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ON A SPECIAL CLASS OF SUBMANIFOLDS IN PSEUDOEUCLIDEAN SPACE E_n^{2n}

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ABSTRACT. A special class of $2m$ dimensional submanifolds $M \to E_n^{2n}$ with structure of double fiber bundle is studied. Using the Cartan's method of exterior forms on manifold the structure equations of M are discovered and the differential geometric structure on M is described. M has the structure of Rashevsky-Einstein space.

Introduction

Differential Geometry of multiple integrals depending on parameters is one of new fields of modern Differential Geometry. Researches in this area were possible because of active applications of the method of modern differential geometric researches based on the method of exterior forms (Cartan 1933; Finikov 1948; Laptev 1953; Vasilyev 1987). Cartan (1936) introduced, for the first time, the concept of differential geometric research of mathematical objects. Studying ordinary integral he associated to the integral the corresponding subintegral form and studied it as geometric object. Such study was considering as a differential geometric study of integral.

Our approach is vaster. In general the corresponding scheme is as follows. On the initial stage we consider the geometrical description of main structures of the given theory and connections between them. The next step is the differential geometric analysis of these structures and identification of their most general characteristic (geometric) properties. Finally on the last stage of research these properties or their part only become the foundation for generalizations and new problems in initial theory. New level research starts. This approach has some similarity with iterations and it gives definitive opportunity to keep the studying objects always in the center of differential geometric study penetrating in the more and more deep areas of investigation. Saying differential geometry of analytical objects we have in mind above mentioned research.

All this is true for the geometry of multiple integral depending on parameters. The study of differential geometric structures defined by such integral on the manifold of integrations variables and parameters to a certain extent is analogical to the study of the integral's geometry (Cartan 1936). Kawaguchi (1931, 1938) realized significant progress in this approach. At the same time the presence of parameters totally changes the circle of arising problems and corresponding results. By systematic study of multiple integrals depending on parameters and also (in a special case when the number of parameters is equal to the number of variables) determined integral transforms one can see a good number of interesting geometric problems connected with the description of invariant properties of such integrals. Solution of the inverse problem of discovering canonical integrals of admissible differential geometric structures on the manifold of double fiber bundle of variables of integration and parameters M (Haroutunian 1998) and further analysis of obtained kernels comes to the necessity to study submanifolds of pseudoeuclidean space E^{2n}_n and calculate canonical integrals on these submanifolds. The present work is devoted to the geometry of submanifolds of dimension $2m$ ($2m > n$) with structure of double fiber bundle.

1. Identification of the problem: Rashevsky space

Rashevsky (1948), from M.V. Lomonosov Moscow State University, introduced a special class of pseudoriemannian spaces (hyperbolic A-space). He studied an invariant scalar field $U(x^1,\dots,x^n,y_1,\dots,y_n)$ with non degenerate matrix of the second order derivatives:

$$\det\left(\frac{\partial^2 U}{\partial x^i \partial y_j}\right) \neq 0$$

and introduced pseudoriemannian metrics of the signature (n,n) on M and corresponding pseudoriemannian connection. This space is known as Rashevsky pseudoriemannian space. It satisfies the following conditions.

(1) The scalar field $U(x^1,\dots,x^n,y_1,\dots,y_n)$ generating the structure of the pseudoriemannian space on M is determined with arbitrarity

$$U(x^i,y_j) \to U(x^i,y_j) + U_1(x^i) + U_2(y_j).$$

(2) Each point of M belongs to one and only one fiber from each of two families of fibers, fibers from different families have intersection in no more than one point.

(3) Fibers of both families are isotropic.

(4) Fibers of each family have the property of absolute parallelism (autoparallelism): vectors tangent to fibers of the given family stay tangent to them after parallel transfer along arbitrary smooth curve.

It follows from each of two last properties that both families of fibers are totally geodesic in M.

Haroutunian (1975) proved that n-tuple integral of the function $K(x^1,\dots,x^n,y_1,\dots,y_n)$ depending on n parameters induces some bilinear form $d\varphi$ and this form induces the structure of Rashevsky pseudoriemannian space on $2n$ dimensional manifold of integration variables and parameters M in an invariant way under condition that the matrix of second order derivatives of the function $\ln K$ is non degenerate. So instead of integral we can study the pair of objects: subintegral form – manifold of integration variables and parameters M. It is clear that the manifold M is a cross product of two n dimensional manifolds M_1 and M_2

$$M = M_1 \times M_2, \quad \dim M_1 = \dim M_2 = n$$

but for maximal generality instead of cross product $M_1 \times M_2$ we will consider $2n$ dimensional double fiber manifold with n dimensional base and n dimensional fibers. Such agreement is acceptable because locally the cross product is a double fiber bundle and our research has a differential geometric character, *i.e.*, local. We consider manifolds M, M_1, M_2 smooth but in reality we can consider them differentiable of the class $C^{(4)}$.

We have a geometric description of our general construction. Let us introduce the family of tangent frames $\{p,(e_i)_0,(e^i)_0,\ i=1,2,\ldots,n\}$ on M adapted to the structure of double fiber bundle, then in the coframe of linear differential forms $\omega^1,\omega^2,\ldots,\omega^n$, $\omega_1,\omega_2,\ldots,\omega_n$ the form $d\varphi$ is reducing to the $d\varphi=\omega^i\wedge\omega_i$ and the structure equations of Rashevsky space can be written as follows:

$$\begin{aligned} d\omega^i &= \omega^i_k\wedge\omega^k \\ d\omega_i &= -\omega^k_i\wedge\omega_k \\ d\omega^i_k &= \omega^i_p\wedge\omega^p_k + R^{ir}_{kp}\omega^p\wedge\omega_r \end{aligned} \tag{1}$$

