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A GENERALIZATION OF GROUPS

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ABSTRACT. In this article we introduce the notion of e-group as a new generalization of a group. The condition for a group to be an e-group is given. The characterization of some properties is established and some results follow.

1. Introduction and preliminaries

In mathematics and abstract algebra group theory studies the algebraic structures known as groups. Groups recur throughout mathematics and the methods of group theory have influenced many parts of algebra (see Hall 1959; Rotman 1995). Group theory and the closely related representation theory have many important applications in physics, chemistry and materials science. Extended and further developed groups give an important approach to a general description of other algebraic structures which have only a binary operation and only a constant.

In this paper, we study the properties of extended group to generalize the notion of e-group by considering the non-empty subset instead of the identity e. In particular, we get a new algebraic structure which, in general, is not a group but every group is an e-group. Suitable morphisms between them, which are generalized, are considered.

Definition 1.1. (Hungerford 1974) *Let G be a non-empty set. By a group we shall mean an algebra $(G;*,e)$ of type $(2,0)$ which satisfies the following axioms:*

(G1) $x*(y*z) = (x*y)*z$ *(associative law).*
(G2) $x*e = e*x = x$ *for all $x \in G$ (the existence of an identity element in G).*
(G3) *For every $x \in G$ there exists an element $x^{-1} \in G$ such that $x*x^{-1} = x^{-1}*x = e$ (the existence of inverses in G).*

for all $x,y,z \in G$.

Definition 1.2. (Hungerford 1974) *A group G is said to be abelian (or commutative) if* $x * y = y * x$*, for all* $x, y \in G$*.*

2. A new extension of groups

In this section we extend the notion of groups, which will be a physical background in the unified gauge theory and has direct relations with isotopies. For this aim we consider the subset A instead of the identity e; axioms (G2) and (G3) are extended.

Definition 2.1. Let G be a non-empty set. By an e-*group* we shall mean an algebra $(G;*;A)$ such that "$*$" is a binary operation on G and A is a non-empty subset of G which satisfies the following axioms:

(G1) $x * (y * z) = (x * y) * z$ for all $x, y, z \in G$ (associative law).
(eG2) For every $x \in G$ there exists an element $a \in A$ such that $x * a = a * x = x$ (the existence of an identity element corresponding to every element of G).
(eG3) For every $x \in G$ there exists an element $y \in G$ such that $x * y,\ y * x \in A$.

Definition 2.2. An e-group $(G;*;A)$ is said to be abelian (or commutative) if $x * y = y * x$, for all $x, y \in G$.

Proposition 2.3. Every group is an e-group.

Proof. Assume that $(G;*,e)$ is a group. Put $A := \{e\}$; we can see that $(G;*;A)$ is an e-group. □

We note that if there exists a unique element for the conditions (eG2) and (eG3), then $(G;*;A)$ is a group. The following example shows that axioms (G1), (eG2) and (eG3) are independent.

Example 2.4. (i) Let $G = \{a, b, c, d\}$ and $A = \{a, b\}$ be two sets with the following tables:

$*_1$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	a
d	d	d	d	d

$*_2$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	c	c	a	d
d	b	d	b	c

Then $(G;*_1,A)$ satisfies (G1) and (eG2) but does not satisfy (eG3). In addition, $(G;*_2;A)$ satisfies (eG2) and (eG3) but does not satisfy (G1), since

$$(c *_2 a) *_2 c = c *_2 c = a \neq c = c *_2 a = c *_2 (a *_2 c).$$

(ii) Let $G = \{a, b, c\}$ and $A = \{a, b\}$ be two sets with the following table:

$*_3$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

Then $(G; *_3; A)$ satisfies (G1) and (eG3) but does not satisfy (eG2).

Example 2.5. Let $G = \{a, b, c, d\}$, $K = \{a, b, c\}$ and $A = \{a, b\}$ be three sets with the following tables:

$*_4$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	a
d	a	d	c	d

$*_5$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

We can easily seen that $(G; *_4; A)$ is a non-abelian e-group and $(K; *_5; A)$ is an abelian e-group.

