32}$$

$$\text{and} \quad \mathbf{B} = (v_1, v_2, v_3, 0, 0, 0, 0, \tilde{P}^*_{11}, \tilde{P}^*_{12}, \tilde{P}^*_{13}, \tilde{P}^*_{22}, \tilde{P}^*_{23}, P'^*)^T, \tag{33}$$

$$\text{where} \quad \tilde{P}^*_{ik} = -R^{(\tau)}_{(d)0}\tilde{P}_{ik}, \qquad \tilde{P}^*_{ik} = -R^{(\tau)}_{(v)0}P'. \tag{34}$$

The symbol $(...)^T$ means that $\mathbf{U}$ and $\mathbf{B}$ are column vectors. The above equations (27) – (31) form a system of thirteen PDEs for the three components of the displacement field, the mass density, the three components of the velocity field, the five independent components of $\tilde{P}_{ik}$ and P'.

4. Propagation into an uniform unperturbed state

An uniform unperturbed state is considered in which

$$\mathbf{U}^0 = (0,0,0,\rho^0,0,0,0,0,0,0,0,0,0). \tag{35}$$

Now, by introducing the quantities $\lambda = -\frac{\partial\varphi/\partial t}{|grad\varphi|}$ and $\mathbf{n} = \frac{grad\varphi}{|grad\varphi|}$ in Eq. (12), it takes the form:

$$(\mathbf{A}_{0n} - \lambda\mathbf{I})\frac{\partial\mathbf{U}^1}{\partial\xi} = 0, \tag{36}$$

with $\mathbf{A}_{0n} = (\mathbf{A}_n)_0$, and

$$\mathbf{A}_n(\mathbf{U}) = \mathbf{A}^i n_i =$$

$$\begin{pmatrix}
v_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & v_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_n & \rho n_1 & \rho n_2 & \rho n_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v_n & 0 & 0 & \frac{n_1}{\rho} & \frac{n_2}{\rho} & \frac{n_3}{\rho} & 0 & 0 & \frac{n_1}{\rho} \\
0 & 0 & 0 & 0 & 0 & v_n & 0 & 0 & \frac{n_1}{\rho} & 0 & \frac{n_2}{\rho} & \frac{n_3}{\rho} & \frac{n_2}{\rho} \\
0 & 0 & 0 & 0 & 0 & 0 & v_n & -\frac{n_3}{\rho} & 0 & \frac{n_1}{\rho} & -\frac{n_3}{\rho} & \frac{n_2}{\rho} & \frac{n_3}{\rho} \\
\frac{2\alpha n_1}{3} & -\frac{\alpha n_2}{3} & -\frac{\alpha n_3}{3} & 0 & \frac{2an_1}{3} & -\frac{an_2}{3} & -\frac{an_3}{3} & v_n & 0 & 0 & 0 & 0 & 0 \\
\frac{\alpha n_2}{2} & \frac{\alpha n_1}{2} & 0 & 0 & \frac{an_2}{2} & \frac{an_1}{2} & 0 & 0 & v_n & 0 & 0 & 0 & 0 \\
\frac{\alpha n_3}{2} & 0 & \frac{\alpha n_1}{2} & 0 & \frac{an_3}{2} & 0 & \frac{an_1}{2} & 0 & 0 & v_n & 0 & 0 & 0 \\
-\frac{\alpha n_1}{3} & \frac{2\alpha n_2}{3} & -\frac{\alpha n_3}{3} & 0 & -\frac{an_1}{3} & \frac{2an_2}{3} & -\frac{an_3}{3} & 0 & 0 & 0 & v_n & 0 & 0 \\
0 & \frac{\alpha n_3}{2} & \frac{\alpha n_2}{2} & 0 & 0 & \frac{an_3}{2} & \frac{an_2}{2} & 0 & 0 & 0 & 0 & v_n & 0 \\
\frac{\beta n_1}{3} & \frac{\beta n_2}{3} & \frac{\beta n_3}{3} & 0 & \frac{bn_1}{3} & \frac{bn_2}{3} & \frac{bn_3}{3} & 0 & 0 & 0 & 0 & 0 & v_n
\end{pmatrix}. \tag{37}$$

From Eq. (36) one has

$$\mathbf{U}^1(x^\alpha,\xi) = u(x^\alpha,\xi)\mathbf{r}^{(+)}(\mathbf{U}^0,\mathbf{n}) + v^1(x^\alpha), \tag{38}$$

where u is a scalar function to be determined and v^1 is an arbitrary vector of integration which can be taken as zero, without loss of generality. Consequently, in order to determine $\mathbf{U}^1$ we must determine u, which gives rise to a phenomenon of distortion of the signals and governs the first-order perturbation obeying a non-linear partial differential equation, that will be studied in Section 6. Of course, equations of asymptotic approximation of higher order can be written and they are affine, but their solution is very difficult.

