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PROBLEMS ON MULTIVARIATE RELIABILITY POLYNOMIAL

CONSTANTIN UDRIŞTE,[a*] IONEL ŢEVY,[a] AND SAAD ABBAS ABED [a]

ABSTRACT. The original results include: (i) homogenization of a reliability polynomial; (ii) compact hypersurfaces attached to homogeneous polynomials; (iii) an affine diffeomorphism that preserves a reliability polynomial; (iv) duality of networks via a diffeomorphism; (v) straight line segments in constant level sets associated to a reliability polynomial.

1. Algebraic-geometric tools

This paper focuses on an algebraic-geometric properties of the multivariate reliability polynomial. *Homogenization of multivariate reliability polynomial* refers to the fact that this process consists in to do homogenizing on each variable separately. The problem of *affine diffeomorphisms* that preserves a reliability polynomial is yet an open geometrical problem. Our model is a symmetry with respect to the origin and translation. The *dual networks via diffeomorphisms* remaind the idea of geometrical global equivalence. The problem of *straight line segments in constant level sets* comes from a private discussion with Prof. Dr. Gheorghiţă Zbăganu.

Now let us underline the tools offered by our references: the papers by Calin and Udrişte (2014) and Aubry and Brinzei (2015) refer to geometric modeling; the papers by Satyanarayana and Prabhakar (1978), Satyanarayana (1982), Satyanarayana and Wood (1985), Colbourn (1987), Gertsbakh and Shpungin (2011), and Sun and Zhou (2012) are dedicated to important reliability problems; Jula and Costin (2012) give details about reliability of electrical circuits; Royle and Sokal (2004) analyse zeros of reliability polynomials; the papers by Rebaiaia *et al.* (2009), Hassan and Balan (2015), Hassan and Udrişte (2015), and Hassan *et al.* (2016) give geometric properties of reliability polynomials; Sarhan *et al.* (2008) discuss equivalent factors of parallel-series systems; Teruggia (2010) presents a reliability analysis of probabilistic networks; the paper by Udrişte (1994) is dedicated to Riemannian convexity and optimization; finally, the papers by Udrişte *et al.* (2016, 2017) study optimal reliability allocation.

2. Homogenization of multivariate reliability polynomial

Let us consider a connected multi-graph $G = (V, E)$ as a communications network with communication channels that do not influence each other, in which each edge e is operational with probability p^e and failed with probability $1 - p^e$, independently for each edge. Let $R_G(p)$ be the probability that every node is capable of communicating with every other node (this is the so-called all-terminal reliability). The terminal reliability is

$$R_G(p) = \sum_{\substack{A \subseteq E \\ (V,A)\ \text{connected}}} \prod_{e \in A} p^e \prod_{e \in E \setminus A} (1 - p^e),$$

where $p = \{p^e\}_{e \in E}$ and the sum runs over all connected spanning subgraphs $\{V, A\}$ of G. The polynomial $R_G(p)$ is called the *multivariate reliability polynomial* for the graph G. It is a multiaffine polynomial, i.e., of degree at most 1 in each variable separately. Writing explicitly, the reliability polynomial $R_G(p)$ takes a new form $P(G, p)$ whose coefficients are integers and $0 \leq P(G, p) \leq 1$. Setting $q^e = 1 - p^e$, we obtain the polynomial

$$R_G(p, q) = \sum_{\substack{A \subseteq E \\ (V,A)\ \text{connected}}} \prod_{e \in A} p^e \prod_{e \in E \setminus A} q^e$$

in $2n$-variables (p, q). Let us consider a connected sub-graph $\mathscr{G}$ of $G = (V, E)$. Let $R_{\mathscr{G}}(p)$ be the probability that every node is capable of communicating with every other node. Let $c_A = 1$ if $A \in V_{\mathscr{G}}$ and $c_A = 0$, otherwise. Then the *multivariate reliability polynomial* for the graph $\mathscr{G}$ is

$$R_{\mathscr{G}}(p) = \sum_{\substack{A \subseteq E \\ (V,A)\ \text{connected}}} c_A \prod_{e \in A} p^e \prod_{e \in E \setminus A} (1 - p^e).$$

These questions also have a close connection with reliability theory. Consider a finite set E of communication channels, which fail independently with probabilities $\{q_e\}_{e \in E}$. Let S be a set system on E, whose members we shall interpret as the sets of failed channels that allow the system as a whole to be operational. Then the probability that the system is operational is given by the multivariate reliability polynomial $Rel_S(q) = \sum_{A \in S} q^A (1 - q)^{E \setminus A}$. In the reliability context, it is natural to assume that S is a complex, i.e., that S contains $\emptyset$ and is closed under taking subsets. Indeed, the simplest case arises when $G = (V, E)$ is a connected graph and we declare the system to be operational if the non-failed edges form a connected spanning subgraph (this is the "all-terminal reliability").