where R^{ir}_{kp} are non zero components of the curvature tensor. If this tensor is vanishing we come to the structute equations of pseudoeuclidean Rashevsky space E^{2n}_n. Rashevsky space is said to be an Einstein space, if the Ricci tensor of this space (*i.e.*, $R^i_k=R^{ip}_{kp}$) is proportional to the metrical tensor with constant coefficient. It was proved that this structure is inducing on $M$by the Fourier integral transform (canonical integral). The problem “what kind of n-tuple integrals depending on n parameters induce the structure of Rashevsky pseudoriemannian space on $2n$ dimensional manifold of integration variables and parameters M in an invariant way”comes to the necessity to study submanifolds of E^{2n}_n with structure of double fiber bundle and calculate corresponding integrals on these submanifolds. n-tuple integral depending on k parameters is said to be a canonical integral of the given admissible differential geometric structure on M if this integral induces this structure on M and besides the total number of integration variables and parameters is equal to the dimension of M. The necessity to study submanifolds of arbitrary dimension $2m$ in $2n$ dimensional pseudoeuclidean Rashevsky space E^{2n}_n is determined also by two other reasons. First the research of canonical integral of submanifold M of codimension two in this space comes to some results when the kernel of canonical integral contains not only the components of the curvature tensor of the space of affine connection but also some components of the torsion tensor of this connection (the torsion of the Rashevsky space is equal to zero). It is interesting to study the content of this phenomena and the character of the presence of these components in canonical kernel. Second in the previous publications we composed some classification of submanifolds of codimension two in pseudoeuclidean Rashevsky space. It is natural to construct the similar classification for submanifolds of arbitrary dimension in this total space.

2. Essential embedding: structure theorems

As it is well known (Whitney 1957), it is possible to insert n dimensional manifold in Euclidean $2n+1$ dimensional space as a submanifold. It means that we can limit our research studying submanifolds of dimension n and more. Therefore we consider the case when $2m>n$ (or $m>n-m$).

The structure equations of pseudoeuclidean $2n$ dimensional space with metrics of half index may be represented in the form (Haroutunian 1975)

$$\begin{aligned} & d\omega^I = \omega^I_K \wedge \omega^K \\ & d\omega_I = -\omega^K_I \wedge \omega_K; \; I,K,P = 1,2,\dots,n, \\ & d\omega^I_K = \omega^I_P \wedge \omega^P_K. \end{aligned} \tag{2}$$

We will start our study by introduction of indices. The preliminary zones of indices are presented in the Figure 1.

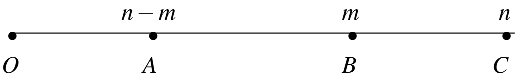

Figure. 1

For the zone OA we will use the indices i, j, k, l. The lengths of zones BC and OA are equal, therefore for the zone BC we have indices $m+i$, $m+k$ etc. The zone AB with the length $2m-n$ will be described by indices $\xi,\eta,\mu,\nu = n-m+1,\dots,m$. Our research is limited by the inequality $2m > n$ or $n-m < m$. Haroutunian (2001) studied the case $m < n-m$ (or $2m < n$) and the case of submanifolds of half dimension ($n-m = m$) (Haroutunian 2004).

In analogy with differential geometry of submanifolds of codimension two we will study one of classes of embeddings in the case of submanifolds of arbitrary codimensions in pseudoeuclidean space E^{2n}_n defined by relations

$$\omega^{m+i} = \omega_i, \;\; \omega_{m+i} = \omega^i, \quad i = 1,2,\dots,n-m. \tag{3}$$

It is not difficult to note that this class is a direct generalization of the embedding (C) (Haroutunian 1991) when codimension of submanifold M is arbitrary. Relations (3) are identities on M. Haroutunian (1991) studied this embedding in the case when $2m = n$. The bilinear form $d\varphi = \omega^I \wedge \omega_I$ can be rewritten as follows:

$$d\varphi = \omega^I \wedge \omega_I = \omega^i \wedge \omega_i + \omega^\xi \wedge \omega_\xi + \omega^{m+i} \wedge \omega_{m+i} = 2\omega^i \wedge \omega_i + \omega^\xi \wedge \omega_\xi$$

First, we must separate indices in structure equations (2) and substiture relations (3) there. We obtain general structure equations of submanifold M:

$$\begin{aligned}
&d\omega^i = \omega^i_k \wedge \omega^k + \omega^i_\xi \wedge \omega^\xi + \omega^i_{m+k} \wedge \omega_k,\\
&d\omega_i = -\omega^k_i \wedge \omega_k - \omega^\xi_i \wedge \omega_\xi + \omega^{m+k}_i \wedge \omega^k,\\
&d\omega^\xi = \omega^\xi_\eta \wedge \omega^\eta + \omega^\xi_i \wedge \omega^i + \omega^\xi_{m+k} \wedge \omega_k,\\
&d\omega_\xi = -\omega^\eta_\xi \wedge \omega_\eta - \omega^i_\xi \wedge \omega_i + \omega^{m+k}_\xi \wedge \omega^k,\\
&d\omega^i_k = \omega^i_p \wedge \omega^p_k + \omega^i_\xi \wedge \omega^\xi_k + \omega^i_{m+p} \wedge \omega^{m+p}_k,\\
&d\omega^i_\xi = \omega^i_p \wedge \omega^p_\xi + \omega^i_\eta \wedge \omega^\eta_\xi + \omega^i_{m+p} \wedge \omega^{m+p}_\xi,\\
&d\omega^\xi_i = \omega^\xi_p \wedge \omega^p_i + \omega^\xi_\eta \wedge \omega^\eta_i + \omega^\xi_{m+p} \wedge \omega^{m+p}_i,\\
&d\omega^\xi_\eta = \omega^\xi_\nu \wedge \omega^\nu_\eta + \omega^\xi_i \wedge \omega^i_\eta + \omega^\xi_{m+i} \wedge \omega^{m+i}_\eta,\\
&d\omega^i_{m+k} = \omega^i_p \wedge \omega^p_{m+k} + \omega^i_\xi \wedge \omega^\xi_{m+k} + \omega^i_{m+p} \wedge \omega^{m+p}_{m+k},\\
&d\omega^{m+i}_k = \omega^{m+i}_p \wedge \omega^p_k + \omega^{m+i}_\xi \wedge \omega^\xi_k + \omega^{m+i}_{m+p} \wedge \omega^{m+p}_k,\\
&d\omega^\xi_{m+i} = \omega^\xi_k \wedge \omega^k_{m+i} + \omega^\xi_\eta \wedge \omega^\eta_{m+i} + \omega^\xi_{m+p} \wedge \omega^{m+p}_{m+i},\\
&d\omega^{m+i}_\xi = \omega^{m+i}_p \wedge \omega^p_\xi + \omega^{m+i}_\eta \wedge \omega^\eta_\xi + \omega^{m+i}_{m+p} \wedge \omega^{m+p}_\xi,\\
&d\omega^{m+i}_{m+k} = \omega^{m+i}_p \wedge \omega^p_{m+k} + \omega^{m+i}_\xi \wedge \omega^\xi_{m+k} + \omega^{m+i}_{m+p} \wedge \omega^{m+p}_{m+k};\\
&i,k,p = 1,2,\dots,n-m; \qquad \xi,\eta,\nu = n-m+1,\dots,m.
\end{aligned} \tag{4}$$