Remark 2.6. In Example **2.5** we can see that, in general, A is not a closed subset, since $a \in A$ but $a *_5 a = c \notin A$.

An e-group which is not a group will be called *proper*. Also, we will call $(G; *; A)$ a *proper* e-group if $|A| > 1$. The following example shows that, in general, every e-group is not a group.

Example 2.7. (i) Let $G := \mathbb{Q}$, $* := .$ and $A := \mathbb{Z}$. Then $(\mathbb{Q}; .; \mathbb{Z})$ is an e-group. Since

(e$\mathbb{Q}$1) $x.(y.z) = (x.y).z$ (associative law) holds for all $x, y, z \in \mathbb{Q}$.

(e$\mathbb{Q}$2) For every $\frac{m}{n} \in \mathbb{Q}$ there exists an element $1 \in \mathbb{Z}$ such that $\frac{m}{n}.1 = 1.\frac{m}{n} = \frac{m}{n}$.

(e$\mathbb{Q}$3) For every $\frac{m}{n} \in \mathbb{Q}$ there exists $kn \in \mathbb{Q}$ such that $\frac{m}{n}.kn,\ kn.\frac{m}{n} \in \mathbb{Z}$ for all $m, n, k \in \mathbb{Z}$ and $n \neq 0$.

(ii) Let $G = \{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{Q}\}$, $* := .$ (*i.e.*, . is a product of matrices) $H = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z}\}$. Then $(G; .; H)$ is an e-group, since:

(eG1) $A.(B.C) = (A.B).C$ (associative law) holds for every $A, B, C \in G$.

(eG2) For every $A = \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{r}{s} \end{pmatrix} \in G$, $\frac{m}{n}, \frac{r}{s} \in \mathbb{Q}$ there exists an element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in H$, such that

$$\begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{r}{s} \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{r}{s} \end{pmatrix} = \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{r}{s} \end{pmatrix}.$$

(eG3) For every $A = \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{r}{s} \end{pmatrix} \in G$, $\frac{m}{n}, \frac{r}{s} \in \mathbb{Q}$ $m, n, r, s \in \mathbb{Z}$ and $n, s \neq 0$ there exists $B = \begin{pmatrix} kn & 0 \\ 0 & ks \end{pmatrix} \in G$, $k \in \mathbb{Z}$ such that $A.B, B.A \in H$.

(iii) Let $G=\mathbb{R}\times(\mathbb{R}\setminus\{0\})$ with operation $(a,b)*(c,d)=(bc,bd)$ and $A=\{(\frac{a}{b},1): a\in\mathbb{R}$ and $b\in\mathbb{R}\setminus\{0\}\}$. Then $(G;*;A)$ is an e-group.

(iv) From Example **2.5**(i) $(G;*_2;A)$ is not a group, because there is not a unique identity element.

(v) Let X be a set and $P(X)$ be the power set of X. If we put $A=\{X,\emptyset\}$, then $(P(X);\cup;A)$, $(P(X);\cap;A)$ and $(P(X);\setminus;A)$ are e-groups, which are not groups.

Remark 2.8. We note that if $(G;*;A_i)$, for $i=1,2$, are e-groups, then, in general, $(G;*;A_1\cap A_2)$ is not an e-group. Indeed, let G be the additive semi-group of non-negative real numbers, and for A_1 (resp. A_2) the set of non-negative integers (resp. of non-negative multiples of $\sqrt{2}$). Then $(G;+;A_i)$, for $i=1,2$, is an e-group but $(G;+;A_1\cap A_2)$ is not, since $A_1\cap A_2=\{0\}$.