A reliability polynomial has terms of degree at most one in each variable. If we want to get a homogenization of this polynomial, preserving the reliability character, we are required to do homogenizing on each variable separately. This means to consider $(P^1)^n$ with homogeneous coordinates $[R^1 : Q^1, ..., R^n : Q^n]$. If $R(R^1, ..., R^n)$ is a reliability polynomial, then its homogenization is a polynomial of the form

$$\rho((R^1, Q^1), ..., (R^n, Q^n)) = Q^1 ... Q^n \, R\left(\frac{R^1}{Q^1}, ..., \frac{R^n}{Q^n}\right).$$

To have explicit form both for the reliability polynomial and its homogenization, we judge in the following way. Let $\mathscr{F}$ be a family of subsets of the set $N_n = \{1, ..., n\}$ whose union is

N_n. Then a reliability polynomial can be written as

$$R(R^1,...,R^n) = \sum_{A\in\mathscr{F}} c_A \prod_{i\in A} R^i,\ c_A \in \mathbb{Z},\ \sum c_A = 1$$

and the corresponding homogenized polynomial is

$$\rho((R^1,Q^1),...,(R^n,Q^n)) = \sum_{A\in\mathscr{F}} c_A \prod_{i\in A} R^i \prod_{i\in N_n\setminus A} Q^i.$$

Constructing $\rho((R^1,Q^1),...,(R^n,Q^n))$ from $R(R^1,...,R^n)$ is called *homogenization*; recovering $R(R^1,...,R^n)$ from $\rho((R^1,Q^1),...,(R^n,Q^n))$ is called *dehomogenization*. The dehomogenization is obtained setting $Q_k = 1$, i.e., $R(R^1,...,R^n) = \rho((R^1,1),...,(R^n,1))$. Explicitly, the homogenized polynomial can be written as

$$\rho((R^1,Q^1),...,(R^n,Q^n)) = c_{a_1...a_n;b_1...b_n} R_1^{a_1}...R_n^{a_n} Q_1^{b_1}...Q_n^{b_n},$$

$$a_i \in \{0,1\},\ b_i \in \{0,1\},\ a_i + b_i = 1,\ \forall i = 1,...,n.$$

The reliability polynomial R_G is the restriction of a homogenized polynomial ρ to the "diagonal" $R^i + Q^i = 1, i = 1,...,n,$ of unit hypercube in R^{2n}. The constant level sets $\rho((R^1,Q^1),...,(R^n,Q^n)) = c$ are algebraic hypersurfaces in $(P^1)^n$.

Theorem 1. *If $f(x_1,...,x_n)$ is a homogeneous polynomial of degree $p \geq 1$ which has at least a strictly positive value, then the set $M : f(x_1,...,x_n) = 1$, is a hypersurface. If $f(x_1,...,x_n) > 0$ excepting $(0,...,0)$, then the hypersurface M is compact and conversely.*

Proof. By hypothesis, there exists a point $P(x_1,...,x_n) \in \mathbb{R}^n$ such that $f(x_1,...,x_n) = a > 0$. In this way, the set M contains the point $\frac{1}{\sqrt[p]{a}}(x_1,...,x_n)$. By Euler formula for homogeneous functions, it follows that any critical point Q satisfies $f(Q) = 0$ and hence $Q \notin M$. Consequently M is a hypersurface.

Suppose $f > 0$, excepting the origin. We consider the sphere S and the natural injection $i : S \to \mathbb{R}^n$. The composition $f \circ i : S \to \mathbb{R}$ is continuous. Since S is compact, the function $f \circ i$ has a minimum value b and $b > 0$. If $Q \in \mathbb{R}^n$, we have $f(Q) = ||Q||^p f\left(\frac{Q}{||Q||}\right) \geq ||Q||^p b$. In this way, the condition $f(Q) = 1$ implies $||Q|| \leq \frac{1}{\sqrt[p]{b}}$. It follows that $M = f^{-1}(1)$ is bounded, and hence M is compact.

Conversely, suppose that M is a compact hypersurface. It follows that M is bounded and hence $f(Q) \geq ||Q||^p b$. That is way, $f > 0$, excepting the origin. □

If f is homogeneous and $c > 0$, then each level set $f(x_1,...,x_n) = c$ is homothetical to the set $f(x_1,...,x_n) = 1$.

3. An affine diffeomorphism that preserves a reliability polynomial

Let us use the diffeomorphism (symmetry with respect to the origin and translation)

$$p^{e'} = 1 - p^e. \tag{1}$$

It is orientation preserving for even dimension and reversing for an odd dimension. The fixed point of this diffeomorphism is $p^e = \frac{1}{2}$, $e = 1,...,n$. The properties of this diffeomorphism make evident the following

Theorem 2. *The diffeomorphism (1) has two properties: (i) it transforms the unit hypercube $[0,1]^n$ into itself (closed invariant set); (ii) it preserves the multivariate reliability polynomial if the graphs (V,A) and $(V,E\setminus A)$ are simultaneously connected.*

Since the Open Unit Hypercube $[0,1]^n$ is diffeomorphic to the Open Unit Ball $B(0,1)$, the multivariate reliability polynomial is identified with a function defined over $B(0,1)$.

