

STEADY CONVECTION IN MHD BÉNARD PROBLEM WITH HALL EFFECTS

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ABSTRACT. In this paper we apply some variants of the classical energy method to study the nonlinear Lyapunov stability of the thermodiffusive equilibrium for a viscous thermo-electroconducting fully ionized fluid in a horizontal layer heated from below. The classical L^2 norm, *too weak* to highlight some stabilizing or unstabilizing effects, can be used to *dominate* the nonlinear terms. A *more fine* Lyapunov function is obtained by reformulating the initial perturbation evolution problem, in terms of some *independent scalar fields*. In such a way, if the principle of exchange of stabilities holds, we obtain the coincidence of linear and nonlinear stability bounds.

1. Introduction

The weakness of the *classical energy method* to study the Lyapunov stability for non-stationary equations has been largely investigated. In literature there are several variants aimed at finding an optimal Lyapunov function and, consequently, the coincidence of the linear and nonlinear stability bounds (Prodi 1962; Joseph 1965; Yudovich 1965; Joseph 1966, 1970, 1976a,b; Galdi and Straughan 1985; Georgescu 1985; Rionero and Mulone 1988; Galdi and Padula 1990; Mulone and Rionero 1997; Mulone 2004; Straughan 2004).

Joseph's idea was the parameter's differentiation method, *i.e.*, the introduction of some suitable constants and additional equations, to take into account the effect of some varying-sign terms in the energy relation (Joseph 1966, 1970, 1976a,b).

Rionero and Mulone (1988) considered another variant obtained by splitting the Lyapunov function V into two parts $V = V_0 + bV_1$, where $b > 0$, and V_0, V_1 are chosen according to some guidelines. In this way the coincidence of linear and nonlinear stability bounds is obtained in the radius of attraction of initial data, as large as possible.

The papers of Georgescu and Palese (1996) and Georgescu *et al.* (2000, 2001) are devoted to an extension of the Joseph's parametric differentiation method based on symmetry properties of the involved operator, applied to the stability of the conduction diffusion state of a binary mixture in the presence of Soret and Dufour currents.

Georgescu and Palese (2010, 2011) changed the given problem for a binary mixture in a plane layer, in presence of chemical surface reactions, to obtain an equivalent one with

better symmetry properties and the equality of linear and nonlinear stability bounds was obtained, in the region of stationary convection of the linear instability theory, without any restriction on initial data. This method generalizes the approach of Joseph (1965, 1966, 1970, 1976a,b) in several aspects.

Palese (2005) studied the nonlinear stability of the thermodiffusive equilibrium for the magnetohydrodynamic anisotropic Bénard problem, using Rionero's idea of the energy splitting (Rionero and Mulone 1988), obtaining that the conduction diffusion solution, if linearly stable, is conditionally nonlinearly asymptotically stable in the radius of attraction of initial data.

Palese (2014a,b,c), Labianca and Palese (n.d.), and Palese (n.d.) reformulated the nonlinear stability problem by splitting the vectorial perturbation evolution equations in their *potential and solenoidal* parts. These authors obtained some additional equations where the unknown are *independent scalar fields*, the poloidal and toroidal fields, to preserve the contribution of some symmetric or skewsymmetric terms, and physically, the contribution of some physical effects, such as the rotation (Palese 2014a,b,c, n.d.) and the magnetic field (Labianca and Palese n.d.). In this way the perturbation energy is a linear combination of the the classical L^2 norm with some additional terms due to the reformulation of the nonlinear stability problem. The nonlinear stability problem becomes a linear one, because all nonlinear terms vanish. If the principle of exchange of stabilities holds, we recover the coincidence of the nonlinear stability bound with the linear one obtained by the classical normal mode technique, for the rotating Bénard problem in the hydrodynamic case (Palese 2014a,b,c, n.d.), for the classical magnetohydrodynamic Bénard problem (Labianca and Palese n.d.).

In this paper we study the nonlinear Lyapunov stability of the thermodiffusive equilibrium of a viscous thermoelectroconducting fully ionized fluid in a plane layer heated from below in the Oberbeck-Boussinesq approximation. We consider a second order effect for the magnetic field, the anisotropic electrical conductivity, that give rise to the Hall effect. Therefore we are forced to consider a *more fine* Lyapunov function to exhibit such an effect. To study the effect of the magnetic field, Labianca and Palese (n.d.) considered only an additional equation, the potential part, derived from the evolution equation of the magnetic field. In this case, we must consider both equations, the potential part and the solenoidal one, of the perturbation evolution equation for the magnetic field, whence a *more fine* Lyapunov function from which we can obtain, in the particular case of an isotropic electroconducting fluid, the results of Labianca and Palese (n.d.). Physically, the Lyapunov function employed by Labianca and Palese (n.d.) is not appropriate in the presence of Hall current because the contribution of the density current vector is neglected.

After formulating the initial boundary value problem (Sec. 2), we derive some additional evolution perturbation equations (Sec. 3) in terms of poloidal and toroidal fields, suitable to represent a solenoidal field in a plane layer. We study (Sec. 4) the Lyapunov stability of the thermodiffusive equilibrium, obtaining (Sec. 5) the coincidence of linear and non linear stability bounds, in the range of validity of the principle of exchange of stabilities. Since Hopf bifurcations and the stabilizing effects of the magnetic field have not been investigated, we are aware that the problem needs a further investigation and, hopefully, a new approach. In this respect we mention a very recent paper by Rionero (2017).

