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ABOUT A CLASS OF THREE-DIMENSIONAL SUBMANIFOLDS IN AFFINE SPACE A^6

OFELYA ARABYAN *

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ABSTRACT. A three-dimensional submanifold M in affine space A^6 is studied by the method of exterior forms. It is proven that the structure of total space generates a special type of affine connection on this submanifold; the structure equations of M are found.

The foundation of the method of exterior forms is placed in classic differential geometry. As it is known, the Segret-Frenet formulas describe all differential geometric properties of smooth curve, because they contain the main geometric invariants - curvature and torsion (Favard 1957). The same way, through the surface derivative formulas we can obtain all differential geometric properties of this surface. In both the first and second cases in every point of the submanifold the moving frame corresponding its structure is constructed in internal way. In the Segret-Frenet moving frame case the unit coordinate vectors are collinear to the curve tangent, main normal and binormal vectors, and in the case of derivative formulas, in the given point only the normal vector of the surface is unitary and orthogonal to the tangent plane of the surface in that point. The rest two vectors are tangent to the coordinate lines in that point and, in general, are not orthogonal and have norms, different of the unit.

When extracting submanifold's equations, from the very beginning the moving frame is not regulated entirely: they introduce an arbitrary frame, then gradually canonize it and accordingly simplify the differential equations of submanifold. This allows to obtain the final equations in maximally general moving frame, and also to find out geometric meaning of every step of the frame's canonization. Submanifolds of the affine space are relatively less-studied: only certain cases were discussed. Compared to the geometry of Euclidean space, the moderation of this space is accomplished by special choice of affine frame or, so-called, the equipment (Norden 1976). Blaschke (1923), Chakmazyan (1990), Arabyan (2015) and some other authors presented examples of studying affine space's submanifolds by the method of exterior forms.

If in the affine space A^6 we consider $M \subset A^6$ three-dimensional submanifold, then we have two bundles - tangent and normal. Let us take $R = \{O, e_1, e_2, e_3, e_4, e_5, e_6\}$ affine frame in this space, where the vectors e_1, e_2, e_3, e_4, e_5, e_6 compose the basis of the tangent vector space for A^6 and $e_1, e_2, e_3 \in T(M)$. Similarily the vectors e_4, e_5, e_6 compose a basis of the normal subspace: $e_4, e_5, e_6 \in N(M)$. Dual concept of these basic vectors are linear differential forms ω^1, ω^2, ω^3, ω^4, ω^5, ω^6 called principal forms. If we take an arbitrary γ smooth curve in submanifold $M \subset A^6$ then moving to the point P' near to point P along this curve, the vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, $\vec{e}_4$, $\vec{e}_5$, $\vec{e}_6$ will be transformed into vectors $\vec{e}'_1$, $\vec{e}'_2$, $\vec{e}'_3$, $\vec{e}'_4$, $\vec{e}'_5$, $\vec{e}'_6$ and for moving frame R will appear so-called infinitesimal deplacement equations, where the vectors dP, $de = e'_I - e_I$, $I = 1, \ldots, 6$ are decomposing by basis e_1, e_2, e_3, e_4, e_5, e_6:

$$
\begin{aligned}
&dP = -\omega^1 e_1 - \omega^2 e_2 - \omega^3 e_3 - \omega^4 e_4 - \omega^5 e_5 - \omega^6 e_6,\\
&de_i = \omega_i^k e_k + \omega_i^4 e_4 + \omega_i^\xi e_\xi,\\
&de_4 = \omega_4^k e_k + \omega_4^4 e_4 + \omega_4^\xi e_\xi,\\
&de_\xi = \omega_\xi^k e_k + \omega_\xi^4 e_4 + \omega_\xi^\eta e_\eta, \qquad i,k = 1,2,3; \qquad \xi, \eta = 5,6.
\end{aligned}
\tag{1}
$$

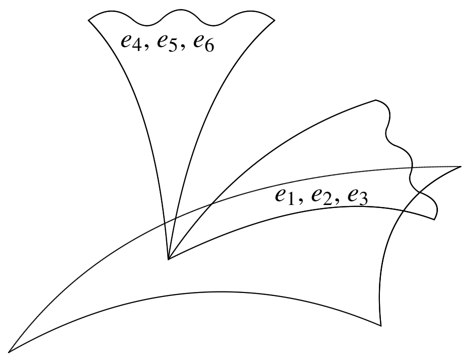
Here the expression ω_i^k describes the rotation of the vector e_i towards vector e_k while moving from the point P to the near point P'. We limit the choice of the moving frame on submanifold $M \subset A^6$ by requirement that in the case of infinitesimal deplacement of this frame, the vector e_4 moves so that the rotations of this vector can be held only towards vectors of the normal fibration. It means there is no rotations toward vectors e_1, e_2, e_3. In this case, in any fixed point of curve γ the forms ω_4^i are expressing by principal forms ω^1, ω^2, ω^3 only: $\omega_4^i = a_{4k}^i \omega^k$. In the system (1) forms ω_i^k, ω_i^4, ω_i^ξ, ω_4^k, ω_4^4, ω_4^ξ, ω_ξ^k, ω_ξ^4, ω_ξ^η are linearly independent on principal forms and on each other: these are calling secondary forms. The last mentioned forms show how basic vectors rotate in case of infinitesimal deplacement of the moving frame on submanifold. The compatibility conditions of the system (1) are said to be a A^6 space structure (derivative) equations, and in general case come to the following (Cartan 1962):