Second, the submanifold M is a double fiber bundle, therefore the systems of linear differential equations $\{\omega^i = 0,\ \omega^\xi = 0,\ i = 1,\dots,n-m;\ \xi = n-m+1,\dots,m\}$ and $\{\omega_i = 0,\ \omega_\xi = 0,\ i = 1,\dots,n-m;\ \xi = n-m+1,\dots,m\}$ are totally integrable. In virtue of the Frobenius Theorem (Vasilyev 1987) the following identities hold

$$\omega^i_{m+k} \wedge \omega_k = 0,\ \omega^\xi_{m+k} \wedge \omega_k = 0,\ \omega^{m+k}_i \wedge \omega^k = 0,\ \omega^{m+k}_\xi \wedge \omega^k = 0$$

and therefore the secondary forms ω^i_{m+k}, ω^ξ_{m+k}, ω^{m+k}_i, ω^{m+k}_ξ are linear combinations of corresponding totalities of principal forms:

$$\begin{aligned}
&\omega^i_{m+k} = C^{ip}_{m+k}\omega_p, \omega^\xi_{m+k} = C^{\xi k}_{m+k}\omega_k,\\
&\omega^{m+k}_i = C^{m+k}_{ip}\omega^p, \omega^{m+i}_\xi = C^{m+i}_{\xi k}\omega^k.
\end{aligned} \tag{5}$$

We will substitute these decompositions in structure equations (4) later, after some simplifications. Exterior differentiation of relations (3) being identities on M, application of structure equations come to the following two exterior identities:

$$\begin{aligned}
&\left(\omega^{m+i}_k - \omega^{m+k}_i\right) \wedge \omega^k + \left(\omega^{m+i}_{m+k} + \omega^k_i\right) \wedge \omega_k + \omega^{m+i}_\xi \wedge \omega^\xi + \omega^\xi_i \wedge \omega_\xi = 0,\\
&\left(\omega^i_{m+k} - \omega^k_{m+i}\right) \wedge \omega_k + \left(\omega^{m+k}_{m+i} + \omega^i_k\right) \wedge \omega^k + \omega^i_\xi \wedge \omega^\xi - \omega^\xi_{m+i} \wedge \omega_\xi = 0.
\end{aligned} \tag{6}$$

Taking in account the decomposition of the form ω_k^{m+i} from (5) it is easy to note that the addend $(\omega_k^{m+i} - \omega_i^{m+k}) \wedge \omega^k$ in the first identity is transforming into expression $\omega^i \wedge \omega^k$ which has not similars in this identity and therefore is equal to zero. It means that the first identity is equivalent to following two identities

$$\begin{aligned} &\left(\omega_{m+k}^{m+i} + \omega_i^k\right) \wedge \omega_k + \omega_i^\xi \wedge \omega_\xi = 0, \\ &\omega_\xi^{m+i} \wedge \omega^\xi = 0. \end{aligned} \tag{7}$$

Using the same reasonings we obtain the following two identities equivalent to the second identity from the system (6)

$$\begin{aligned} &\left(\omega_{m+i}^{m+k} + \omega_k^i\right) \wedge \omega^k + \omega_\xi^i \wedge \omega^\xi = 0, \\ &\omega_{m+i}^\xi \wedge \omega_\xi = 0. \end{aligned} \tag{8}$$

It is easy to note from first relations of the systems (7) and (8) that in (7) the form $\omega_{m+k}^{m+i} + \omega_i^k$ is a linear combination of principal forms $\omega_1, \ldots, \omega_{n-m}$ only, but simultaneously the first relation from the system (8) shows that it has a decomposition by forms $\omega^1, \ldots, \omega^{n-m}$ only. It is possible if and only if

$$\omega_{m+k}^{m+i} + \omega_i^k = 0.$$

Other side the second identities of these two systems show that the secondary forms ω_ξ^{m+i} and ω_{m+i}^a are vanishing. Indeed it follows from these identities that these forms are linear combinations of principal forms $\omega^{n-m+1}, \ldots, \omega^m$ and $\omega_{n-m+1}, \ldots, \omega_m$ respectively. But relations (5) show that these forms are linear combinations of principal forms $\omega^1, \ldots, \omega^{n-m}$ and $\omega_1, \ldots, \omega_{n-m}$ only. It means that these forms are equal to zero:

$$\omega_\xi^{m+i} = 0, \qquad \omega_{m+i}^\xi = 0. \tag{9}$$

We come to the following decompositions for secondary forms $\omega_k^{m+i} - \omega_i^{m+k}$, $\omega_{m+k}^i - \omega_{m+i}^k$, ω_i^ξ, ω_ξ^i by principal forms

$$\begin{aligned} &\omega_k^{m+i} - \omega_i^{m+k} = \left(C_{kp}^{m+i} - C_{ip}^{m+k}\right) \omega^p, \\ &\omega_{m+k}^i - \omega_{m+i}^k = \left(C_{m+k}^{ip} - C_{m+i}^{kp}\right) \omega_p, \\ &\omega_i^\xi = C_i^{\xi\eta} \omega_\eta, \\ &\omega_\xi^i = C_{\xi\eta}^i \omega^\eta. \end{aligned} \tag{10}$$