2. Mathematical model

Let us consider a Newtonian thermo-electroconducting viscous fluid in a horizontal layer S , bounded by the planes $z = 0$ and $z = 1$. in a Cartesian frame of reference (O, i, j, k) , with k vertical upwards unit vector. The fluid, heated from below, is subject to a vertical temperature gradient, in an external constant magnetic field $H_0 = H_0 k$. In the Oberbeck-Boussinesq approximation the (dimensionless) mathematical model is the following (Joseph 1965; Chandrasekhar 1968; Georgescu and Palese 2009):

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + M^2 \mathbf{H} \cdot \nabla \mathbf{H} - [1 - \mathcal{R}(T - T_0)] \mathbf{k} + \Delta \mathbf{v}, \\ \frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}) + \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta \mathbf{H} + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times (\mathbf{H} \times \nabla \times \mathbf{H}), \\ \mathcal{P}_r \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \Delta T, \\ \nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{H} = 0, \end{array} \right. \quad (1)$$

with the boundary conditions for stress free, thermal conducting and electrically non conducting planes (Chandrasekhar 1968):

$$\left\{ \begin{array}{l} \mathbf{v} \cdot \mathbf{n} = 0, \quad \mathbf{n} \times \mathbf{D} \cdot \mathbf{n} = 0, \quad \mathbf{H} = \mathbf{H}_0, \quad \mathbf{n} \cdot \nabla \times \mathbf{H} = 0, \quad z = 0, 1, \\ T = T^0, \quad z = 0, \\ T = T^1, \quad z = 1, \end{array} \right. \quad (2)$$

where \mathbf{v} , \mathbf{H} , T , P are velocity, magnetic, temperature and pressure fields, respectively. T_0 represents a reference temperature, \mathbf{D} is the strain rate tensor, \mathbf{n} is the outer (unit) normal to the boundary ∂S of S . M^2 , \mathcal{R}^2 , \mathcal{P}_r , \mathcal{P}_m , β_H denote dimensionless Hartmann, Rayleigh, Prandtl, magnetic Prandtl and Hall numbers, respectively. Moreover, Eqs. (1)_{1,3,4,1} are, in the Oberbeck-Boussinesq approximation, the balance equation for momentum, energy and mass, respectively. In addition, for the electromagnetic part, Eq. (1)₂ follows by taking into account the Maxwell equations Galileo invariant and the generalized Ohm’s law.

A layer of fluid heated from below, for a not too large temperature gradient β , is in mechanical equilibrium, called *conduction state* (Koschmieder 1993). When β increases the fluid has a stationary motion, periodic in the x and y directions, *i.e.*, the thermal horizontal convection that, for increasing gradient, becomes non stationary, till the turbulence (Koschmieder 1993). We consider the conduction state

$$(\bar{\mathbf{v}} = \mathbf{0}, \bar{\mathbf{H}} = \mathbf{H}_0, \bar{T} = T^0 - \beta z, \bar{P} = P(z)), \quad (3)$$

in the periodicity cell $V = \mathcal{V} \times [0, 1]$, where $\mathcal{V} = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right]$ and $a^2 = a_x^2 + a_y^2$ is the wave number.

3. Perturbation model

Let us denote with $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{u}$, $\mathbf{H} = \bar{\mathbf{H}} + \mathbf{h}$, $P = \bar{P} + p$, $T = \bar{T} + \vartheta$ the perturbed fields around the conduction state (3). Then the initial boundary value problem governing the evolution of the perturbation $(\mathbf{u}, \mathbf{h}, p, \vartheta)$ of Eq. (3) is the following:

$$\left\{ \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + M^2 \left(\frac{\partial \mathbf{h}}{\partial z} + \mathbf{h} \cdot \nabla \mathbf{h} \right) + \mathcal{R} \vartheta \mathbf{k} + \Delta \mathbf{u}, \\ \frac{\partial \mathbf{h}}{\partial t} &= \frac{\partial \mathbf{u}}{\partial z} + \nabla \times (\mathbf{u} \times \mathbf{h}) + \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta \mathbf{h} + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times \left((\mathbf{h} + \mathbf{k}) \times \nabla \times \mathbf{h} \right), \\ \mathcal{P}_r \left(\frac{\partial \vartheta}{\partial t} + \mathbf{u} \cdot \nabla \vartheta \right) &= \Delta \vartheta + \mathcal{R} w, \\ \nabla \cdot \mathbf{u} &= 0, \quad \nabla \cdot \mathbf{h} = 0, \end{aligned} \right. \quad (4)$$

in the space \mathcal{N} given by:

$$\mathcal{N} = \left\{ (\mathbf{u}(\cdot, t), \mathbf{h}(\cdot, t)) \in [L^2(V)]^2, (w(\cdot, t), h_3(\cdot, t), \vartheta(\cdot, t), \Delta h_3(\cdot, t)) \in [W^{2,2}(V)]^4, \forall t \geq 0 \right. \\ \left. \left| \frac{\partial \mathbf{u}}{\partial z} = \frac{\partial v}{\partial z} = w = 0, \mathbf{h} = \mathbf{0}, \vartheta = 0 \text{ on } (\partial V)_0 \cup (\partial V)_1 \right. \right\}, \quad (5)$$

with

$$(w(\mathbf{x}, \cdot), h_3(\mathbf{x}, \cdot), \vartheta(\mathbf{x}, \cdot)) \in (C^1[0, +\infty])^3, \forall \mathbf{x} \in V; \quad (6)$$

furthermore, $\mathbf{x}(x, y, z)$, $\mathbf{u}(u, v, w)$, $\mathbf{h}(h_1, h_2, h_3)$ and

$$(\partial V)_0 = \left\{ \mathbf{x} \in \mathbb{R}^3 \left| 0 \leq x \leq \frac{2\pi}{a_x}, 0 \leq y \leq \frac{2\pi}{a_y}, z = 0 \right. \right\}, \\ (\partial V)_1 = \left\{ \mathbf{x} \in \mathbb{R}^3 \left| 0 \leq x \leq \frac{2\pi}{a_x}, 0 \leq y \leq \frac{2\pi}{a_y}, z = 1 \right. \right\}.$$