Theorem 2.9. Let $(G;*_1;A)$ and $(K;*_2;B)$ be two e-groups. The direct product of G and K denoted by

$$G\times K=\{(g,k): g\in G \text{ and } k\in K\}$$

is an e-group under the binary operation $\star$ such that

$$(g_1,k_1)\star(g_2,k_2)=(g_1*_1 g_2, k_1*_2 k_2).$$

Proof. This is achieved by investigating the axioms of e-group for the pair $(G\times K;\star;A\times B)$. □

Given any group $(G;*,e)$, we can construct an e-group as follows. Let Λ, I be nonempty sets. Let $P=(g_{\lambda i})$ be a $\Lambda\times I$ matrix with entries in G. Define a binary operation "$\circ$" on the set $I\times G\times\Lambda$ by

$$(i,g,\lambda)\circ(j,h,\mu)=(i,g*(g_{\lambda j}*h),\mu).$$

It is seen that the operation "$\circ$" is associative (G1). Thus $(I\times G\times\Lambda;\circ)$ is a semigroup, which will be denoted by $M(G;I,\Lambda;P)$. We claim that $(M(G;I,\Lambda;P);\circ;G_{\lambda i})$ is an e-group for all $i\in I$ and $\lambda\in\Lambda$ in which $G_{\lambda i}=\{(i,g,\lambda): g\in G\}$.

To prove condition (eG2), let $(i,g,\lambda)\in M(G;I,\lambda;P)$ and consider the element $(i,g_{\lambda i}^{-1},\lambda)\in M(G;I,\Lambda;P)$. Hence, we have:

$$(i,g,\lambda)\circ(i,g_{\lambda i}^{-1},\lambda)=(i,g*(g_{\lambda i}*g_{\lambda i}^{-1}),\lambda)=(i,g,\lambda),$$

and

$$(i,g_{\lambda i}^{-1},\lambda)\circ(i,g,\lambda)=(i,g_{\lambda i}^{-1}*(g_{\lambda i}*g),\lambda)=(i,g,\lambda).$$

To prove condition (eG3), let $(i,g,\lambda)\in M(G;I,\lambda;P)$ and consider the element $(i,g_{\lambda i}^{-1}*(g^{-1}*g_{\lambda i}^{-1}),\lambda)\in M(G;I,\Lambda;P)$. Hence, we have:

$$(i,g,\lambda)\circ(i,g_{\lambda i}^{-1}*(g^{-1}*g_{\lambda i}^{-1}),\lambda)=(i,g*(g_{\lambda i}*(g_{\lambda i}^{-1}*(g^{-1}*g_{\lambda i}^{-1}))),\lambda)=(i,g_{\lambda i}^{-1},\lambda)\in G_{\lambda i}.$$

and

$(i,g_{\lambda i}^{-1}*(g^{-1}*g_{\lambda i}^{-1}),\lambda)\circ(i,g,\lambda)=(i,(g_{\lambda i}^{-1}*(g^{-1}*g_{\lambda i}^{-1}))*(g_{\lambda i}*g),\lambda)=(i,g_{\lambda i}^{-1},\lambda)\in G_{\lambda i}.$

Theorem 2.10. Let $(G;*;A)$ be an e-group and $A\subseteq B$. Then $(G;*;B)$ is an e-group.

Proof. The proof is clear. □

The following example shows that, in general, the converse of Theorem **2.10** is not valid.

Example 2.11. It is routine to see that $(\mathbb{Z};+;\mathbb{Z})$ is an e-group, but $(\mathbb{Z};+;\mathbb{N})$ is not an e-group. Since for every $x\in\mathbb{Z}$ there is not an element $n\in\mathbb{N}$ such that $x+n=n+x=x$.

Corollary 2.12. If $(G;*;A_i)$, for $i\in\Lambda$, is a family of e-groups, then $(G;*;\bigcup_{i\in\Lambda}A_i)$ is, too.

Proof. Since $A_i\subseteq\bigcup_{i\in\Lambda}A_i$ the proof is obvious by Theorem **2.10**. □

Remark 2.13. Let $(G;*;A)$ be an e-group. By Theorem **2.10**, since $A\subseteq G$, we can see that $(G;*;G)$ is an e-group.