5. Matrix $\mathbf{A}_n$ and its eigenvalues and eigenvectors

The eigenvalues of $\mathbf{A}_n(\mathbf{U})$ were found to be (Ciancio and Restuccia 1987):

$$\lambda_1 = \mathbf{v}\cdot\mathbf{n} = v_n, \quad \lambda_2^{(\pm)} = v_n \pm \sqrt{\frac{a}{2\rho}}, \quad \lambda_3^{(\pm)} = v_n \pm \sqrt{\frac{2a+b}{3\rho}}, \tag{39}$$

where the multiplicity of λ_1 is equal to 7, both $\lambda_2^{(+)}$ and $\lambda_2^{(-)}$ have multiplicity 2 and $\lambda_3^{(\pm)}$ are simple eigenvalues. Ciancio and Restuccia (1987) showed that among the eigenvalues (39) λ_1 and $\lambda_2^{(\pm)}$ satisfy the Lax-Boillat exceptionality condition $\nabla\lambda\cdot\mathbf{r} = 0$, and for this reason the results that will be obtained are valid only for $\lambda_3^{(\pm)}$, for which $\nabla\lambda_3^{(\pm)} \cdot \mathbf{r}_3^{(\pm)} \neq 0$. We focus our attention on $\lambda = \lambda_3^{(+)}$, which corresponds to a progressive fast longitudinal wave traveling to the right (Ciancio and Restuccia 1987).

Denoting $\gamma = \sqrt{\frac{2a+b}{3\rho}}$, we have $\lambda_3^{(+)} = v_n + \gamma$ and the corresponding eigenvectors read

$$\mathbf{l}_3^{(+)} = \frac{1}{\gamma}\left(\gamma\delta\frac{\mathbf{n}}{n_3}0, \gamma\frac{\mathbf{n}}{n_3}, \frac{(n_1^2-n_3^2)}{\rho n_3}, 2\frac{n_1n_2}{\rho n_3}, 2\frac{n_1}{\rho}, \frac{n_2^2-n_3^2}{\rho n_3}, 2\frac{n_2}{\rho}, \frac{1}{\rho n_3}\right), \tag{40}$$

$$\mathbf{r}_3^{(+)} = \frac{1}{\gamma}\left(\mathbf{0},\ \rho,\ \gamma\mathbf{n},\ \frac{a}{3}(3n_1^2-1),\ an_1n_2,\ an_1n_3,\ \frac{a}{3}(3n_2^2-1),\ an_2n_3,\ \frac{b}{3}\right), \tag{41}$$

and they satisfy the relation

$$\mathbf{l}\cdot\mathbf{r} = \frac{2}{n_3}. \tag{42}$$

This condition ensures the hyperbolicity of the system (36). In Eq. (40) δ is given by $\delta = \frac{2R_{(d)0}^{(\varepsilon)}+R_{(v)0}^{(\varepsilon)}}{2R_{(d)1}^{(\varepsilon)}+R_{(v)1}^{(\varepsilon)}}$.