Exterior differentiation of relations (9) being identities on the manifold M with further application of structure equations come to the following algebraic relations:

$$C_{kp}^{m+i} C_{\xi\eta}^k = 0, \quad C_k^{\xi\eta} C_{m+i}^{kp} = 0. \tag{11}$$

Substituting relations (9), (10) in general structure equations of the manifold M we obtain much more simple system

$$\begin{aligned}
&d\omega^i = \omega^i_k \wedge \omega^k,\\
&d\omega_i = -\omega^k_i \wedge \omega_k,\\
&d\omega^\xi = \omega^\xi_\eta \wedge \omega^\eta + C^{\xi\eta}_i \omega_\eta \wedge \omega^i,\\
&d\omega_\xi = -\omega^\eta_\xi \wedge \omega_\eta - C^i_{\xi\eta} \omega^\eta \wedge \omega_i,\\
&d\omega^i_k = \omega^i_p \wedge \omega^p_k + C^i_{\xi\eta} C^{\xi\eta}_k \omega^\nu \wedge \omega_\eta - C^{it}_{m+p} C^{m+p}_{kr} \omega^r \wedge \omega_t,\\
&d\omega^\xi_\eta = \omega^\xi_\nu \wedge \omega^\nu_\eta - C^{\xi\nu}_i C^i_{\eta\mu} \omega^\mu \wedge \omega_\nu.
\end{aligned} \tag{12}$$

This system of structure equations is not final. Indeed it is necessary to establish the system of differential equations for coefficients of this system. Starting this procedure for coefficients $C^i_{\xi\eta}$ and $C^{\xi\eta}_i$, we differentiate the decompositions of secondary forms ω^i_ξ and ω^ξ_i respectively (see last relations of the system (10)) in exterior way. Applying structure equations (10) we obtain after some evident modifications:

$$\begin{aligned}
&\left(dC^i_{\xi\eta} - C^k_{\xi\eta}\omega^i_k + C^i_{\xi\nu}\omega^\nu_\eta + C^i_{\nu\eta}\omega^\nu_\xi\right) \wedge \omega^\eta + C^i_{\xi\eta} C^{\eta\nu}_k \omega_\nu \wedge \omega^k = 0,\\
&\left(dC^{\xi\eta}_i - C^{\xi\nu}_i \omega^\eta_\nu - C^{\nu\eta}_i \omega^\xi_\nu + C^{\xi\eta}_k \omega^k_i\right) \wedge \omega_\eta - C^{\xi\eta}_i C^k_{\eta\nu} \omega^\nu \wedge \omega_k = 0.
\end{aligned}$$

The following two algebraic relations follow directly from these identities:

$$C^i_{\xi\eta} C^{\eta\nu}_k = 0, \qquad C^{\xi\eta}_i C^k_{\eta\nu} = 0. \tag{13}$$

Applying Cartan's lemma we obtain differential equations for coefficients $C^i_{\xi\eta}$, $C^{\xi\eta}_i$ and exterior differentiation of decompositions $\omega^i_{m+k} = C^{ip}_{m+k}\omega_p$, $\omega^{m+k}_i = C^{m+k}_{ip}\omega^p$ with further application of this lemma comes to the differential equations for coefficients C^{ip}_{m+k} and C^{m+k}_{ip}:

$$\begin{aligned}
&dC^i_{\xi\eta} = C^k_{\xi\eta}\omega^i_k - C^i_{\xi\nu}\omega^\nu_\eta - C^i_{\nu\eta}\omega^\nu_\xi + C^i_{\xi\eta\nu}\omega^\nu,\\
&dC^{\xi\eta}_i = C^{\xi\nu}_i \omega^\eta_\nu + C^{\nu\eta}_i \omega^\xi_\nu - C^{\xi\eta}_k \omega^k_i + C^{\xi\eta\nu}_i \omega_\nu,\\
&dC^{ip}_{m+k} = C^{it}_{m+k}\omega^p_t + C^{tp}_{m+k}\omega^i_t - C^{ip}_{m+t}\omega^{m+t}_{m+k} + C^{ipt}_{m+k}\omega_t,\\
&dC^{m+k}_{ip} = -C^{m+k}_{it}\omega^t_p - C^{m+k}_{tp}\omega^t_i + C^{m+t}_{ip}\omega^{m+k}_{m+t} + C^{m+k}_{ipt}\omega^t,
\end{aligned} \tag{14}$$

where new coefficients $C^i_{\xi\eta\nu}$, $C^{\xi\eta\nu}_i$ and C^{ipt}_{m+k}, C^{m+k}_{ipt} are symmetric with respect to indices η, ν and p, t respectively. Substitution of conditions (13) in (12) comes to the final system of structure equations of submanifold M:

$$\begin{aligned}
&d\omega^i = \omega^i_k \wedge \omega^k,\\
&d\omega_i = -\omega^k_i \wedge \omega_k,\\
&d\omega^\xi = \omega^\xi_\eta \wedge \omega^\eta + C^{\xi\eta}_i \omega_\eta \wedge \omega^i,\\
&d\omega_\xi = -\omega^\eta_\xi \wedge \omega_\eta - C^i_{\xi\eta} \omega^\eta \wedge \omega_i,\\
&d\omega^i_k = \omega^i_p \wedge \omega^p_k - C^{it}_{m+p} C^{m+p}_{kr} \omega^r \wedge \omega_t,\\
&d\omega^\xi_\eta = \omega^\xi_\nu \wedge \omega^\nu_\eta - C^{\xi\nu}_i C^i_{\eta\mu} \omega^\mu \wedge \omega_\nu.
\end{aligned} \tag{15}$$