If the mean values of the components of velocity and magnetic fields vanish over \mathcal{V} , that is, if the conditions (Joseph 1976a,b; Schmitt and von Wahl 1992)

$$\int_{\mathcal{V}} u(x, y, z) dx dy = \int_{\mathcal{V}} v(x, y, z) dx dy = \int_{\mathcal{V}} w(x, y, z) dx dy = 0, \quad \forall z \in [0, 1], \quad (7)$$

$$\int_{\mathcal{V}} h_1(x, y, z) dx dy = \int_{\mathcal{V}} h_2(x, y, z) dx dy = \int_{\mathcal{V}} h_3(x, y, z) dx dy = 0, \quad \forall z \in [0, 1], \quad (8)$$

are satisfied, then the velocity \mathbf{u} and the magnetic field \mathbf{h} have the unique decomposition (Joseph 1976a,b; Schmitt and von Wahl 1992):

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \quad \mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2, \quad (9)$$

with

$$\nabla \cdot \mathbf{u}_1 = \nabla \cdot \mathbf{u}_2 = \mathbf{k} \cdot \nabla \times \mathbf{u}_1 = \mathbf{k} \cdot \mathbf{u}_2 = 0, \quad (10)$$

$$\nabla \cdot \mathbf{h}_1 = \nabla \cdot \mathbf{h}_2 = \mathbf{k} \cdot \nabla \times \mathbf{h}_1 = \mathbf{k} \cdot \mathbf{h}_2 = 0, \quad (11)$$

$$\mathbf{u}_1 = \nabla \frac{\partial \chi}{\partial z} - \mathbf{k} \Delta \chi \equiv \nabla \times \nabla \times (\chi \mathbf{k}), \quad \mathbf{u}_2 = \mathbf{k} \times \nabla \psi = -\nabla \times (\mathbf{k} \psi), \quad (12)$$

$$\mathbf{h}_1 = \nabla \frac{\partial \chi'}{\partial z} - \mathbf{k} \Delta \chi' \equiv \nabla \times \nabla \times (\chi' \mathbf{k}), \quad \mathbf{h}_2 = \mathbf{k} \times \nabla \psi' = -\nabla \times (\mathbf{k} \psi'). \quad (13)$$

In Eqs. (12) and (13), χ , χ' and ψ , ψ' , called *poloidal and toroidal potentials*, are doubly periodic functions satisfying (Joseph 1976a,b; Schmitt and von Wahl 1992):

$$\Delta_1 \chi = -\mathbf{k} \cdot \mathbf{u} = -w, \quad \Delta_1 \psi = \mathbf{k} \cdot \nabla \times \mathbf{u}, \quad (14)$$

$$\Delta_1 \chi' = -\mathbf{k} \cdot \mathbf{h} = -h_3, \quad \Delta_1 \psi' = \mathbf{k} \cdot \nabla \times \mathbf{h}, \tag{15}$$

where $\Delta_1 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The boundary conditions (5) written in terms of χ, ψ, χ', ψ' become (Joseph 1976a,b):

$$\chi = \frac{\partial^2 \chi}{\partial z^2} = \frac{\partial \psi}{\partial z} = 0, \quad \chi' = \frac{\partial \chi'}{\partial z} = \Delta_1 \psi' = 0, \quad z = 0, 1. \tag{16}$$

From the third component of Eq. (4)₂, we obtain:

$$\frac{\partial h_3}{\partial t} = \frac{\partial w}{\partial z} + \nabla \cdot [(\mathbf{u} \times \mathbf{h}) \times \mathbf{k}] + \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta h_3 + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \cdot [((\mathbf{h} + \mathbf{k}) \times \nabla \times \mathbf{h}) \times \mathbf{k}]. \tag{17}$$

Taking into account Eqs. (14) and (15), and the embedding of $W^{2,2}(V)$ in $C(\bar{\Omega})$ (Sobolev 1963; Mikhlin 1970), Eq. (17) can be written as follows:

$$\nabla \cdot \left[\frac{\partial}{\partial t} \nabla_1 \chi' - \nabla_1 \frac{\partial \chi}{\partial z} + (\mathbf{u} \times \mathbf{h}) \times \mathbf{k} - \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla_1 \Delta \chi' + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} ((\mathbf{h} + \mathbf{k}) \times \nabla \times \mathbf{h}) \times \mathbf{k} \right] = 0, \tag{18}$$

where $\nabla_1 \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$. It follows that there exists a vector field \mathbf{B} such that

$$\frac{\partial}{\partial t} \nabla_1 \chi' - \nabla_1 \frac{\partial \chi}{\partial z} + (\mathbf{u} \times \mathbf{h}) \times \mathbf{k} - \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla_1 \Delta \chi' + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} ((\mathbf{h} + \mathbf{k}) \times \nabla \times \mathbf{h}) \times \mathbf{k} = \nabla \times \mathbf{B}. \tag{19}$$

By using the identity

$$(\mathbf{k} \times \nabla \times \mathbf{h}) \times \mathbf{k} = -\nabla \times (\Delta \chi' \mathbf{k}) - \nabla_1 \frac{\partial \psi'}{\partial z}, \tag{20}$$

Eq. (19) becomes

$$\begin{aligned} \frac{\partial}{\partial t} \nabla_1 \chi' - \nabla_1 \frac{\partial \chi}{\partial z} + (\mathbf{u} \times \mathbf{h}) \times \mathbf{k} - \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla_1 \Delta \chi' - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times (\Delta \chi' \mathbf{k}) - \\ \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla_1 \frac{\partial \psi'}{\partial z} + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} (\mathbf{h} \times \nabla \times \mathbf{h}) \times \mathbf{k} = \nabla \times \mathbf{B}. \end{aligned} \tag{21}$$