6. First approximation of wave front and U

In this section we show how the wave front $\varphi(t,x^1,x^2,x^3) = 0$ can be determined. Following the general theory (Boillat 1965, 1976), we introduce the quantity

$$\Psi(\mathbf{U},\Phi_\alpha) = \varphi_t + |grad\varphi|\lambda(\mathbf{U},\mathbf{n}). \tag{43}$$

The characteristic equations for (43) are

$$\frac{dx_\alpha}{d\sigma} = \frac{\partial\Psi}{\partial\Phi_\alpha}, \qquad \frac{d\Phi_\alpha}{d\sigma} = -\frac{\partial\Psi}{\partial x_\alpha} \quad (\alpha = 0,1,2,3), \tag{44}$$

where σ is the time along the rays. The $i-$th component of the *radial velocity* Λ is defined by

$$\Lambda_i(\mathbf{U},\mathbf{n}) \equiv \frac{dx_i}{d\sigma} = \frac{\partial\Psi}{\partial\phi_i} = \lambda n_i + \frac{\partial\lambda}{\partial n_i} - \left(\mathbf{n}\cdot\frac{\partial\lambda}{\partial\mathbf{n}}\right)n_i = \lambda n_i + v_i - (n_kv_k)n_i \ \ (i=1,2,3). \tag{45}$$

Hence,

$$\Lambda(\mathbf{U},\mathbf{n}) = \mathbf{v} - (v_n - \lambda)\mathbf{n}. \tag{46}$$

For $\lambda = \lambda_3^{(+)}$ we have

$$\Lambda(\mathbf{U},\mathbf{n}) = \mathbf{v} + \sqrt{\frac{2R_{(d)1}^{(\varepsilon)} + R_{(v)1}^{(\varepsilon)}}{3\rho}}\mathbf{n}. \tag{47}$$

Since we are considering the propagation into a uniform unperturbed state, it is known that the wave front φ satisfies the partial differential equation

$$\Psi(\mathbf{U}^0,\Phi_\alpha) = \varphi_t + |grad\varphi|\lambda_3^{(+)}(\mathbf{U}^0,\mathbf{n}^0) = \Psi^0 = 0, \tag{48}$$

and so

$$\Lambda_i^0(\mathbf{U}^0,\mathbf{n}^0) = \frac{\partial\Psi^0}{\partial\Phi_i} = \sqrt{\frac{2R_{(d)1}^{(\varepsilon)} + R_{(v)1}^{(\varepsilon)}}{3\rho^0}}n_i^0, \tag{49}$$

where $\mathbf{n}^0$ is the constant value of $\mathbf{n}$ and represents the normal vector at the point $(x^i)^0$ defined by

$$\mathbf{n}^0 = \left(\frac{grad\varphi}{|grad\varphi|}\right)_{t=0} = \frac{grad^0\varphi^0}{|grad^0\varphi^0|}, \quad \text{with} \quad (grad^0)_i \equiv \frac{\partial}{\partial(x^i)^0} \quad (i=1,2,3). \tag{50}$$

The characteristic equations for (48) are

$$\frac{dx_\alpha}{d\sigma} = \frac{\partial\Psi^0}{\partial\Phi_\alpha} \quad (\alpha=1,2,3), \tag{51}$$

$$\frac{d\Phi_\alpha}{d\sigma} = -\frac{\partial\Psi^0}{\partial x_\alpha} \quad (\alpha=1,2,3), \tag{52}$$

where σ is the time along the rays. By integration one obtains

$$x^0 = t = \sigma, \quad x^i = (x^i)^0 + \Lambda_i^0 t = (x^i)^0 + \sqrt{\frac{2R^{(\varepsilon)}_{(d)1} + R^{(\varepsilon)}_{(v)1}}{3\rho^0}}\, n_i^0 t, \tag{53}$$

with $(x^i)^0 = (x^i)_{t=0}$ $(i=1,2,3)$. Let $\varphi^0((x^i)^0) = (\varphi)_{t=0}$, be the given initial surface. If the Jacobian of the transformation $\mathbf{x} \to \mathbf{x}|_{t=0}$ is non vanishing: $J = det|t\frac{\partial\Lambda_k^0}{\partial(x^i)^0} + \delta_{ik}| \neq 0$ $(i,k = 1,2,3)$, x_i^0 can be deduced from (53) and φ takes the form:

$$\varphi(t,x^i) = \varphi^0\left(x^i - \Lambda_i^0 t\right) = \varphi^0\left(x^i - \sqrt{\frac{2R^{(\varepsilon)}_{(d)1} + R^{(\varepsilon)}_{(v)1}}{3\rho^0}}\, n_i^0 t\right). \tag{54}$$