Definition 2.14. An e-group $(G;*;A)$ is said to be a good e-group if $(G;*;B)$ is not an e-group, for all $B\subset A$.

Example 2.15. In Example **2.7**(v), $(P(X);\cup;A)$, $(P(X);\cap;A)$ and $(P(X);\setminus;A)$ are good e-groups.

Theorem 2.16. Let $(G;*;A)$ be a good e-group. Then $(G;*)$ is a group if and only if A is a singleton set.

Proof. Let $A=\{a\}$ be a singleton set. If we put $e:=a$, then $(G;*,e)$ is a group. Conversely, let $(G;*;A)$ be a group. By (eG2) and the definition of identity element e, it follows that $e\in A$. It is obvious that $(G;*;\{e\})$ is an e-group. Since $(G;*;A)$ is a good e-group, we get $A=\{e\}$ which, thus, is a singleton set. □

Theorem 2.17. Every e-group of order 3 is abelian.

Definition 2.18. Let $(G_1;*_1;A_1)$ and $(G_2;*_2;A_2)$ be two e-groups. A mapping $f:G_1\to G_2$ is called a homomorphism if it satisfies the following conditions:

(i) $f(A_1)\subseteq A_2$;
(ii) $f(x*_1 y)=f(x)*_2 f(y)$, for all $x,y\in G_1$.

Example 2.19. Let $G_1:=G$ in Example **2.7**(i) and $G_2:=G$ in Example **2.7**(ii). Define $f:G_1\to G_2$ by $f(\frac{m}{n})=\begin{pmatrix} m & 0\\ 0 & n\end{pmatrix}$. Then f is a homomorphism. Since for every $k\in\mathbb{Z}$,

$$f(k)=f(\frac{k}{1})=\begin{pmatrix} k & 0\\ 0 & 1\end{pmatrix}\in H \text{ and so } f(\mathbb{Z})\subseteq H.$$

For every $\frac{m}{n}, \frac{r}{s} \in \mathbb{Q}$, we also have

$$f(\frac{m}{n}.\frac{r}{s}) = f(\frac{mr}{ns}) = \begin{pmatrix} mr & 0 \\ 0 & ns \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \cdot \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = f(\frac{m}{n}).f(\frac{r}{s}).$$

Let $(G_1;*_1;A_1)$ and $(G_2;*_2;A_2)$ be two e-groups and $f : G_1 \to G_2$ be a homomorphism. Define $\ker(f) = \{x \in G_1 : f(x) \in G_2\}$. Now, let $x \in G_1$. Then

$$f(x) \in f(G_1) \subseteq G_2 \text{ and so } f(x) \in G_2.$$

Hence, $G_1 \subseteq \ker(f)$. Therefore, $\ker(f) \neq \emptyset$.

Proposition 2.20. Let $(G_1;*_1;A_1)$ and $(G_2;*_2;A_2)$ be two e-groups and $f : G_1 \to G_2$ be a homomorphism. Then $(G_1;*_1;\ker(f))$ is an e-group.

Proof. The proof is clear by Theorem **2.10**. □

Definition 2.21. A nonempty subset H of an e-group $(G;*;A)$ is said to be a *sub-e-group* of G if, under the operation $*$ on G, $(H;*;A)$ itself forms an e-group.

We note that for every e-group $(G;*;A)$, itself and $(A;*;A)$ are sub-e-groups, which are called trivial sub-e-groups.

Example 2.22. (i) In Example **2.5**, put $H := \{a,b,c\}$, then $(H;*_4;A)$ is a sub-e-group of $(G;*_4;A)$.

(ii) In Example **2.7**, $(\mathbb{Z};.;\mathbb{Z})$ is a sub-e-group of $(\mathbb{Q};.;\mathbb{Z})$.

Remark 2.23. Let G be a set and $|G| = 2$. If $(G;*;A)$ is an e-group for some $\emptyset \neq A \subseteq G$, then A is a singleton set or $A = G$. Hence, if $|A| = 1$, then $(G;*;A)$ is a group and $(A;*;A)$ is a trivial subgroup. In addition, if $A = G$, then $(G;*;G)$ is a trivial sub-e-group.