From the Weyl decomposition theorem of $L^2(V)$ (Sobolev 1963; Georgescu 1985), the vector fields $-(\mathbf{u} \times \mathbf{h}) \times \mathbf{k}$ and $\mathbf{h} \times \nabla \times \mathbf{h}$ can be written as

$$-(\mathbf{u} \times \mathbf{h}) \times \mathbf{k} = \nabla U + \nabla \times \mathbf{A}, \tag{22}$$

$$(\mathbf{h} \times \nabla \times \mathbf{h}) \times \mathbf{k} = \nabla U_1 + \nabla \times \mathbf{A}_1, \tag{23}$$

U, U_1 and \mathbf{A}, \mathbf{A}_1 being scalar and vector fields, respectively. If we define the scalar and vector fields

$$\begin{aligned} \Phi = \nabla \cdot (-(\mathbf{u} \times \mathbf{h}) \times \mathbf{k}), \quad \mathbf{W} = \nabla \times (-(\mathbf{u} \times \mathbf{h}) \times \mathbf{k}), \\ \Phi_1 = \nabla \cdot ((\mathbf{h} \times \nabla \times \mathbf{h}) \times \mathbf{k}), \quad \mathbf{W}_1 = \nabla \times ((\mathbf{h} \times \nabla \times \mathbf{h}) \times \mathbf{k}), \end{aligned}$$

the embedding Sobolev theorems of $W^{2,2}(V)$ in the space of continuous functions $C(\bar{V})$ (Sobolev 1963; Mikhlin 1970; Georgescu 1985) allow us to prove the following identities:

$$\nabla \times (-(\mathbf{u} \times \mathbf{h}) \times \mathbf{k}) \equiv \nabla \times (-(\mathbf{u} \times \mathbf{h}) \times \mathbf{k}) - \nabla U,$$

$$\nabla \times \left((\mathbf{h} \times \nabla \times \mathbf{h}) \times \mathbf{k} \right) \equiv \nabla \times \left((\mathbf{h} \times \nabla \times \mathbf{h}) \right) \times \mathbf{k} - \nabla U_1.$$

Let us define

$$\mathbf{B} = -(\mathbf{u} \times \mathbf{h}) \times \mathbf{k} - \nabla U, \quad (24)$$

$$\mathbf{B}_1 = (\mathbf{h} \times \nabla \times \mathbf{h}) \times \mathbf{k} - \nabla U_1, \quad (25)$$

by choosing $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{B}_1 = 0$; the scalar functions U and U_1 are, respectively, (up to a constant) the solutions of the interior Neumann problems (Mikhlin 1970) in the periodicity cell V :

$$\begin{cases} \Delta U = \Phi, \\ \frac{\partial U}{\partial \mathbf{n}} = \Gamma, \end{cases} \quad (26)$$

$$\begin{cases} \Delta U_1 = \Phi_1, \\ \frac{\partial U_1}{\partial \mathbf{n}} = \Gamma_1, \end{cases} \quad (27)$$

where $\frac{\partial U}{\partial \mathbf{n}}$, $\frac{\partial U_1}{\partial \mathbf{n}}$ are the normal derivatives of U and U_1 on the boundary ∂V of the periodicity cell V and $\Gamma = -\mathbf{B} \cdot \mathbf{n}$, $\Gamma_1 = -\mathbf{B}_1 \cdot \mathbf{n}$. The relations

$$\int_V \Phi dv - \int_{\partial V} \Gamma dv = \int_{\partial V} -(\mathbf{u} \times \mathbf{h}) \times \mathbf{k} \cdot \mathbf{n} d\sigma + \int_V \nabla \cdot \mathbf{B} dv = 0,$$

$$\int_V \Phi_1 dv - \int_{\partial V} \Gamma_1 dv = \int_{\partial V} (\mathbf{h} \times \nabla \times \mathbf{h}) \times \mathbf{k} \cdot \mathbf{n} d\sigma + \int_V \nabla \cdot \mathbf{B}_1 dv = 0,$$

which are necessary conditions for the existence of a solution of Eq. (27), are fulfilled, otherwise the interior Neumann problem in the general case has no solution. Taking into account the solenoidality of \mathbf{B} and \mathbf{B}_1 , it follows that two vector fields \mathbf{A} and \mathbf{A}_1 exist such that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$ (i.e., Eqs. (24) and (25)).

From Eqs. (19) and (22), taking into account that the only vector belonging to both the subspaces of potential and solenoidal vectors is zero (Sobolev 1963; Georgescu 1985), it follows that:

$$\frac{\partial}{\partial t} \nabla_1 \chi' = \nabla_1 \frac{\partial \chi}{\partial z} + \nabla U + \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla_1 \Delta \chi' + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla_1 \frac{\partial \psi'}{\partial z} + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla U_1. \quad (28)$$

If we apply the operator $\mathbf{I} - \mathbf{k} \oplus \mathbf{k}$ to Eqs. (4), where \mathbf{I} is the identity operator, we have:

$$\frac{\partial \mathbf{h}^\perp}{\partial t} = \frac{\partial \mathbf{u}^\perp}{\partial z} + [\nabla \times (\mathbf{u} \times \mathbf{h})]^\perp + \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta \mathbf{h}^\perp + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} [\nabla \times ((\mathbf{h} + \mathbf{k}) \times \nabla \times \mathbf{h})]^\perp, \quad (29)$$

where \mathbf{f}^\perp denotes the projection of a vector field \mathbf{f} on the plane normal to \mathbf{k} . From the identities

$$\mathbf{u}^\perp = \nabla_1 \frac{\partial \chi}{\partial z} - \nabla \times (\Psi \mathbf{k}) \quad \mathbf{h}^\perp = \nabla_1 \frac{\partial \chi'}{\partial z} - \nabla \times (\Psi' \mathbf{k}), \quad (30)$$

$$[\nabla \times (\mathbf{k} \times \nabla \times \mathbf{h})]^\perp = -\frac{\partial (\nabla \times \mathbf{h})^\perp}{\partial z}, \quad (31)$$

