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A HARD ANALYSIS OF THE CONCEPT OF IMPULSE IN THE FOUNDATIONS OF CLASSICAL IMPULSIVE MECHANICS

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ABSTRACT. We show how a detailed analysis, mainly but not only based on the point of view of causality, of the usual definition of mechanical impulse of Classical Mechanics of free systems discloses theoretical unclear assumptions and practical applicability problems of very basic nature. We prove that Classical Impulsive Mechanics of constrained systems is not subject to these criticisms, being then a well founded theory. We propose then to step out of the concept of mechanical impulse acting on a free system and to consider impulsive aspects of Classical Mechanics as peculiar of constrained systems.

Introduction

Classical Mechanics is one of the older branch of Classical Physics, being it studied with modern mathematical methods from more than three centuries. However, the in-depth analysis of the basis of Classical Mechanics found its first cornerstones at the turn of the 20th century in the fundamental works of Mach (1933) and Painlevé (1922), where both a critical collation of the main results and techniques of Classical Mechanics along its historical development and a first attempt of giving a formal definition to the basic concepts of Classical Mechanics (such as causality, inertia and action–reaction principles, the definition of mass and force...) were presented.[1]

The formal axiomatization of Classical Mechanics, pushed and directed by the emergence and development of modern Mathematical Logic, enjoyed a major step forward in the middle of the 20th century (let us cite, among the wide bibliography, the papers of Noll (1959), Adams (1959), the even more important works of Bressan (1962, 1974), with his Followers (*e.g.*, Bressan and Montanaro 1982; Montanaro 1987), when both the construction of the theoretical aspects of Classical Mechanics starting from a strictly logically defined basic concepts and a critical comparison of similar constructions built on different basic concepts and inference rules were discussed.

[1] It is known that the contents of the works of Mach (1933) and Painlevé (1922) played a significant role in determining the range of applicability of Classical Mechanics viewed as a useful but approximated theory, and then opening a crack to Special Relativity.

It is however important to point out that, if on the one side the formal axiomatization of Classical Mechanics (for instance, the one presented by Bressan (1962)) gives firm foundations to the "naive" axiomatic approach to Classical Mechanics (for instance, the one presented by Painlevé (1922)), on the other side the results and techniques of formal axiomatization are background of very few Researchers in Classical Mechanics. The expertise of the majority of the Researchers is closer to the "naive" axiomatic approach of Mach and Painlevé. This paper is pointed to this second Readership.

Classical Impulsive Mechanics is commonly considered the subbranch of Classical Mechanics dealing with discontinuous mechanical phenomena, such as collisions between parts of the same mechanical system or between parts of the mechanical system and external constraints.[2] For this reason, some axiomatic aspects of Classical Impulsive Mechanics, for example the concept of causality, the action–reaction principle and the definition of mass are those of Classical Mechanics. Nevertheless, due to the peculiar characteristics of the impulsive phenomena, it is clear that a pedestrian translation of the whole set of axioms is not possible: for instance, the concept of force is not appropriate and it must be "translated" in the concept of impulse. We state that some of these translations, in the form commonly belonging to the expertise of the majority of Researchers in Mechanics, are not sufficiently accurate and present some unclear assumptions and applicability problems.

A possible reason of this inaccuracy could be found in the fact that Classical Smooth Mechanics and Classical Impulsive Mechanics have a common origin but rapidly diverging methodologies of study. The basic concepts (such as indeed causality, action–reaction principle and mass) that are common ground of both Smooth and Impulsive Mechanics must then be taken into account when the (more developed) ideas and techniques of Smooth Mechanics are in parallel translated in the context of Impulsive Mechanics. Moreover, the possible appearance of inconsistencies must be checked every time a new concept is introduced.