Fusco (1979) showed that, by using (12) and (13), the following equation for $u(x_\alpha,\xi)$ can be obtained:

$$\frac{\partial u}{\partial\sigma} + \left(\nabla\Psi\cdot\mathbf{r}_3^{(+)}\right)_0 u\frac{\partial u}{\partial\xi} + \frac{1}{\sqrt{J}}\frac{\partial\sqrt{J}}{\partial\sigma}u + \mu^0\frac{\partial^2 u}{\partial\xi^2} = \nu^0 u, \tag{55}$$

where

$$\left(\nabla\Psi\cdot\mathbf{r}_3^{(+)}\right)_0 = (|grad\varphi|)^0\left(\nabla\lambda_3^{(+)}\cdot\mathbf{r}_3^{(+)}\right)_0, \tag{56}$$

$$\mu_0 = \frac{\left[\mathbf{l}_3^{(+)}\cdot\left(\mathbf{H}^k\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x^k} + \mathbf{H}^{ik}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k}\right)\left(\mathbf{r}_3^{(+)}\right)\right]_0}{\left(\mathbf{l}_3^{(+)}\cdot\mathbf{r}_3^{(+)}\right)_0}, \tag{57}$$

$$\nu^0 = \frac{\left(\mathbf{l}_3^{(+)}\cdot\nabla\mathbf{B}\,\mathbf{r}_3^{(+)}\right)_0}{\left(\mathbf{l}_3^{(+)}\cdot\mathbf{r}_3^{(+)}\right)_0}. \tag{58}$$

Straightforward computations yield:

$$\left(\nabla\Psi\cdot\mathbf{r}_3^{(+)}\right)_0 = \frac{1}{2}(|grad\varphi|)_0, \tag{59}$$

where

$$\nabla\lambda_3^{(+)} = \frac{\partial\lambda_3^{(+)}}{\partial\mathbf{U}} \equiv \left(0,0,0,-\frac{1}{2\rho}\sqrt{\frac{2R^{(\varepsilon)}_{(d)1}+R^{(\varepsilon)}_{(v)1}}{3\rho}},n_1,n_2,n_3,0,0,0,0,0,0\right)$$

$$\text{and} \quad \nabla\lambda_3^{(+)}\cdot\mathbf{r}_3^{(+)} = \frac{1}{2}, \tag{60}$$

$$\mu^0 = \frac{\left(\frac{\partial\varphi}{\partial t}\right)_0 |grad\varphi|_0 \left(2R'^{(\varepsilon)}_{(d)2}+R'^{(\varepsilon)}_{(v)2}\right)}{2\sqrt{3\rho^0\left(2R^{(\varepsilon)}_{(d)1}+R^{(\varepsilon)}_{(v)1}\right)}}, \tag{61}$$

$$\nu^0 = -\frac{\left(2R^{(\varepsilon)}_{(d)1}R^{(\tau)}_{(d)0}-R^{(\varepsilon)}_{(d)0}\right)+\left(R^{(\varepsilon)}_{(v)1}R^{(\tau)}_{(v)0}-R^{(\varepsilon)}_{(v)0}\right)}{2\left(2R^{(\varepsilon)}_{(d)1}+R^{(\varepsilon)}_{(v)1}\right)}, \tag{62}$$

and where we have used Eq. (42) and the relations $R^{(\varepsilon)}_{(d)2} = \omega^{-1}R'^{(\varepsilon)}_{(d)2}$ and $R^{(\varepsilon)}_{(v)2} = \omega^{-1}R'^{(\varepsilon)}_{(v)2}$. By using the transformation of variables (Fusco 1979):

$$u = \frac{z}{\sqrt{J}}e^w, \qquad \kappa = \int_0^\sigma \frac{e^w}{\sqrt{J}}\left(\nabla\Psi\cdot\mathbf{r}_3^{(+)}\right)d\sigma, \quad \text{with} \quad w = -\int_0^\sigma \nu^0 d\sigma, \tag{63}$$