$$\nabla \times \nabla \times \mathbf{h}^\perp = \nabla \nabla \cdot \mathbf{h}^\perp - \Delta \mathbf{h}^\perp, \quad (32)$$

taking into account the Weyl decomposition theorem:

$$[\nabla \times (\mathbf{u} \times \mathbf{h})]^\perp = \nabla U' + \nabla \times \mathbf{A}', \quad [\nabla \times ((\mathbf{h} + \mathbf{k}) \times \nabla \times \mathbf{h})]^\perp = \nabla U'_1 + \nabla \times \mathbf{A}'_1 \quad (33)$$

where U' , U'_1 , and \mathbf{A}' , \mathbf{A}'_1 are scalar and vector fields, respectively, the solenoidal part of Eq. (29) gives us

$$-\frac{\partial \nabla \times (\Psi' \mathbf{k})}{\partial t} = -\frac{\partial \nabla \times (\Psi \mathbf{k})}{\partial z} + \nabla \times \mathbf{A}' - \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times \nabla \times \mathbf{h}^\perp + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times \mathbf{A}'_1 + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times \nabla \times \left[\frac{\partial^2 \chi'}{\partial z \partial x} \mathbf{j} - \frac{\partial^2 \chi'}{\partial z \partial y} \mathbf{i} \right]. \quad (34)$$

It follows that exists a scalar field F_1 such that

$$-\frac{\partial \Psi' \mathbf{k}}{\partial t} = -\frac{\partial \Psi \mathbf{k}}{\partial z} + \mathbf{A}' - \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times \mathbf{h}^\perp + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \mathbf{A}'_1 + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \nabla \times \left[\frac{\partial^2 \chi'}{\partial z \partial x} \mathbf{j} - \frac{\partial^2 \chi'}{\partial z \partial y} \mathbf{i} \right] + \nabla F_1. \quad (35)$$

We can assume the third components of the vector fields \mathbf{A}' and \mathbf{A}'_1 equal to zero, and $\nabla F_1 = 0$, taking into account that \mathbf{A}' and \mathbf{A}'_1 are defined up to the gradient of a scalar function. From Eq. (35) we have

$$\frac{\partial \Psi'}{\partial t} = \frac{\partial \Psi}{\partial z} + \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta_1 \Psi' - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta_1 \frac{\partial \chi'}{\partial z}. \quad (36)$$

4. Lyapunov stability

If we multiply Eq. (4)₁ by \mathbf{u} , Eq. (4)₂ by $M^2 \mathbf{h}$, and Eq. (4)₃ by $\frac{b}{\mathcal{P}_r} \vartheta$, where b is a scalar parameter, upon adding the resulting equations and integrating over V we obtain the classical energy relation:

$$\frac{dE}{dt} = \mathcal{I} - \mathcal{D}, \quad (37)$$

where

$$E(t) = \frac{1}{2} \left(\|\mathbf{u}\|^2 + M^2 \|\mathbf{h}\|^2 + b \|\vartheta\|^2 \right), \quad (38)$$

$$\mathcal{I} = \mathcal{R} \left(1 + \frac{b}{\mathcal{P}_r} \right) (\vartheta, w), \quad (39)$$

$$\mathcal{D} = \|\nabla \mathbf{u}\|^2 + M^2 \frac{\mathcal{P}_m}{\mathcal{P}_r} \|\nabla \mathbf{h}\|^2 + \frac{b}{\mathcal{P}_r} \|\nabla \vartheta\|^2, \quad (40)$$

and $\|\cdot\|$ and (\cdot, \cdot) are the norm and the scalar product in $L^2(V)$, respectively. Indeed, from the boundary conditions (5) it follows that the non linear terms vanish and the Hall current has no influence on the nonlinear stability of the conduction state.

From Eq. (28) we have:

$$\frac{\partial}{\partial z} \left(U + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} U_1 \right) = 0, \quad (41)$$

and

$$\nabla_1 \left(\frac{\partial \chi'}{\partial t} - \frac{\partial \chi}{\partial z} - \left(U + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} U_1 \right) - \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta \chi' - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \frac{\partial \Psi'}{\partial z} \right) = \mathbf{0}. \quad (42)$$

Therefore the function in parentheses does not depend on the variables x and y . We can introduce a function $\mathcal{F}(z)$ such that

$$\frac{\partial \mathcal{X}'}{\partial t} = \frac{\partial \mathcal{X}}{\partial z} + \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta \mathcal{X}' + (U + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} U_1)(x, y) + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \frac{\partial \Psi'}{\partial z} + \mathcal{F}(z). \quad (43)$$

If we consider the scalar product of $\Delta_1 \mathcal{X}'$ with the second derivative of Eq. (43) with respect to z , owing to the boundary conditions (16) we obtain:

$$\frac{1}{2} \frac{d}{dt} \left\| \nabla_1 \frac{\partial \mathcal{X}'}{\partial z} \right\|^2 = \left(\Delta_1 \mathcal{X}', \frac{\partial^3 \mathcal{X}}{\partial z^3} \right) + \frac{\mathcal{P}_m}{\mathcal{P}_r} \left(\Delta \frac{\partial^2 \mathcal{X}'}{\partial z^2}, \Delta_1 \mathcal{X}' \right) + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \left(\frac{\partial^3 \Psi'}{\partial z^3}, \Delta_1 \mathcal{X}' \right), \quad (44)$$

because, by using Eqs (15)₁ and (8)₃, it follows:

$$\left(\frac{d^2 F}{dz^2}, \Delta_1 \mathcal{X}' \right) = \int_0^1 \frac{d^2 F}{dz^2} dz \int_{\mathcal{V}} \Delta_1 \mathcal{X}'(x, y, z) d\mathcal{V} = 0. \quad (45)$$