In this paper, we analyze in depth, especially but not only from the point of view of causality, the inference of the "equation of motion" of an impulsive mechanical system (the so–called integrated Newton's second law $\underline{\mathbf{I}} = m\Delta\underline{\mathbf{v}}$ as presented, for example, by Pérès (1962), Whittaker (1988), and Levi Civita and Amaldi (2013)) and the definition of impulse acting on a mechanical system (the LHS of the integrated Newton's second law).[3] In particular, following a line of thought parallel to that usually accepted for Classical Smooth Mechanics, we highlight that the concept of active impulse acting on a free mechanical system, although admitting a well founded mathematical definition, does not seem to be effectively applicable to study the impulsive behavior of a mechanical system. The absence of concrete examples of active impulses in literature strengthens this observation. On the contrary, we highlight that the concept of reactive impulse acting on a constrained

[2]Such a naive definition of Impulsive Mechanics is based on the finding that, due to the shortness of the duration and the smallness of the location of an impulsive mechanical phenomenon, this can hardly be studied by using the standard techniques of Classical "Smooth" Mechanics. It goes without saying that the choice between impulsive and smooth approach is guided *a priori* by the characteristics of the mechanical systems and *a posteriori* by the correctness of the results.

[3]It is remarkable that even in the axiomatization proposed by Bressan (1962) the possibility of motions with discontinuities in the velocity is explicitly dealt (see Axiom 19.1 pts. b,c and Theorem 27.1). Nevertheless a formal definition of impulse and a formal inference of the integrated Newton's second law were not presented.

mechanical system admits both a well founded mathematical definition, and, by way of the concept of constitutive characterization of constraint, it is also effectively applicable to study the impulsive behavior of mechanical systems.

Once the applicability problem of the concept of active impulse is detected, solutions can be looked for on the basis of different criteria. We propose a gordian solution that capitalizes on the fundamental difference between active and reactive impulses: we propose to drop out the concept itself of active impulse acting on a free mechanical system, so that every system whose mechanical evolution is framed in the context of Impulsive Mechanics must be considered as a constrained one. This choice of course wipes away all the foundational problems of Classical Impulsive Mechanics of free systems, also legitimizing the lack of examples of active impulses in literature.

With the aim of focussing the attention of the Reader on the foundational aspects of the paper, we choose to approach and propose its contents using the most simple mathematical instruments sufficient for a clear and self–contained dissertation. Therefore all the mathematical aspects presented in the paper are not new and they can be easily found in the almost endless literature on Impulsive Mechanics (see, *e.g.*, Johnson 1985; Stronge 2000; Brogliato 2016, for a wide but not exhaustive lists of references). Nevertheless, although presented in a simple setup, the extension of the contents to more refined setups, for instance to mechanical systems with a finite number of degrees of freedom, is a very simple process. For this reason, we think that the problems arisen in the paper are far from being only of epistemological nature and that they could have repercussions in the actual research work on the argument.

For the sake of brevity, we use the following acronyms: CSM for Classical Smooth Mechanics, CIM for Classical Impulsive Mechanics, DNSL for Differential Newton's Second Law and INSL for Integrated Newton's Second Law.

In Section 1 we discuss the inference of the INSL from the DNSL by using the integration–limit process. We focus attention on some logical critical points that weaken the accuracy of the integration–limit process, but we however recover a meaningful INSL by using an axiomatic point of view.

In Section 2 we discuss causality in the INSL, using arguments parallel to the standard ones usually considered and accepted for the DNSL. We compare the concepts of active impulse acting on a free mechanical system, the so called "percussion" described by Levi Civita and Amaldi (2013), and the concept of reactive impulse acting on a constrained mechanical system, focussing attention on a crucial difference, dependent on the causal structure of the INSL, between the active and reactive impulses: the first must be known *a priori*, while the second is an unknown of the problem. This difference points up the applicability problem of determining physically meaningful expressions for active impulses, about which the whole literature of CIM, less or more recent, seems to be full of blanks.

In Section 3 we propose and discuss a gordian solution of the problem of causality in INSL as presented in Section 2: we claim that the concept of active impulse is meaningless, and only reactive impulses have a physical meaning. On the basis of the action–reaction principle applied to impulsive systems, we also give an heuristic justification of this postulation.

In Section 4 we briefly analyse the main consequences of the choice of Section 3, with a final remark. If on the one side, the choice brings to the advantage of solving the problem of causality in CIM, on the other side it opens new problems. The first is about the possibility of rewriting the results of free CIM structurally related to the concept of active impulse in the context of constrained CIM. The second is about the increased importance of the concept of constitutive characterization for impulsive constraints and its effective application.