It is easy to check that the system of linear differential forms ω^i, ω^ξ, ω_i, ω_ξ, ω^i_k, ω^ξ_η, $i = 1,\dots,n-m$; $\xi = n-m+1,\dots,m$ and functions $C^{\xi\eta}_i$, $C^i_{\xi\eta}$, C^{ik}_{m+p}, C^{m+i}_{kp}, satisfying differential equations (14)-(15), is closed: exterior differentiation of these structure equations doesn't give new forms and conditions. Following the Laptev's Theorem (Laptev 1953) this system determines an affine connection on submanifold M. The following result is established.

Theorem 2.1. *Metrical connection of the pseudoeuclidean Rashevsky space* E^{2n}_n *induces affine connection* γ *defined by differential forms* ω^i, ω^ξ, ω_i, ω_ξ, ω^i_k, ω^ξ_η, $i = 1,\dots,n-m$; $\xi = n-m+1,\dots,m$, *satisfying structure equations* (15) *and functions* $C^{\xi\eta}_i$, $C^i_{\xi\eta}$, C^{ik}_{m+p}, C^{m+i}_{kp}, *satisfying differential equations* (14) *on* $2m$ *dimensional submanifold* M *determined by embedding* (3).

It is not difficult to note that the systems of Pfaff's equations

$$\omega^i = 0, \quad i = 1,\dots,n-m, \qquad \omega_i = 0, \quad i = 1,\dots,n-m$$

are totally integrable and therefore determine $(3m-n)$ dimensional submanifolds L_1 and L_2. Moreover in the system of differential forms ω^i, ω^ξ, ω_i, ω_ξ, ω^i_k, ω^ξ_η, $i = 1,\dots,n-m$; $\xi = n-m+1,\dots,m$ and functions $C^{\xi\eta}_i$, $C^i_{\xi\eta}$, C^{ik}_{m+p}, C^{m+i}_{kp}, satisfying structure equations (15) and differential equations (14) respectively, the subsystem of forms ω^i, ω^ξ, ω^i_k is closed too. It means that the manifold M contains a $2(n-m)$ dimensional submanifold $N_1 \subset M$ determined by structure equations

$$\begin{aligned}
&d\omega^i = \omega^i_k \wedge \omega^k,\\
&d\omega_i = -\omega^k_i \wedge \omega_k,\\
&d\omega^i_k = \omega^i_p \wedge \omega^p_k - C^{it}_{m+p} C^{m+p}_{kr} \omega^r \wedge \omega_t.
\end{aligned}$$

Therefore submanifold N_1 is $2(n-m)$ dimensional Rashevsky space with curvature tensor

$$R^{it}_{kp} = -C^{it}_{m+p} C^{m+p}_{kr}.$$

It is easy to see that the submanifold N_1 is the intersection of submanifolds L_1 and $L_2 : N_1 = L_1 \bigcap L_2$. Other side the system of Pfaff's equations $\omega^i = 0$, $\omega_i = 0$, $i = 1,\dots,n-m$ is totally integrable and determines $2(2m-n)$ dimensional submanifold N_2 with structure equations

$$\begin{aligned} d\omega^{\xi} &= \omega^{\xi}_{\eta} \wedge \omega^{\eta}, \\ d\omega_{\xi} &= -\omega^{\eta}_{\xi} \wedge \omega_{\eta}, \\ d\omega^{\xi}_{\eta} &= \omega^{\xi}_{\nu} \wedge \omega^{\nu}_{\eta} - C^{\xi\nu}_{i} C^{i}_{\eta\mu} \omega^{\mu} \wedge \omega_{\nu}. \end{aligned} \tag{16}$$

It means that the submanifold N_2 is $2(2m-n)$ dimensional Rashevsky space with curvature tensor

$$R^{\xi\nu}_{\eta\mu} = -C^{\xi\nu}_{i} C^{i}_{\eta\mu}.$$

Taking in account the relations (13) it is easy to see that

$$R^{\xi}_{\eta} = R^{\xi\nu}_{\eta\nu} = -C^{\xi\nu}_{i} C^{i}_{\eta\nu} = 0,$$

i.e., the Ricci tensor of submanifold N_2 is vanishing. Manifolds with vanishing Ricci tensors compose a subclass of Einstein spaces. It follows from here that the submanifold N_2 is $2(2m-n)$ dimensional Rashevsky-Einstein space with vanishing Ricci tensor. Therefore locally the submanifold M is a cross product $N_1 \times N_2$. The following statement is established.

Theorem 2.2. *The submanifold* $M \subset E^{2n}_{n}$ *determined by embedding* (3) *is a bundle with* $2(n-m)$ *dimensional base having structure of pseudoriemannian Rashevsky-Einstein space with curvature tensor* $R^{it}_{kp} = -C^{it}_{m+p} C^{m+p}_{kr}$ *and fibers having the structure of* $2(2m-n)$ *dimensional Rashevsky space with vanishing Ricci tensor.*

Other side, the system of differential forms ω^{ξ}, ω_{ξ}, ω^{ξ}_{η}, $\xi, \eta = n-m+1, \ldots, m$, satisfying structure equations (15) is not closed: exterior differentials of basic forms ω^{ξ} and ω_{ξ}, $\xi, \eta = n-m+1, \ldots, m$ contain forms ω^{i} and ω_{i}. It means that generally the manifold M has non trivial torsion tensor.