Theorem 2.24. *Let $(G;*,e)$ be a group and A be a set such that $A \cap G = \emptyset$. Define the operation "$\circ$" on $H = G \cup A$ as follows:*

$$x \circ y = \begin{cases} x & \text{if } x \in A \\ y & \text{if } x \notin A \text{ and } y \in A \\ x * y & x,y \notin A \end{cases}$$

Then $(H;\circ;B)$ is an e-group, where $B = A \cup \{e\}$.

Proof. Let $x,y,z \in H$. If $x \in A$, then $x \circ (y \circ z) = x$ and $(x \circ y) \circ z = x \circ z = x$. Thus $x \circ (y \circ z) = (x \circ y) \circ z = x$. Now let $x \notin A$. If $y \in A$, then $(x \circ y) \circ z = y \circ z = y$ and $x \circ (y \circ z) = x \circ y = y$. Hence $(x \circ y) \circ z = x \circ (y \circ z)$. If $x \notin A$ and $y \notin A$, then we have:

Case 1. If $x,y \notin A$ and $z \in A$, then $(x \circ y) \circ z = (x * y) \circ z = z$ and $x \circ (y \circ z) = x \circ z = z$. Thus $(x \circ y) \circ z = x \circ (y \circ z) = z$.

Case 2. If $x,y,z \notin A$, then $(x \circ y) \circ z) = (x * y) * z = x * (y * z) = x \circ (y \circ z)$.

Therefore, H satisfies axiom (G1).

Let $x \in H$. If $x \in A$, then $x \circ y = y \circ x = x$, for all $y \in G$. If $x \notin A$, then $x \in G$ and then $x \circ e = x * e = x = e * x = e \circ x$. Thus, $(H; \circ; A)$ satisfies axiom (eG2).

Finally, if $x \in A$, then $x \circ y = y \circ x = x \in A$, for all $y \in G$. If $x \notin A$, then $x \circ y = y \circ x = y$, for all $y \in A$. Hence, $(H; \circ; A)$ satisfies axiom (eG3). Therefore, $(H; \circ; A)$ is an e-group. □

Example 2.25. Let $A = \{a, b\}$ and $(Z_3; \oplus, \overline{0})$ be the cyclic group of order 3. Then according the Theorem **2.24**, $H = \{a, b, \overline{0}, \overline{1}, \overline{2}\}$ and $(H; \circ; A)$ is an e-group with the following table:

$*_6$	a	b	$\overline{0}$	$\overline{1}$	$\overline{2}$
a	a	a	a	a	a
b	b	b	b	b	b
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{2}$	$\overline{2}$	$\overline{0}$	$\overline{1}$

Definition 2.26. *We call an e-group $(G; *; A)$ simple if $x * y = y * x = x$, for all $x \in G \setminus A$ and $y \in A$.*

Example 2.27. Let $G = \{a, b, c, d\}$ and $A = \{a, b\}$. Define the operation "$*_7$" on G as follows:

$*_7$	a	b	c	d
a	a	a	c	d
b	a	b	c	d
c	c	c	d	a
d	d	d	a	c

We can see that $(G; *_7; A)$ is a simple e-group. We note that $(H; *_6; A)$ is not simple and the extension by Theorem **2.24** is not simple, because $x \circ y = x \neq y$, for all $x \in A$ and $y \in G \setminus A$. Furthermore, in Example **2.5**, $(G; *_4; A)$ is not simple because $a *_4 c = a \neq c$.

Theorem 2.28. Let $(G; *; A)$ be a simple e-group. Then every element of $G \setminus A$ has a unique element such that the condition (eG2) is satisfied.