1. The origin of INSL

The INSL (Integrated Netwton's Second Law) in its simplest form has its genesis from the DNSL (Differential Newton's Second Law): $\underline{\mathbf{F}} = m\underline{\mathbf{a}}$ (see Pérès 1962; Whittaker 1988; Levi Civita and Amaldi 2013). The usual deduction procedure consists in considering the law, with expressed time dependence, applied for the sake of simplicity to a single point particle supposed of constant mass

$$\underline{\mathbf{F}}(t) = m\underline{\mathbf{a}}(t)\,, \tag{1}$$

integrating both members of the equation in a time interval $[t_0, t_1]$

$$\int_{t_o}^{t_1} \underline{\mathbf{F}}(t)\,\mathrm{d}t = m\int_{t_o}^{t_1} \underline{\mathbf{a}}(t)\,\mathrm{d}t = m\left(\underline{\mathbf{v}}(t_1) - \underline{\mathbf{v}}(t_0)\right),$$

and then applying a limit process, so that, with standard notation, we have

$$\begin{aligned}\lim_{t_1\to t_o}\int_{t_o}^{t_1} \underline{\mathbf{F}}(t)\,\mathrm{d}t &= m\lim_{t_1\to t_o}\left(\underline{\mathbf{v}}(t_1) - \underline{\mathbf{v}}(t_0)\right)\\ &= m\left(\underline{\mathbf{v}}^+(t_0) - \underline{\mathbf{v}}^-(t_0)\right).\end{aligned}$$

CIM deals with the mechanical problems where the left and right hand sides are not null. The LHS is then defined as the impulse $\underline{\mathbf{I}}(t_0)$ acting on the point particle at the instant t_0, so that the final form of the INSL is

$$\underline{\mathbf{I}}(t_0) = m\left(\underline{\mathbf{v}}^+(t_0) - \underline{\mathbf{v}}^-(t_0)\right) = m\Delta\underline{\mathbf{v}}(t_0)\,. \tag{2}$$

The mathematical correctness of the procedure is undeniable, possibly using suitable mathematical instruments such as bounded variation functions, distributions and so on. Nevertheless an effective application of Eq. (2) in the study of the motion of a point particle requires, roughly speaking, an *a priori* knowledge of one member of Eq. (2) to determine the other. The study of the "jump of velocity" on the RHS in order to determine the impulse on the LHS is a standard argument of experimental CIM but it is also the conceptually easiest aspect of a mechanical problem (see, once again, Pérès 1962, p. 1). Vice versa, the effective knowledge *a priori* of the impulse on the LHS in order to determine the jump of velocity on the RHS would give to Eq. (2) the character of "equation of motion" for the point particle. However, we claim that the knowledge of the LHS is problematic in the CIM context, for two main reasons: the first is a theoretical reason, the second is a practical one. We begin describing the theoretical one.

A more detailed formulation of the DNSL (1) exhibits the general dependence of the force $\underline{\mathbf{F}}$ on the kinematic quantities time, position and velocity of the point particle:

$$\underline{\mathbf{F}}(t, P(t), \underline{\mathbf{v}}(t)) = m\underline{\mathbf{a}}(t)\,. \tag{3}$$

The knowledge of $\underline{\mathbf{F}}(t)$ in the time interval $[t_0, t_1]$ is then in general subject to the knowledge of the function $P(t)$, that is the motion of the particle, but if $P(t)$ is known, the interest in going on with the integration–limit process is very scarce. If, differently, $P(t)$ is not known, but the dependence of $\underline{\mathbf{F}}$ on $P, \underline{\mathbf{v}}$ is effective, it should be proved that the impulse

$$\underline{\mathbf{I}}(t_0) = \lim_{t_1 \to t_o} \int_{t_o}^{t_1} \underline{\mathbf{F}}(t, P, \underline{\mathbf{v}})\, \mathrm{d}t$$

is well defined, independent of $P(t)$ and $\underline{\mathbf{v}}(t)$ whatever they will be. This is a hard task, that seems to be ignored in the current literature on CIM.