$$
\begin{cases}
d\omega^I = \omega_K^I \wedge \omega^K,\\
d\omega_K^I = \omega_P^I \wedge \omega_K^P,
\end{cases}
\qquad I, K, P = 1,2,3,4,5,6
\tag{2}
$$

where only the principal ω^1, ω^2, ω^3 and secondary ω_i^k forms directly describe the structure of submanifold $M \subset A^6$. It means, our nearest goal is to exclude the rest of principal and secondary forms or express them through these forms. In submanifold M the forms ω^4, ω^5, ω^6 are expressing by forms ω^1, ω^2, ω^3:

$$
\omega^\alpha = a_i^\alpha \omega^i, \qquad \alpha = 4,5,6; \qquad i = 1,2,3.
\tag{3}
$$

These relations are identities on three-dimensional submanifold $M \subset A^6$. It means, that the exterior differentiation of these relations gives identities on this submanifold. It might be presented in the following way:



$$\left(da_i^\alpha + a_k^\alpha \omega_i^k - a_i^\beta \omega_\beta^\alpha - \omega_i^\alpha + a_k^\alpha a_i^\beta \omega_\beta^k\right) \wedge \omega^i \equiv 0. \tag{4}$$

The rank of the third order matrix (a_i^α) can be equal to 0, 1, 2, and 3:

$$\operatorname{rank}(a_i^\alpha) = 0, 1, 2, 3.$$

There is not any metrics in affine space, because of which various equipments are importing in the space, so the structure (derivative) equations are simplifying. If we have three-dimensional submanifold M in affine space A^6 then naturally appear two bundles - tangent and normal. In tangent bundle are acting differential forms ω^i, ω_k^i, $i,k = 1,2,3$ and in normal bundle – ω^α, ω_β^α, $\alpha,\beta = 4,5,6$. For description of the tangent bundle they are using basic vectors – e_i, $i = 1,2,3$ and for the normal bundle the basic vectors – e_α, $\alpha = 4,5,6$. Linear differential forms ω_i^α, ω_α^i, $i = 1,2,3$ and $\alpha = 4,5,6$, establish connection between tangent and normal bundles. If these forms simultaneously are equal to the zero, then the whole space turns into cross product of tangent and normal bundles. In practice these differential forms are expressing by basic forms and through which the curvature and torsion tensors components of submanifold are forming.

Let us consider the case when rank $(a_i^\alpha) = 1$. We suppose that $a_1^4 = 1$ and other coefficients are equal to zero. Therefore

$$\begin{cases} \omega^4 = \omega^1, \\ \omega^5 = 0, \\ \omega^6 = 0. \end{cases} \tag{5}$$

If we will differentiate in exterior way relations of the system (5) being identities on submanifold M and apply general structural equations (2) of M, then after some identical modifications we will obtain two differential identities

$$\begin{cases} \left(\omega_1^4 + \omega_4^4 - \omega_1^1 - \omega_4^1\right) \wedge \omega^1 + \left(\omega_2^4 - \omega_2^1\right) \wedge \omega^2 + \left(\omega_3^4 - \omega_3^1\right) \wedge \omega^3 \equiv 0, \\ \left(\omega_1^\xi + \omega_4^\xi\right) \wedge \omega^1 + \omega_2^\xi \wedge \omega^2 + \omega_3^\xi \wedge \omega^3 \equiv 0. \end{cases} \tag{6}$$

It follows from the first identity that the secondary forms ω_2^4, ω_3^4 are expressing by forms ω_2^1 and ω_3^1 and the form ω_1^4 might be expressed by secondary form ω_1^1. It is not difficult to check that the system of differential equations $\omega_1^4+\omega_4^4-\omega_1^1-\omega_4^1=0$, $\omega_1^\xi+\omega_4^\xi=0$, $\omega_2^\xi=0$, $\omega_3^\xi=0$, $\xi=5,6$, is totally integrable. For example, differentiating in exterior way the expression $\omega_1^4+\omega_4^4-\omega_1^1-\omega_4^1$ and applying primary derivative formulas we obtain

$$d\left(\omega_1^4+\omega_4^4-\omega_1^1-\omega_4^1\right)=\omega_4^1\wedge\left(\omega_1^4+\omega_4^4-\omega_1^1-\omega_4^1\right)+\omega_1^4\wedge\left(\omega_1^4+\omega_4^4-\omega_1^1-\omega_4^1\right).$$