Proof. Let $(G; *; A)$ be a simple e-group and $x \in G \setminus A$. By (eG3), there exists an element $y \in G$ such that $x * y, y * x \in A$. Since G is simple, $y \in A$ implies $x * y = y * x = x \in A$, which is a contradiction. Hence $y \notin A$ and so $y \in G \setminus A$. If there are y_1 and y_2 such that $x * y_1, y_1 * x, x * y_2, y_2 * x \in A$, then we have:

$$y_1 = y_1 * (x * y_2) = (y_1 * x) * y_2 = y_2.$$

□

Theorem 2.29. Let $(G;*;A)$ be a simple e-group. Define the operation "$\circ$" on H as follows:

$$x\circ y=\begin{cases} x*y & \text{if } x*y\notin A \\ e & \text{if } x*y\in A \\ y & x=e \\ x & y=e \end{cases}$$

Where $H=(G\setminus A)\cup\{e\}$. Then $(H;\circ;e)$ is a group.

Proof. Let $(G;*;A)$ be a simple group. Then, by definition of "$\circ$" it is obvious that H satisfies axiom (G2). Also, by Theorem **2.28**, it follows that that H satisfies axiom (G3). To prove that $(H;\circ,e)$ is a group it is sufficient to show that H satisfies axiom (G1). Let $x,y,z\in H$. If $x=e$ or $y=e$ or $z=e$, then we get $(x\circ y)\circ z=x\circ(y\circ z)$. Now, let $x,y,z\in G\setminus A$. Then we have the following cases:

Case 1. Let $x*y\in A$. Then, $(x\circ y)\circ z=e\circ z=z$. We note that, since G is simple, $x*y\in A$ implies $(x*y)*z=z=x*(y*z)$. Now, if $y*z\in A$, then it means that $z=y^{-1}$ and $x*y\in A$ means that $x=y^{-1}$. By Theorem **2.28**, $x=z$. Thus, $x\circ(y\circ z)=x\circ e=x=z=(x\circ y)\circ z$. If $y*z\notin A$, then $y\circ z=y*z$. We note that $x*(y*z)\in A$ means that $z=x*(y*z)\in A$, which is a contradiction. Thus, $x*(y*z)\notin A$ which implies $x\circ(y\circ z)=x\circ(y*z)=x*(y*z)=z=(x\circ y)\circ z$.

Case 2. Let $x*y\notin A$ and $(x*y)*z\in A$. Then, $x\circ y=x*y$ and $(x\circ y)\circ z=e$. Also, $x*(y*z)=(x*y)*z\in A$. Now, if $y*z\in A$, then, since G is simple, it follows that $x=x*(y*z)=(x*y)*z\in A$, which is a contradiction. Hence, $y*z\notin A$ and so $y\circ z=y*z$. Since $x*(y*z)\in A$, we have: $x\circ(y\circ z)=x\circ(y*z)=e=(x\circ y)\circ z$.

Case 3. Let $x*y\notin A$ and $(x*y)*z\notin A$. Then, $(x\circ y)\circ z=(x*y)*z=x*(y*z)$. If $y*z\in A$, then $x*(y*z)=x$ (since G is simple). Hence, $(x\circ y)\circ z=(x*y)*z=x*(y*z)=x$ and so $x\circ(y\circ z)=x\circ e=x=(x\circ y)\circ z$. Now, if $y*z\notin A$, then $y\circ z=y*z$. We note that $x*(y*z)\in A$ implies $(x*y)*z\in A$, which is a contradiction. Hence, $x\circ(y\circ z)=x\circ(y*z)=x*(y*z)=(x*y)*z=(x\circ y)\circ z$. Therefore, $(H;*,e)$ satisfies (G1) and it is a group. □

Remark 2.30. The following example shows that in Theorem **2.29**, an e-group must be simple and we cannot remove this condition.