Of course, there is the theoretical possibility that the function $\underline{\mathbf{F}}$ does not depend on position and velocity, and in this case the integration–limit procedure determining $\underline{\mathbf{I}}(t_0)$ acquires a significant usefulness. However this possibility should be supported by physically meaningful examples of functions $\underline{\mathbf{F}}(t)$ and impulses $\underline{\mathbf{I}}(t_0)$: regarding this argument, once again the literature in CIM seems to be almost empty.

The absence in literature of detailed examples of evaluation of $\underline{\mathbf{I}}(t_0)$ through the integration–limit process is probably related to the (second) practical reason: the forces involved in an impact problem are very hard to be analyzed, depending on very complex phenomena (deformation, elasticity, thermodynamics, wave propagation...) happening in very small and inaccessible locations for very short time intervals. The experimental difficulties discourage a microscopical analysis of the forces involved in an impulsive phenomenon with the mere scope of determining the impulse $\underline{\mathbf{I}}(t_0)$: if and when a detailed analysis of the forces is done (such as, for example, in Hertzian contact theories), then the DNSL gets back in being the main instrument for determining the motion of the systems, and the impulsive aspects of the phenomenon are lost.

A possible way to avoid all these logical and practical problems is to state the INSL by an axiomatic point of view. To this aim, we parallelize in CIM the axiomatic introduction of the DNSL for CSM. Like CSM have its cornerstone in the differential equation

$$\underline{\mathbf{F}}(t, P, \underline{\mathbf{v}}) = m\,\underline{\mathbf{a}}, \tag{4}$$

we assume as cornerstone of CIM the finite equation

$$\underline{\mathbf{I}}(t_0, P(t_0), \underline{\mathbf{v}}^-(t_0)) = m\,\underline{\mathbf{v}}^+(t_0), \tag{5}$$

the integration–limit process being only an heuristic motivation for the structure of Eq. (5).

It goes without saying that Eq. (5), conceived for a free single point particle, is the starting point for generalizations to analogous equations applicable in more refined contexts (such as CIM of systems formed by one or more point particles and rigid bodies). Note that Eq. (5) is not an innovative way to introduce the INSL in a formal way, even in the above–mentioned more refined contexts (see, *e.g.*, Ballard 2001; Pasquero 2005). However, in order to keep the mathematical aspects of the paper as straight as possible, we avoid the technicalities of mechanical models such as the rigid body, or general mechanical systems with a finite number of degrees of freedom. Taking always in mind that the most arguments hold even in these more refined contexts, we essentially restrict our discussion to the case of a single point particle. This will be sufficient for the aim of the paper.

2. Causality in INSL

Newton's Second Law in the form (4) has a well known and universally accepted causal structure: the active force in the LHS is the cause of the acceleration in the RHS. This is the reason why in Classical Mechanics forces can depend on time, position and velocity, but they are independent of acceleration, since otherwise acceleration would be cause and effect of itself. Moreover, this causal structure is coherent with the determination principle of Classical Mechanics (for free systems), since it allows to make the DNSL a second order differential equation in normal form, that is one of the hypotheses for the application of Cauchy's existence and uniqueness theorem.[4] Then the knowledge of the function $\underline{\mathbf{F}}(t,P,\underline{\mathbf{v}})$ giving the active force acting on the free system, together with the knowledge of the kinetic data $P(t_0),\underline{\mathbf{v}}(t_0)$ in an instant t_0, univocally determines (if the function expressing the force is sufficiently regular) the motion of the system for the instants following (but near to) t_0.

Applying a parallel line of thinking in CIM, we also require both a clear causal structure for Eq. (5) and the applicability (possibly under some additional requirement) of the determination principle of Classical Mechanics. In this case, the way is easier to go through. The causal structure of Eq. (5) can be chosen analogously to the causal structure of (4), by saying that the impulse in the LHS is the cause of the (variation of) velocity in the RHS. Moreover, since Eq. (5) has a finite, and not differential, nature, the knowledge of the active impulse $\underline{\mathbf{I}}(t_0,P(t_0),\underline{\mathbf{v}}^-(t_0))$ as function of the kinetic data $P(t_0),\underline{\mathbf{v}}^-(t_0)$ clearly determines the motion of the system.