This means that, without loss of generality, we can assume

$$\omega_1^4+\omega_4^4-\omega_1^1-\omega_4^1=0,\ \ \omega_i^\xi=0,\ \ \omega_4^\xi=0,\quad i=1,2,3;\ \xi=5,6.$$

Moreover, for simplicity we will observe only the case when

$$\omega_4^4=\omega_1^1,\ \ \omega_1^4=\omega_4^1,\ \ \omega_4^4=\omega_1^4,\ \ \omega_4^1=\omega_1^1. \tag{7}$$

The part of these relations expresses the equality of forms ω^4, ω^1. According to Cartan's lemma we obtain the following decomposition from the first identity of the system (7):

$$\omega_i^4=\omega_i^1+b_{ik}^4\omega^k,\quad b_{ik}^4=0. \tag{8}$$

Thanks to the choice of moving frame, the secondary forms are expressed by forms ω^1, ω^2, ω^3:

$$\omega_4^i=a_{4k}^i\omega^k. \tag{9}$$

Thereby, the primary structure equations are come to the following:

$$\begin{aligned}
d\omega^1&=2\omega_1^1\wedge\omega^1+\omega_2^1\wedge\omega^2+\omega_3^1\wedge\omega^3,\\
d\omega^2&=2\omega_1^2\wedge\omega^1+\omega_2^2\wedge\omega^2+\omega_3^2\wedge\omega^3,\\
d\omega^3&=2\omega_1^3\wedge\omega^1+\omega_2^3\wedge\omega^2+\omega_3^3\wedge\omega^3,\\
d\omega_k^i&=\omega_p^i\wedge\omega_k^p+a_{4p}^ib_{kt}^4\omega^p\wedge\omega^t.
\end{aligned} \tag{10}$$

The structure equation of the secondary form ω_1^1 has a speciality

$$d\omega_1^1=\omega_p^1\wedge\omega_1^p$$

Exterior differentiation of relations (9) and (8) with further application of structure equations and then Cartan's lemma leads to the following differential equations:

$$\begin{aligned}
&da_{4k}^i+a_{4p}^i\omega_k^p+a_{4k}^i\omega_4^4-a_{4k}^p\omega_p^i-a_{4p}^ia_{4k}^p\omega^1=a_{4kt}^i\omega^t,\\
&db_{ik}^4+b_{ip}^4\omega_k^p+b_{pk}^4\omega_i^p-b_{ik}^p\omega_4^4+b_{ik}^4\omega_4^1-b_{ip}^4a_{4k}^p\omega^1=b_{ikt}^4\omega^t,
\end{aligned} \tag{11}$$

where the coefficients a_{4kt}^i and b_{ikt}^4 are symmetric with respect to indices k and t. It is not difficult to check now that the system (10)–(11) is closed, that is, exterior differentiation of these relations does not lead to new relations. According to Cartan-Laptev's theorem (Laptev 1966) there exists an affine connection on submanifold M determined by the system of structure equations (10).

Theorem 1. *If three-dimensional submanifold M in affine space A^6 is given by imbedding* (5) *and the conditions* (7) *hold, then the connection of the total space induces a special type affine connection with non zero curvature tensor such that the structure equations of M come to the form* (10) *and coefficients of this system satisfy the differential equations* (11).

The non-zero components of the curvature tensor of submanifold M are:

$$R^{i4}_{4k} = a^{i}_{4p} b^{4}_{kt}.$$

Thereby, this affine connection has vanishing torsion and non-zero curvature. The specificity of this affine connection is seen from structure equations. The result of the Theorem 1 is the three-dimensional analogue of the result established by Arabyan (2015).

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References

Arabyan, O. (2015). “About one class of two dimensional submanifolds in four dimensional affine space”. *Scientific Publications of Armenian State Pedagogical University* **2-3**(24-25), 92–96. (in Armenian).

Blaschke, W. (1923). *Vorlesungen über Differentialgeometrie und Geometrische Grundlagen von Einsteins Relativitätstheorie II. Affine Differentialgeometrie*. Ed. by K. Reidemeister. Vol. 7. Die Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg. DOI: 10.1007/978-3-642-47392-0.

Cartan, E. (1962). “Spaces of Affine, Projective and Conform Connection”. *Kazan University Press*. (in Russian).

Chakmazyan, A. V. (1990). *Normal Connection in the Geometry of Submanifolds*. Yerevan: ASPU Press. (in Russian).

Favard, J. (1957). *Cours de Géométrie Différentielle Locale*. Paris: Gauthier-Villars.

Laptev, G. F. (1966). “Fundamental infinitesimal structures of higher orders on a smooth manifold”. In: *Proceedings of the Geometrical Seminar. Problems of Geometry*. Vol. 1. Moscow: VINITI, Russian Academy of Sciences, pp. 139–190. (in Russian).

Norden, A. P. (1976). *Spaces with Affine Connection*. Moscow: Nauka. (in Russian).

* Armenian State Pedagogical University
Chair of Mathematics and Methodology of its Teaching
5, Khandjan str., 375010, Yerevan, Republic of Armenia

Email: ofelya.arabyan@mail.ru

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