Eq. (55) can be reduced to an equation of the type

$$\frac{\partial z}{\partial\kappa}+z\frac{\partial z}{\partial\xi}+\hat{\mu}^0\frac{\partial^2 z}{\partial\xi^2} = 0, \quad \text{with} \quad \hat{\mu}^0 = \frac{\mu^0\sqrt{J}e^{-w}}{\left(\nabla\Psi\cdot\mathbf{r}_3^{(+)}\right)_0}, \tag{64}$$

which is similar to Burger's equation and is valid along the characteristic rays. Using the obtained results, κ, w and $\hat{\mu}^0$ are given by

$$\kappa = \int_0^\sigma \frac{1}{2}|grad\varphi|\frac{e^w}{\sqrt{J}}d\sigma, \quad w = -\nu^0\sigma \quad \text{and} \qquad \hat{\mu}^0 = \frac{2\mu^0\sqrt{J}e^{\nu^0\sigma}}{(|grad\varphi|)_0}.$$

Equation $(64)_1$ can be reduced to the semilinear heat equation

$$\frac{\partial h}{\partial\kappa} = \hat{\mu}^0\frac{\partial^2 h}{\partial\xi} - hlog\frac{h}{\hat{\mu}^0}\frac{d\hat{\mu}^0}{d\kappa}, \tag{65}$$

which has been extensively studied by many authors and for which the solution is known, using the Hopf transformation (Hopf 1950):

$$v(\xi,\kappa) = \hat{\mu}_0\frac{\partial}{\partial\xi}logh(\xi,\kappa). \tag{66}$$

7. Conclusions

In this article we presented a system of PDEs describing isotropic viscoanelastic media with memory. Since a thermodynamical model has an added value if possible solutions of the derived theory are found, and since the closed-form of solutions of non-linear PDEs are rare, the author had already studied the asymptotic waves in these media in a classical way. Here, applying the double-scale method, asymptotic approximated solutions were re-interpreted as asymptotic sequences of powers of a small parameter. This parameter is related to the thickness of internal layers of a hypersurface S, across which the solutions or/and some of their derivatives vary steeply and permit to introduce a new and very fast variable. This means that around S there exist internal layers such that the order of magnitude (*i.e.*, the scale) of some derivatives of the solutions across these layers is different then that along S. A matrix formulation of the equations governing the motion of viscoanelastic media with memory was introduced, the derivation of the approximation of the first order of the wave front of a particular solution was given in full detail and revisited by the double-scale method. Other original results were obtained.

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Appendix

In the system (3) the matrices $\mathbf{H}^i$ and $\mathbf{H}^{ik} (i,k = 1,2,3)$ have the following form:

$$\mathbf{H}^1 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&\frac{2}{3}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0\\ 0&0&0&0&-\frac{1}{3}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{3}R'^{(\varepsilon)}_{(v)2}&0&0&0&0&0&0&0&0 \end{pmatrix}, \tag{67}$$

$$\mathbf{H}^2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-\frac{1}{3}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{2}{3}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{1}{3}R'^{(\varepsilon)}_{(v)2}&0&0&0&0&0&0&0 \end{pmatrix}, \tag{68}$$

$$\mathbf{H}^3 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-\frac{1}{3}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-\frac{1}{3}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\frac{1}{3}R'^{(\varepsilon)}_{(v)2}&0&0&0&0&0&0 \end{pmatrix}, \tag{69}$$

$$
\mathbf{H}^{11} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{3}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{70}
$$

$$
\mathbf{H}^{12} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2}{3}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{71}
$$

$$
\mathbf{H}^{13} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{72}
$$

$$\mathbf{H}^{21}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\frac{2}{3}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0\\
0&0&0&0&-\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_2&0&0&0&0&0&0&0&0
\end{pmatrix}, \tag{73}$$

$$\mathbf{H}^{22}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0&0\\
0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\frac{2}{3}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0\\
0&0&0&0&0&\frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_2&0&0&0&0&0&0&0
\end{pmatrix}, \tag{74}$$

$$\mathbf{H}^{23}=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0\\
0&0&0&0&0&\frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_2&0&0&0&0&0&0
\end{pmatrix} \tag{75}$$

$$\mathbf{H}^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{76}$$

$$\mathbf{H}^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{77}$$

$$\mathbf{H}^{33} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}R'^{(\varepsilon)}_{(d)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}R'^{(\varepsilon)}_{(v)2}v_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{78}$$

where $R^{(\varepsilon)}_{(d)2} = \omega^{-1}R'^{(\varepsilon)}_{(d)2}$ nd $R^{(\varepsilon)}_{(v)2} = \omega^{-1}R'^{(\varepsilon)}_{(v)2}$.

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