2.1. Causality in INSL applied to a free particle. Unfortunately, in the case of free particle, the pleasantness of the parallel thinking described above has here an abrupt end. The reason is clearly described for the DNSL by Pérès (1962) (Ch. 1, Sect. III-10):

Il est essentiel de noter que (...) la loi fondamentale (DNSL)*, à elle seule, est insuffisante pour construire une théorie mécanique. Elle donne un cadre général, que l'on ne peut remplir sans établir préalablement des représentations valables pour les forces qui sont en jeu. C'est la recherche de telles représentations, par l'analyse et l'interprétation des données expérimentales, qui constitue la première démarche de toute théorie mécanique. Et la théorie sera d'autant meilleure que le schéma obtenu pour les forces serrera de plus près la réalité.*

In CSM, the usable representations of forces to which Pérès (1962) refers were deeply studied, and several physically meaningful functions $\underline{\mathbf{F}}(t,P,\underline{\mathbf{v}})$ were determined to describe active forces (Newton's attraction, Coulomb's attraction/repulsion, Lorentz's force and so on...). Likewise, in CIM, the knowledge of the active impulse $\underline{\mathbf{I}}(t_0,P(t_0),\underline{\mathbf{v}}^-(t_0))$ is required for a correct causal application of Eq. (5) in order to determine the motion. In absence of meaningful expressions for $\underline{\mathbf{I}}$, the effective applicability to physical problems of several results of CIM, notwithstanding their mathematical correctness, is compromised. However, it seems that a similar analysis were not performed on the concept of active impulse and the literature on this argument is clearly unsatisfactory.

Once again, theoretical and practical reasons could motivate this lack. The practical one is similar to that regarding the analysis of the forces involved in an impact: impulses depend on very complex phenomena, generally happening for very short times and in very small

[4]It is known that some peculiar contexts of CSM, for instance Control Theory, include the possibility of the assumption $\underline{\mathbf{a}} = \underline{\mathbf{G}}(t,P,\underline{\mathbf{v}},\underline{\mathbf{a}})$. However this is not related to an axiomatic definition of the force acting on the system. Such an assumption, or the analogous expression $\underline{\mathbf{H}}(t,P,\underline{\mathbf{v}},\underline{\mathbf{a}}) = 0$, is more related to the necessity of "overriding" the Cauchy's theorem, in order to have more solutions amongst which choosing the most suited for the problem in study.

locations. However, once again, the theoretical reason is more subtle and rich of conceptual enhancements: it is very hard to conceive an impulsive phenomenon, in the context of CIM, concerning a free system, even and especially for the simplest system formed by a single point particle. To support this statement, in the next section a heuristic motivation will be flanked to the absence of significant examples of active impulse in literature.

2.2. Causality in INSL applied to a constrained particle. Once again parallelizing the line of reasoning of CSM, the INSL for a constrained system presents a distinction among the impulses acting on the system explicitly separating the impulses, called active, acting on the system independently of the presence of the constraint and the impulses, called reactive, exerted by the constraint in order to limit the possible motions to those admitted by the constraint itself. Eq. (5) then becomes

$$\underline{\mathbf{I}}^{act}(t_0, P(t_0), \underline{\mathbf{v}}^-(t_0)) + \underline{\mathbf{I}}^{react}(t_0, P(t_0), \underline{\mathbf{v}}^-(t_0)) = m\underline{\mathbf{v}}^+(t_0)\,. \tag{6}$$

Although the causal structure of the INSL remains unchanged, being the LHS the cause and the RHS the effect, there is a crucial difference between the two addenda of the LHS: while active impulses must be known as an *a priori* data of the problem, reactive impulses are part of the unknowns of the problem itself. Of course this implies that, in principle, determinism of Classical Mechanics is lost, and it can be restored only by the introduction of a suitable additional information, the so called constitutive characterization of the constraint, that specifies what reactive impulses the constraint can or cannot exert on the system. However, the unknown nature of reactive impulses avoid them the critiques raised in the previous sections to impulses acting on a free mechanical system. In fact, although the physical genesis of reactive impulses is the same of the active ones (involving the same complex phenomena such as deformation, elasticity, thermodynamics...), in order to study the evolution of the system, the *a priori* knowledge of the reactive impulses in the LHS of Eq. (5) is senseless, becoming instead a (mandatory, if we hope to restore determinism) consequence of the choice of a suitable constitutive characterization of the constraints acting on the system.