In analogy with (Haroutunian 2005, 2009), the relations (11) and (13) can be used for classification of different types of submanifolds M. For example it is easy to check that the following statements are true.

Theorem 2.3. *If at least for one value of the index* $i(=1,\ldots,n-m)$ *the matrices* (C^{m+i}_{kp}) *and* (C^{kp}_{m+i}) *are nondegenerate then affine connection in the second factor of the local product* $M = N_1 \times N_2$ *is flat and the bundle* M *is a cross product of two submanifolds* N_1 *and* N_2*.*

Theorem 2.4. *If at least for one value of the index* $i(=1,\ldots,n-m)$ *the matrix* $(C^{i}_{\xi\eta})$ *is nondegenerate then affine connection in the second factor of the local product* $M = N_1 \times N_2$ *is flat.*

Theorem 2.5. *If both matrices* (C^{m+i}_{kp}), (C^{kp}_{m+i}) *are vanishing then the submanifold* M *is a bundle with* $2(n-m)$ *dimensional base with structure of pseudoeuclidean Rashevsky-Einstein space and fibers with structure of* $2(2m-n)$ *dimensional pseudoriemannian Rashevsky-Einstein space.*

It is evident that if all coefficients in structure equations (15) are equal to zero then the submanifold M is a cross product of two pseudoeuclidean Rashevsky spaces of dimensions $2(n-m)$ and $2(2m-n)$. We can interpret these results as generalizations of corresponding theorems (Haroutunian 1996).

3. Canonical integrals

Let us study the canonical integral of submanifold N_2. The dimension of this submanifold is equal to $2(2m-n)$. If $2m-n<n-m$, which is equivalent to the inequality $3m<2n$, then $\dim N_2$ is more than two times less than $\dim E_n^{2n}$. It means that the submanifold N_2 belongs to the category of submanifolds of moderate dimension. The calculation of canonical integral for such submanifold is much more simple than in general case (Haroutunian 2001). Structure equations of submanifold N_2 are (16). We will consider a special case when for arbitrary value of the index $i(=1,\dots,n-m)$

$$\operatorname{rank}(C^i_{\xi\eta})=u, \quad \operatorname{rank}(C_i^{\xi\eta})=v, u+v=2m-n.$$

Let us introduce now indices $\alpha,\beta=n-m+1,\dots,n-m+u$; $a,b=n-m+u+1,\dots,m$. We can assume without any loss of generality that for any value of the index i matrices $(C^i_{\xi\eta})$ and $(C_i^{\xi\eta})$ are presented in the form

$$\left(C^i_{\xi\eta}\right)=\begin{pmatrix}0 & 0\\ 0 & C^i_{ab}\end{pmatrix}, \qquad \left(C_i^{\xi\eta}\right)=\begin{pmatrix}C_i^{\alpha\beta} & 0\\ 0 & 0\end{pmatrix}.$$

Conditions (13) will be held automatically. Using these notations, *i.e.*, replacing indices ξ, η by a, b and α, β, we can rewrite the structure equations of submanifold N_2 in the form

$$\begin{aligned}
d\omega^\alpha &= \omega^\alpha_\beta\wedge\omega^\beta+\omega^\alpha_b\wedge\omega^b,\\
d\omega^a &= \omega^a_b\wedge\omega^b+\omega^a_\alpha\wedge\omega^\alpha,\\
d\omega_\alpha &= -\omega^\beta_\alpha\wedge\omega_\beta-\omega^b_\alpha\wedge\omega_b,\\
d\omega_a &= -\omega^b_a\wedge\omega_b-\omega^\alpha_a\wedge\omega_\alpha,\\
d\omega^\alpha_a &= \omega^\alpha_\beta\wedge\omega^\beta_a+\omega^\alpha_b\wedge\omega^b_a-C_i^{\alpha\beta}C^i_{ab}\omega^b\wedge\omega_\beta,\\
d\omega^a_\alpha &= \omega^a_b\wedge\omega^b_\alpha+\omega^a_\beta\wedge\omega^\beta_\alpha,\\
d\omega^a_b &= \omega^a_c\wedge\omega^c_b+\omega^a_\beta\wedge\omega^\beta_b,\\
d\omega^\alpha_\beta &= \omega^\alpha_\gamma\wedge\omega^\gamma_\beta+\omega^\alpha_b\wedge\omega^b_\beta.
\end{aligned}\tag{17}$$

It is easy to see that the following systems of linear differential equations

$$\begin{array}{ll}
\omega^\alpha_\beta=0, & \alpha,\beta=n-m+1,\dots,n-m+u;\\
\omega^a_b=0, & a,b=n-m+u+1,\dots,m;\\
\omega^a_\alpha=0, & \alpha=n-m+1,\dots,n-m+u;\ \ a=n-m+u+1,\dots,m
\end{array}$$

are totally integrable and then the structure equations (17) come to the form

$$\begin{aligned}
&d\omega^\alpha=\omega^\alpha_b\wedge\omega^b, \quad d\omega^a=0,\\
&d\omega_\alpha=0, \quad d\omega_a=-\omega^\alpha_a\wedge\omega_\alpha,\\
&d\omega^\alpha_a=-C_i^{\alpha\beta}C^i_{ab}\omega^b\wedge\omega_\beta.
\end{aligned}\tag{18}$$

Coefficients of this system satisfy differential equations

$$\begin{aligned} dC^i_{ab} &= C^k_{ab}\omega^i_k + C^i_{\xi\eta\nu}\omega^\nu, \\ dC^{\alpha\beta}_i &= -C^{\xi\eta}_k\omega^k_i + C^{\xi\eta\nu}_i\omega_\nu, \end{aligned} \tag{19}$$