Example 2.31. It is easily seen that $(\mathbb{R}^+;+;\mathbb{Z}^+)$ is an e-group, but it is not simple. Put $e:=0$. Then $(\mathbb{R}^+\setminus\mathbb{Z}^+)\cup\{0\}$ is not a group ((G1) is not valid), since

$$(\frac{1}{2}+\frac{1}{3})+\frac{2}{3}=\frac{9}{6}=\frac{3}{2}\neq\frac{1}{2}+(\frac{1}{3}+\frac{2}{3})=\frac{1}{2}+0=\frac{1}{2}.$$

Theorem 2.32. Let $(G;*,e)$ be a group and $A = \{...,a_1,a_2,a_3,...\}$ be a countable set such that $A \cap G = \emptyset$. Define the operation "$\circ$" on $H = G \cup A$ as follows:

$$x \circ y = \begin{cases} x & \text{if } x \notin A \text{ and } y \in A \\ y & \text{if } x \in A \text{ and } y \notin A \\ a_k & \text{if } x,y \in A, x = a_i, y = a_j \text{ and } k = min\{i,j\} \\ x * y & x,y \notin A \end{cases}$$

Then $(H;\circ;B)$ is a simple e-group, where $B = A \cup \{e\}$.

Proof. The proof is straightforward. □

Remark 2.33. Let $(G;*;A)$ be a simple e-group. Then according to the Theorem **2.29**, we can make a group which we call the center of a simple e-group. Furthermore, if $(G;*,e)$ is a group and A is a countable set such that $A \cap G = \emptyset$, then, by Theorem **2.32**, we can make a simple e-group which we call a simple extension by A.

Theorem 2.34. Let $(G;*,e)$ be a group and A be a countable set such that $A \cap G = \emptyset$. Then, the center of the simple extension by A of $(G;*,a)$ is equal to itself.

Proof. Let $(G;*,e)$ be a group and $A = \{...,a_1,a_2,a_3,...\}$ be a countable set such that $A \cap G = \emptyset$. Then, by Theorem **2.32**, $(H;\circ;B)$ is a simple e-group, where $B = A \cup \{e\}$. $H = G \cup A$ and "$\circ$" defined as follows:

$$x \circ y = \begin{cases} x & \text{if } x \notin A \text{ and } y \in A \\ y & \text{if } x \in A \text{ and } y \notin A \\ a_k & \text{if } x,y \in A, x = a_i, y = a_j \text{ and } k = min\{i,j\} \\ x * y & x,y \notin A \end{cases}$$

Now, by Theorem **2.29**, the center of $(H;\circ;B)$ is
$H \setminus B \cup \{e\} = ((G \cup A) \setminus (A \cup \{e\})) \cup \{e\} = G$, with the operation $*'$ defined as follows:

$$x *' y = \begin{cases} x \circ y & \text{if } x * y \notin A \\ e & \text{if } x * y \in A \\ y & x = e \\ x & y = e \end{cases}$$

It follows that $x *' y = x * y$, for all $x, y \in G$. Therefore, the center of $(H;\circ;B)$ is $(G;*,e)$. □

The next example shows that a simple extension of a center of an e-group in general is not equal to itself.

Example 2.35. From Example **2.27**, the center of simple e-group $(G;*_7;A)$ is $(H;*_8,e)$ with the following table:

$*_8$	e	c	d
e	e	c	d
c	c	d	e
d	d	e	c

where $H = \{e,c,d\}$. We note that $(H;*_8,e) \cong (Z_3;\oplus,\overline{0})$. Now, by Theorem **2.32**, a simple extension $(H;*_8,e)$ by $A = \{a,b\}$ is $G' = \{a,b,e,c,d\}$ with the following table:

$*_9$	a	b	e	c	d
a	a	a	e	c	d
b	a	b	e	c	d
e	e	e	e	c	d
c	c	c	c	d	e
d	d	d	d	e	c

We can see that $(G';*_9;B)$ is a simple e-group, where $B = \{a,b,e\}$ and $(G';*_9;B) \neq (G;*_7;A)$.

3. Conclusions

In this paper we generalize the notion of groups to a new algebraic structure as e-groups. Sub-e-groups are defined and some examples are given. Suitable morphisms between such groups, which are generalized, are considered. These concepts can be further generalized and we hope that they may have some applications in the real world.

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