3. Solutions proposed for the applicability problem

It is clear that Eq.(5) applied to free systems and Eq.(6) applied to constrained systems present the same applicability problem: the lacking of the *a priori* knowledge of active impulses. It is equally clear that we have (at least) three simple solutions of the problem.

A crystal clear solution would consist in the exhibition of physically meaningful examples of active impulses (in the same way as Newton's attraction is a physically meaningful example of active force): this would show that there actually is no problem.

A formally flawless solution relies on the unassailable consistency of the mathematical aspects of the theory, considering then CIM of free systems as a coherent abstract theory, and branding the applicability problem as insignificant.

We support a third solution: we postulate that active impulses do not exist and only reactive impulses do. In this way, we wipe away from the logical context of CIM the theoretical and practical weak spots that we previously showed for CIM of free systems. The INSL assumes the explicit form

$$\underline{\mathbf{I}}^{react}(t_0, P(t_0), \underline{\mathbf{v}}^-(t_0)) = m\underline{\mathbf{v}}^+(t_0)\,, \tag{7}$$

preserving its clear causal structure and obtaining an effective applicability once a meaningful constitutive characterization of the impulsive constraint is assigned for the physical system in study.

Let us also add an heuristic motivation to support the assertion of deeming active impulses inconceivable in CIM. If we assume that the action–reaction Newton's Third Law holds for impulses too, for every impulse acting on a mechanical system there should exist the corresponding opposite impulse acting somewhere.

In the case of constrained systems it is clear who undergoes to the opposite impulse: the external constraint if the system is formed by a single point particle or by a single body (for example, the external wall on which the particle or the body collides); the system itself if the system is formed by two or more subsystems (for example two colliding particles or two colliding bodies. Note that in this case the system is subject to one or more constraints on the distance of the subsystems).

In the case of a system subject to an active impulse, the "counterpart" can be imagined with difficulty. In fact in this case there should exist a "part of the universe" that is not under investigation where the corresponding opposite impulse acts: if this part is at close distance to the system in study, it is quite unusual to exclude it by the study. If it is far enough to justify the exclusion, then the active impulse should be caused by a "remote action", such as a sort of "impulse field" acting on the system in study in an assigned position and time and not in the nearby positions and the instants immediately before and after. Surely we could define such an impulse field in a correct mathematical way but a meaningful physical interpretation of the mathematical concept is lacking.

4. Conclusions

The choice of framing every impulsive system in the context of constrained Mechanics instead of (possibly) free Mechanics has, as main consequence, the satisfactory removal of the theoretical unclear assumptions and practical applicability problems illustrated above but it opens two classes of problems.

The first class is related to the wide sea of formally correct results of CIM of free systems: which of them could be rewritten in the context of CIM of constrained systems, and which of them are indissolubly related to the concept of free impulse and then preserve only a formal correctness? For example, the writing of the so called Lagrangian Equations of impulsive motion requires the knowledge of the components of the total impulse (and in particular the free one) acting on the system (see, *e.g.*, Pérès 1962; Whittaker 1988). Can an analogous of these equations be written in the context of constrained CIM?

The second class of problems is related to the increase of importance of the impulsive constitutive characterization of constraints. It is in fact well known that the assignment of an effective constitutive characterization can raise serious problems of both experimental and theoretical nature even for very simple mechanical systems. Impacts with frictional contacts, or multipoint impacts of rigid bodies are two types of impulsive problems for which the assignment of effective constitutive characterizations has nowadays only partially satisfactory answers.

Let us conclude the paper remarking that we think that a wide consent for the impulsive setup proposed and described in Section 3 would have important consequences also in the current common work on CIM, and that we support the solution presented in Section 3 to the foundational problem about active impulses as the most satisfactory we have actually found. This is the reason why we think that a broad debate on the argument could greatly contribute to a better understanding of CIM. Then we hope that this paper will be only a starting point for a collective discussion on the argument.

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