Using structure equations (18) we can check easily that the expression $C^{\alpha\beta}_i C^i_{ab}$ is a total differential, therefore for the sake of simplicity we can consider coefficients $C^i_{\xi\eta\nu}$ and $C^{\xi\eta\nu}_i$ equal to zero: $C^i_{\xi\eta\nu} = 0$, $C^{\xi\eta\nu}_i = 0$. We are searching the integral of the following exterior form

$$\Omega = \lambda\omega^{n-m+1} \wedge \cdots \wedge \omega^{n-m+u} \wedge \omega^{n-m+u+1} \wedge \cdots \wedge \omega^m.$$

If the integral of the form Ω induces the differential geometric structure (18)) on N_2 then the following differential equations hold (Haroutunian 1975)

$$\begin{aligned} d\ln\lambda &= \lambda^\alpha\omega_\alpha + \lambda^a\omega_a + \lambda_\alpha\omega^\alpha + \lambda_a\omega^a, \\ d(\lambda^\alpha\omega_\alpha + \lambda^a\omega_a) &= \omega^\alpha \wedge \omega_\alpha + \omega^a \wedge \omega_a. \end{aligned} \tag{20}$$

Let us define the forms ω^α, ω^a, ω_α, ω_a, ω^α_a by the following way

$$\begin{aligned} &\omega^\alpha = dx^\alpha - \frac{1}{2}C^{\alpha\beta}_k C^k_{ab} x^b y_\beta dx^a, \quad \omega^a = dx^a, \\ &\omega_\alpha = dy_\alpha, \omega_a = dy_a - \frac{1}{2}C^{\alpha\beta}_k C^k_{ab} x^b y_\beta dy_\alpha, \\ &\omega^\alpha_a = \frac{1}{2}C^{\alpha\beta}_k C^k_{ab}(y_\beta dx^b - x^b dy_\beta). \end{aligned}$$

It is easily seen that these forms satisfy the structure equations (18). If we will introduce the following denotations for decompositions of differentials of coefficients λ^α, λ^a, λ_α, λ_a

$$\begin{aligned} d\lambda^\alpha &= \lambda^\alpha_\beta\omega^\beta + \lambda^\alpha_a\omega^a + \lambda^{\alpha\beta}\omega_\beta + \lambda^{\alpha b}\omega_b, \\ d\lambda^a &= \lambda^a_\alpha\omega^\alpha + \lambda^a_b\omega^b + \lambda^{a\beta}\omega_\beta + \lambda^{ab}\omega_b, \\ d\lambda_\alpha &= \lambda_{\alpha\beta}\omega^\beta + \lambda_{\alpha b}\omega^b + b^\beta_\alpha\omega_\beta + b^b_\alpha\omega_b, \\ d\lambda_a &= \lambda_{a\beta}\omega^\beta + \lambda_{ab}\omega^b + b^\beta_a\omega_\beta + b^b_a\omega_b, \end{aligned}$$

substitute them in the second relation and in the result of exterior differentiation of the first relation from (20), we will obtain the following algebraic relations

$$\begin{aligned} &b^\alpha_\beta = \lambda^\alpha_\beta = \delta^\alpha_\beta, \quad b^\alpha_a = \lambda^\alpha_a - \frac{1}{2}\lambda^b C^{\alpha\beta}_i C^i_{ab} y_\beta + \frac{1}{2}\lambda_\beta C^{\alpha\beta}_i C^i_{ab} x^b, \\ &\lambda^{\alpha\beta} = \lambda^{\beta\alpha}, \quad \lambda^{ab} = \lambda^{ba}, \quad b^a_\alpha = \lambda^a_\alpha, \quad b^a_b = \lambda^a_b = \delta^a_b, \quad \lambda_{\alpha b} = \lambda_{b\alpha}, \quad \lambda^a_\alpha = 0, \\ &\lambda_{\alpha\beta} = \lambda_{\beta\alpha}, \quad \lambda_{ab} = \lambda_{ba}, \quad \lambda^\alpha_a = \frac{1}{2}\lambda^b C^{\alpha\beta}_i C^i_{ab} y_\beta. \end{aligned}$$

Therefore the following system of differential equations holds

$$\frac{\partial\lambda^{\alpha}}{\partial x^{\beta}}=\delta^{\alpha}_{\beta},\frac{\partial\lambda^{\alpha}}{\partial x^{a}}=-\frac{1}{2}C^{\alpha\beta}_{i}C^{i}_{ab}x^{b}y_{\beta}+\frac{1}{2}\lambda^{b}C^{\alpha\beta}_{i}C^{i}_{ab}y_{\beta},\frac{\partial\lambda^{\alpha}}{\partial y_{\beta}}=\lambda^{\alpha\beta}-\frac{1}{2}\lambda^{\alpha a}C^{\beta\gamma}_{k}C^{k}_{ab}x^{b}y_{\gamma},$$
$$\frac{\partial\lambda^{\alpha}}{\partial y_{a}}=\lambda^{\alpha a};$$
$$\frac{\partial\lambda^{a}}{\partial x^{\beta}}=0,\ \frac{\partial\lambda^{a}}{\partial x^{a}}=\delta^{a}_{b},\ \frac{\partial\lambda^{a}}{\partial y_{\beta}}=\lambda^{\alpha a}-\frac{1}{2}\lambda^{\alpha b}C^{\alpha\beta}_{i}C^{i}_{bc}x^{c}y_{\beta},\ \frac{\partial\lambda^{a}}{\partial y_{a}}=\lambda^{\alpha b};$$
$$\frac{\partial\lambda_{\alpha}}{\partial x^{\beta}}=\lambda_{\alpha\beta},\ \frac{\partial\lambda_{\alpha}}{\partial x^{a}}=\lambda_{\alpha a}-\frac{1}{2}\lambda_{\alpha\beta}C^{\beta\gamma}_{i}C^{i}_{ab}x^{b}y_{\gamma},\ \frac{\partial\lambda_{\alpha}}{\partial y_{\beta}}=\delta^{\beta}_{\alpha},\ \frac{\partial\lambda_{\alpha}}{\partial y_{a}}=0;$$
$$\frac{\partial\lambda_{a}}{\partial x^{\beta}}=\frac{1}{2}\lambda_{\alpha a},\frac{\partial\lambda_{a}}{\partial x^{a}}=\lambda_{ab}-\frac{1}{2}\lambda_{\alpha a}C^{\alpha\beta}_{i}C^{i}_{bc}x^{c}y_{\beta},\ \frac{\partial\lambda_{a}}{\partial y_{\beta}}=\frac{1}{2}\lambda_{\beta}C^{\alpha\beta}_{i}C^{i}_{ab}x^{b}-\frac{1}{2}C^{\alpha\beta}_{i}C^{i}_{ab}x^{b}y_{\beta},$$
$$\frac{\partial\lambda_{a}}{\partial y_{a}}=\delta^{b}_{a}.$$