The scalar product of $\Delta_1 \Psi'$ with Eq. (36), owing to the boundary conditions (16) yields:

$$-\frac{1}{2} \frac{d}{dt} \left\| \nabla_1 \Psi' \right\|^2 = \left(\Delta_1 \Psi', \frac{\partial \Psi}{\partial z} \right) + \frac{\mathcal{P}_m}{\mathcal{P}_r} (\Delta_1 \Psi', \Delta_1 \Psi') - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \left(\Delta_1 \Psi', \Delta_1 \frac{\partial \mathcal{X}'}{\partial z} \right). \quad (46)$$

The scalar product of $\frac{\partial \Psi'}{\partial z}$ with the derivative of Eq. (36) with respect to z , owing to the boundary conditions (16), gives us:

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial \Psi'}{\partial z} \right\|^2 = \left(\frac{\partial \Psi'}{\partial z}, \frac{\partial^2 \Psi}{\partial z^2} \right) + \frac{\mathcal{P}_m}{\mathcal{P}_r} \left(\Delta_1 \frac{\partial \Psi'}{\partial z}, \frac{\partial \Psi'}{\partial z} \right) + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \left(\frac{\partial \Psi'}{\partial z}, \Delta_1 \frac{\partial^2 \mathcal{X}'}{\partial z^2} \right). \quad (47)$$

From now on, for the sake of simplicity, we will use the subscript notation for the partial differentiation, *i.e.*, $f_x \equiv \frac{\partial f}{\partial x}$ for a function f depending on x .

Now we consider the function

$$E^*(t) = E(t) + d \frac{M^2}{2} \left\| \nabla_1 \mathcal{X}'_z \right\|^2 - \frac{M^2}{2} \left\| \nabla_1 \Psi' \right\|^2 + d \frac{M^2}{2} \left\| \Psi'_z \right\|^2, \quad (48)$$

which is positive definite for $d > 0$; namely, in terms of poloidal and toroidal fields, it can be written as

$$E^*(t) = \frac{1}{2} \left[\left\| \mathcal{X}_{xz} \right\|^2 + \left\| \mathcal{X}_{yz} \right\|^2 + \left\| \Delta_1 \mathcal{X} \right\|^2 + \left\| \Psi_x \right\|^2 + \left\| \Psi_y \right\|^2 + (1+d)M^2 \left(\left\| \mathcal{X}'_{xz} \right\|^2 + \left\| \mathcal{X}'_{yz} \right\|^2 \right) + M^2 \left\| \Delta_1 \mathcal{X}' \right\|^2 + dM^2 \left\| \Psi'_z \right\|^2 + b \left\| \vartheta \right\|^2 \right], \quad (49)$$

where d is a constant to be determined later. From Eqs. (37), (38), (39), (40), (44), (47) and (48) we have:

$$\frac{dE^*}{dt} = \mathcal{I}^* - \mathcal{D}^* = -\mathcal{D}^* \left(1 - \frac{\mathcal{I}^*}{\mathcal{D}^*} \right), \quad (50)$$

where, in terms of poloidal and toroidal fields:

$$\mathcal{I}^* = -\mathcal{R} \left(1 + \frac{b}{\mathcal{P}_r} \right) (\vartheta, \Delta_1 \mathcal{X}) + dM^2 (\mathcal{X}_{zzz}, \Delta_1 \mathcal{X}') + M^2 \frac{\mathcal{P}_m}{\mathcal{P}_r} d \alpha (\Delta_1 \mathcal{X}', \Delta \mathcal{X}'_{zz}) + \quad (51)$$

$$\begin{aligned} & \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} dM^2(\Psi'_{zzz}, \Delta_1 \mathcal{X}') + M^2(\Psi_z, \Delta_1 \Psi') - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} M^2(\Delta_1 \mathcal{X}'_z, \Delta_1 \Psi') \\ & \quad + dM^2(\Psi_{zz}, \Psi'_z) - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} dM^2(\Delta_1 \mathcal{X}'_{zz}, \Psi'_z). \\ \mathcal{D}^* &= \|\nabla \mathcal{X}_{xz}\|^2 + \|\nabla \mathcal{X}_{yz}\|^2 + \|\nabla \Delta_1 \mathcal{X}\|^2 + \|\nabla \Psi_x\|^2 + \|\nabla \Psi_y\|^2 + \frac{b}{\mathcal{P}_r} \|\nabla \vartheta\|^2 + \\ & + M^2 \frac{\mathcal{P}_m}{\mathcal{P}_r} \left\{ [1 + d(1 - \alpha)] \left[\|\nabla \mathcal{X}'_{xz}\|^2 + \|\nabla \mathcal{X}'_{yz}\|^2 \right] + \|\nabla \Delta_1 \mathcal{X}'\|^2 + (1 + d) \|\nabla_1 \Psi'_z\|^2 \right\}, \end{aligned} \tag{52}$$

taking into account that $(\Delta_1 \mathcal{X}', \Delta \mathcal{X}'_{zz}) = - \left[\|\nabla \mathcal{X}'_{xz}\|^2 + \|\nabla \mathcal{X}'_{yz}\|^2 \right]$ and that α is an arbitrary parameter which we shall determine later in order to have \mathcal{D}^* positive definite. Let us define

$$\frac{1}{\sqrt{\mathcal{R}_a^*}} = \max_{\mathcal{K}} \frac{\mathcal{I}^*}{\mathcal{D}^*} \tag{53}$$

in the class \mathcal{K} of the kinematically admissible functions. From Eq. (50) it follows, if $E^*(t)$ is positive definite, that the inequality

$$\sqrt{\mathcal{R}_a^*} \geq 1 \tag{54}$$

is a sufficient condition of the linear and nonlinear Lyapunov stability of the conduction state. In the following section, we shall determine, explicitly, the region of the parameters space where the inequality (54) is satisfied.