4. Dual networks via diffeomorphisms

Definition. *Let G and G' be two networks. They are called dual if there exists a diffeomorphism that change the multivariate reliability polynomial of G into the multivariate reliability polynomial of G'.*

Particularly, the diffeomorphism can be an affine one but not only (see Hassan and Udrişte 2015, for a nonlinear diffeomorphism).

Theorem 3. *A series system G is dual to a parallel system G' via the diffeomorphism $R_{G'} = 1 - R_G$, $p^{e'} = 1 - p^e$, only if they have the same number of components.*

Proof The multivariate reliability polynomial $R_G(p) = \prod_{e=1}^n p^e$ is changed into $1 - R_G(p') = \prod_{e'=1}^n (1 - p^{e'})$.

5. Straight line segments in constant level sets

We ask if the constant level sets of reliability polynomial there exist straight line segment. For the beginning, let us consider a series system with the multivariate reliability polynomial $R_G(p) = \prod_{e=1}^n p^e$. We attach the constant level sets (algebraic hypersurfaces) $c = \prod_{e=1}^n p^e$. Obviously, $0 \leq c \leq 1$. For $c > 0$, we have Tzitzeica hypersurfaces.

Theorem 4. *For $c > 0$, there exist no straight line segment in the constant level set $c = \prod_{e=1}^n p^e$.*

Proof. Let $p^e = a^e t + b^e$. We impose $c = \prod_{e=1}^n (a^e t + b^e), \forall t$. The system

$$\prod_{e=1}^n a^e = 0, \quad \sum_{perm\{1,\ldots,n\}} a^1 a^2 \ldots a^{n-1} b^n = 0, \ldots, \quad \prod_{e=1}^n b^e = c$$

has no solution. Indeed, the situation $a^e = 0$, when $e = 1$ and $a^e \neq 0$, otherwise, leads to $b^n = 0$ which contradicts the last equation. More precisely, since $b^e \neq 0, e = 1, \ldots, n$, let us divide each equation (excepting the last equation) by the product $b^1 \ldots b^n$. Then we obtain the Viéte relations

$$\prod_{e=1}^n \frac{a^e}{b^e} = 0, \quad \sum_{perm\{1,\ldots,n\}} \frac{a^1}{b_1} \cdots \frac{a^{n-1}}{b^{n-1}} = 0, \ldots, \frac{a_1}{b_1} + \ldots + \frac{a_n}{b_n} = 0,$$

for the equation $t^n = 0$. So $a^1 = \ldots = a^n = 0$, i.e., any segment included in the constant level set is reduced to a point. □

There are models for which constant- level of multivariate reliability polynomials include straight line segments.

Example. *Three hard drives in a computer system are configured reliability-wise in parallel. At least two of them must function in order for the computer to work properly. Each hard drive is of the same size and speed, but they are made by different manufacturers and have different reliabilities. The reliability of HD −1 is p^1, HD −2 is p^2 and HD −3 is p^3, all at the same mission time. Since at least two hard drives must be functioning at all times, only one failure is allowed. This is a 2-out-of-3 configuration. The multivariate reliability polynomial can now be expressed as*

$$R_s(p) = p^1p^2 + p^1p^3 + p^2p^3 - 2p^1p^2p^3.$$

We attach the constant level sets $c = p^1p^2 + p^1p^3 + p^2p^3 - 2p^1p^2p^3$. If $p^1 = constant,\ p^2 = a^2t + b^2,\ p^3 = a^3t + b^3$, then the identity is reduced to

$$c = p^1b^2 + p^1b^3 + b^2b^3 - 2p^1p^2p^3,$$
$$p^1a^2 + p^1a^3 + a^2b^3 + b^2a^3 - 2p^1(a^2b^3 + b^2a^3) = 0,$$
$$a^2a^3 - 2p^1a^2a^3 = 0.$$

A constant level set contains, for instance, the family of segments satisfying the conditions $p^1 = \frac{1}{2},\ b^2 + b^3 = 2c,\ a^2 + a^3 = 0.$

6. Conclusions and future work

Problems with multivariate reliability polynomial have been considered: homogenization, geometric properties of attached constant level sets, equivalence of reliability systems via diffeomorphisms, optimal mean time to failure, *etc*.

Future work concerns: (i) nonlinear optimization problems formulated for multitime reliability polynomials; (ii) optimal control problems regarding the reliability and its optimal allocation. Applications will be given in the area of robot control, control of chemical batch reactions and drug design of modeling HIV.

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[a] University Politehnica of Bucharest,
Faculty of Applied Sciences, Department of Mathematics-Informatics,
Splaiul Independentei 313, Bucharest 060042, Romania

[*] To whom correspondence should be addressed | email: udriste@mathem.pub.ro

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