It is easy to present the solution of this system in the form

$$\begin{aligned}
\lambda^{\alpha}&=x^{\alpha}+\frac{1}{2}\psi^{b}(y_{n-m+1},\dots,y_{m})C^{\alpha\beta}_{i}C^{i}_{a}y_{\beta}+\psi^{\alpha}(y_{n-m+1},\dots,y_{m}),\\
\lambda^{a}&=x^{a}+\psi^{a}(y_{n-m+1},\dots,y_{m}),\\
\lambda_{\alpha}&=y_{\alpha}+\varphi_{\alpha}(x^{n-m+1},\dots,x^{m}),\\
\lambda_{a}&=y_{a}+\frac{1}{2}\varphi_{\beta}(x^{n-m+1},\dots,x^{m})C^{\beta}_{i}C^{i}_{ab}x^{b}+\varphi_{a}(x^{n-m+1},\dots,x^{m}),
\end{aligned}$$

where C^{i}_{a} and C^{β}_{i} are integrals of forms $C^{i}_{ab}dx^{b}$ and $C^{\alpha\beta}_{i}dy_{\beta}$ respectively. Substituting these expressions in the first equation of the system (20), we obtain

$$\begin{aligned}
d\ln\lambda&=d(x^{\alpha}y_{\alpha}+x^{a}y_{a}-\frac{1}{2}C^{\alpha}_{i}C^{i}_{a}x^{a}y_{\alpha})+[\varphi_{\alpha}(x^{n-m+1},\dots,x^{m})dx^{\alpha}+\\
&+\varphi_{a}(x^{n-m+1},\dots,x^{m})dx^{a}+\psi^{\alpha}(y_{n-m+1},\dots,y_{m})dy_{\alpha}+\psi^{a}(y_{n-m+1},\dots,y_{m})dy_{a}].
\end{aligned}$$

Denoting $\varphi(x)$ and $\psi(y)$ the integrals of forms $\varphi_{\alpha}(x^{n-m+1},\dots,x^{m})dx^{\alpha}+\varphi_{a}(x^{n-m+1},\dots,x^{m})dx^{a}$ and $\psi^{\alpha}(y_{n-m+1},\dots,y_{m})dy_{\alpha}+\psi^{a}(y_{n-m+1},\dots,y_{m})dy_{a}$ respectively, we come to the following result.

Theorem 3.1. *Differential geometric structure* (18)-(19) *is inducing on submanifold* N_2 *by integral of the form*

$$\Omega=P(x)Q(y)\exp\left(x^{\alpha}y_{\alpha}+x^{a}y_{a}-\frac{1}{2}C^{\alpha}_{i}C^{i}_{a}x^{a}y_{\alpha}\right)\omega^{n-m+1}\wedge\dots\wedge\omega^{m},$$

where $P(x)=P(x^{n-m+1},\dots,x^{m})$ *and* $Q(y)=Q(y_{n-m+1},\dots,y_{m})$ *are integrals of positive smooth functions* $\exp\varphi(x)$ *and* $\exp\psi(y)$ *respectively.*

It is easy to note that here canonical integral does not contain directly the components of the curvature tensor of submanifold N_2 but these components are constructed on derivatives of functions C^{i}_{a} and C^{β}_{i}. In the case when $C^{\alpha}_{i}=C^{\alpha\beta}_{i}y_{\beta}$ and $C^{i}_{a}=C^{i}_{ab}x^{b}$ canonical integral of submanifold N_2 is reducing to the integral of the following exterior form of degree $2(2m-n)$:

$$\Theta = P(x)Q(y)\exp\left(x^{\alpha}y_{\alpha} + x^{a}y_{a} - C_{i}^{\alpha\beta}C_{ab}^{i}x^{a}x^{b}y_{\alpha}y_{\beta}\right)\omega^{n-m+1}\wedge\cdots\wedge\omega^{m}.$$

In this special case components of curvature tensor of N_2 enter in canonical integral directly. Taking in account that $\dim N_2 = 2(2m-n)$ and in the case $4m < 3n$ is a submanifold of moderate dimension we can consider this statement as a generalisation of the corresponding result (Haroutunian 2001).

Canonical integral of submanifold N_1 is reducing to the integral of the following exterior form of degree $2(n-m)$:

$$\Omega = P\left(x^{1},\dots,x^{n-m}\right)Q(y_{1},\dots,y_{n-m})\exp\left(x^{i}y_{i} - C_{m+p}^{it}C_{kr}^{m+p}x^{r}x^{k}y_{i}y_{t}\right)\omega^{1}\wedge\dots\wedge\omega^{n-m}$$

and here components of curvature tensor of submanifold N_1 enter in canonical integral directly.

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