5. The nonlinear stability bound

To deduce, in the parameter space, the nonlinear stability bound, we study now the variational problem (53) in terms of the independent fields $(\mathcal{X}, \mathcal{X}', \Psi, \Psi', \vartheta)$ verifying the boundary conditions (5) and (16).

The Euler equations associated to the maximum problem (53) are:

$$\left\{ \begin{aligned} & -\mathcal{R} \left(1 + \frac{b}{\mathcal{P}_r} \right) \Delta_1 \vartheta - M^2 d \Delta_1 \mathcal{X}'_{zzz} + \frac{2}{\sqrt{\mathcal{R}_a^*}} \Delta \Delta \Delta_1 \mathcal{X} = 0, \\ & -\mathcal{R} \left(1 + \frac{b}{\mathcal{P}_r} \right) \Delta_1 \mathcal{X} + \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{b}{\mathcal{P}_r} \Delta \vartheta = 0, \\ & d \Delta_1 \mathcal{X}_{zzz} + 2 \frac{\mathcal{P}_m}{\mathcal{P}_r} d \alpha \Delta \Delta_1 \mathcal{X}'_{zz} + \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta_1 \Delta_1 \Psi'_z + \\ & \quad + \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} \left[\Delta \Delta \Delta_1 \mathcal{X}' + d(1 - \alpha) \Delta \Delta_1 \mathcal{X}'_{zz} \right] = 0, \\ & -M^2 \Delta_1 \Psi'_z + dM^2 \Psi'_{zzz} - \frac{2}{\sqrt{\mathcal{R}_a^*}} \Delta \Delta_1 \Psi = 0, \\ & \Delta_1 \Psi_z - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} \Delta_1 \Delta_1 \mathcal{X}'_z - d \Psi'_{zzz} - \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} (1 + d) \Delta_1 \Delta_1 \Psi'_{zz} = 0. \end{aligned} \right. \tag{55}$$

In the class of normal mode perturbations

$$(\mathcal{X}, \mathcal{X}', \vartheta, \Psi, \Psi') = (X(z), K(z), \Theta(z), \Psi(z), \Psi'(z)) \exp[i(a_x x + a_y y) + \sigma t], \tag{56}$$

with $\sigma \in \mathbb{C}$, the Euler equations (55) become:

$$\left\{ \begin{array}{l} \mathcal{R} \left(1 + \frac{b}{\mathcal{P}_r} \right) a^2 \Theta + M^2 d a^2 D^3 K - \frac{2}{\sqrt{\mathcal{R}_a^*}} (D^2 - a^2)^2 a^2 X = 0, \\ \mathcal{R} \left(1 + \frac{b}{\mathcal{P}_r} \right) a^2 X + \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{b}{\mathcal{P}_r} (D^2 - a^2) \Theta = 0, \\ d a^2 D^3 X + 2 \frac{\mathcal{P}_m}{\mathcal{P}_r} \left[d \left(\alpha + \frac{1}{\sqrt{\mathcal{R}_a^*}} (1 - \alpha) \right) D^2 + \frac{1}{\sqrt{\mathcal{R}_a^*}} (D^2 - a^2) \right] (D^2 - a^2) a^2 K \\ - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} a^4 D \Psi' = 0, \\ M^2 (d D^3 + a^2 D) \Psi' + \frac{2}{\sqrt{\mathcal{R}_a^*}} (D^2 - a^2) a^2 \Psi = 0 \\ - (d D^3 + a^2 D) \Psi' - \beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r} a^4 D K - (1 + d) \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} a^4 D^2 \Psi' = 0, \end{array} \right. \tag{57}$$

where $D = \frac{d}{dz}$. From Eq. (57) we have:

$$\left(\left[\mathcal{R}^2 \left(1 + \frac{b}{\mathcal{P}_r} \right)^2 a^2 \frac{\mathcal{P}_r \sqrt{\mathcal{R}_a^*}}{b} + \frac{2}{\sqrt{\mathcal{R}_a^*}} (D^2 - a^2)^3 \right] \cdot \right. \\ \left. \left\{ 2 \frac{\mathcal{P}_m}{\mathcal{P}_r} \left[-(d D^3 + a^2 D)^2 \frac{\sqrt{\mathcal{R}_a^*} M^2}{2 a^2} + \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} (1 + d) a^4 (D^2 - a^2) D^2 \right] \cdot \right. \right. \\ \left. \left[d \left(\alpha + \frac{1}{\sqrt{\mathcal{R}_a^*}} (1 - \alpha) \right) D^2 + \frac{1}{\sqrt{\mathcal{R}_a^*}} (D^2 - a^2) \right] (D^2 - a^2) + \right. \\ \left. \left. + \beta_H^2 \left(\frac{\mathcal{P}_m}{\mathcal{P}_r} \right)^2 a^6 (D^2 - a^2) D^2 \right\} - \left[d^2 M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} (D^2 - a^2) D^6 \right] \cdot \right. \\ \left. \left[(d D^3 + a^2 D)^2 \frac{\sqrt{\mathcal{R}_a^*} M^2}{2 a^2} - \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} (1 + d) a^4 (D^2 - a^2) D^2 \right] \right) X = 0. \tag{58}$$

Owing to the boundary conditions (16) we can assume

$$X(z) = \sum_{n=0}^{\infty} X_n \sin(n\pi z), \tag{59}$$

where $X_n \equiv (X, \sin(n\pi z))$ are the Fourier coefficients. Upon substituting Eq. (59) into Eq. (58) we obtain:

$$\begin{aligned} \mathcal{R}^2 \frac{\sqrt{\mathcal{R}_a^*}}{2} \left(1 + \frac{b}{\mathcal{P}_r}\right)^2 \frac{\mathcal{P}_r}{b} a^2 &= \frac{2}{\sqrt{\mathcal{R}_a^*}} (n^2 \pi^2 + a^2)^3 + \frac{2}{\sqrt{\mathcal{R}_a^*}} M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2 (n^2 \pi^2 + a^2) \\ &\left(d_*^2 \frac{\mathcal{P}_m \sqrt{\mathcal{R}_a^*}}{\mathcal{P}_r} n^6 \pi^6 \left[(-d_* n^2 \pi^2 + 1)^2 \frac{\sqrt{\mathcal{R}_a^*}}{2} M^2 - \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} (1 + d_* a^2) a^2 \right] \right) \\ &\left\{ 2 \frac{\mathcal{P}_m}{\mathcal{P}_r} n^2 \pi^2 \left[-(-d_* n^2 \pi^2 + 1)^2 \frac{\sqrt{\mathcal{R}_a^*}}{2} M^2 + \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} (1 + d_* a^2) a^2 \right] \right. \\ &\frac{(n^2 \pi^2 + a^2)}{a^4} \left(d_* \left(\alpha + \frac{1}{\sqrt{\mathcal{R}_a^*}} (1 - \alpha) \right) n^2 \pi^2 a^2 + \frac{2}{\sqrt{\mathcal{R}_a^*}} (n^2 \pi^2 + a^2) \right) + \\ &\left. + (\beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r})^2 (n^2 \pi^2 + a^2) n^2 \pi^2 \right\}^{-1} \end{aligned} \tag{60}$$

where $d_* a^2 = d$. If α is given by

$$\alpha = -(X_n \sqrt{\mathcal{R}_a^*} + 1) (\sqrt{\mathcal{R}_a^*} - 1)^{-1}, \tag{61}$$

where

$$X_n = \left(\frac{\sqrt{\mathcal{R}_a^*}}{2} n^4 \pi^4 d_*^2 a^4 + 2(n^2 \pi^2 + a^2)^2 \frac{1}{\sqrt{\mathcal{R}_a^*}} \right) \left(2d_* a^2 n^2 \pi^2 (n^2 \pi^2 + a^2) \right)^{-1}, \tag{62}$$

Eq. (60) becomes

$$\begin{aligned} \mathcal{R}^2 \frac{\sqrt{\mathcal{R}_a^*}}{2} \left(1 + \frac{b}{\mathcal{P}_r}\right)^2 \frac{\mathcal{P}_r}{b} a^2 &= \frac{2}{\sqrt{\mathcal{R}_a^*}} (n^2 \pi^2 + a^2)^3 + \frac{2}{\sqrt{\mathcal{R}_a^*}} M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2 (n^2 \pi^2 + a^2) \\ &\left(d_*^2 \frac{\mathcal{P}_m \sqrt{\mathcal{R}_a^*}}{\mathcal{P}_r} n^6 \pi^6 \left[(-d_* n^2 \pi^2 + 1)^2 \frac{\sqrt{\mathcal{R}_a^*}}{2} M^2 a^2 - \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} (1 + d_* a^2) a^4 \right] \right) \\ &\left\{ d_*^2 \frac{\mathcal{P}_m \sqrt{\mathcal{R}_a^*}}{\mathcal{P}_r} n^6 \pi^6 \left[(-d_* n^2 \pi^2 + 1)^2 \frac{\sqrt{\mathcal{R}_a^*}}{2} M^2 a^2 - \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} (1 + d_* a^2) a^4 \right] \right. \\ &\left. + (\beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r})^2 (n^2 \pi^2 + a^2) n^2 \pi^2 \right\}^{-1}. \end{aligned} \tag{63}$$

If we choose d_* equal to the positive root of the equation

$$\begin{aligned} d_*^2 \frac{\mathcal{P}_m \sqrt{\mathcal{R}_a^*}}{\mathcal{P}_r} n^6 \pi^6 \left[(-d_* n^2 \pi^2 + 1)^2 \frac{\sqrt{\mathcal{R}_a^*}}{2} M^2 a^2 - \frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{\mathcal{P}_m}{\mathcal{P}_r} (1 + d_* a^2) a^4 \right] &= \\ (n^2 \pi^2 + a^2)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2, \end{aligned} \tag{64}$$

Eq. (63) becomes:

$$\mathcal{R}^2 \frac{\sqrt{\mathcal{R}_a^*}}{2} \left(1 + \frac{b}{\mathcal{P}_r}\right)^2 \frac{\mathcal{P}_r}{b} a^2 = \frac{2}{\sqrt{\mathcal{R}_a^*}} (n^2 \pi^2 + a^2)^3 + \frac{2}{\sqrt{\mathcal{R}_a^*}} M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2 (n^2 \pi^2 + a^2) \frac{(n^2 \pi^2 + a^2)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2}{(n^2 \pi^2 + a^2)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2 + (\beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r})^2 (n^2 \pi^2 + a^2) n^2 \pi^2} \tag{65}$$

and, after differentiating the l.h.s. of Eq. (65) with respect to b , we obtain $b = \mathcal{P}_r$. From Eq. (65) we have:

$$\mathcal{R}^2 = \frac{1}{\mathcal{R}_a^*} \frac{(n^2 \pi^2 + a^2)}{a^2} \left(\frac{(n^2 \pi^2 + a^2)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2}{(n^2 \pi^2 + a^2)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2} \frac{(n^2 \pi^2 + a^2)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2}{(n^2 \pi^2 + a^2)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^2 \pi^2 + (\beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r})^2 (n^2 \pi^2 + a^2) n^2 \pi^2} \right). \tag{66}$$

We observe that, if $\sqrt{\mathcal{R}_a^*} \geq 1$, d and α , given by (69) and (70), satisfy

$$1 + d > 0 \quad \wedge \quad 1 + d(1 - \alpha) > 0;$$

whence both $E^*(t)$ and \mathcal{D}^* are definite positive. The minimum of Eq. (66) with respect to $n \in \mathbb{N}$ is attained for $n = 1$; therefore, as a function of $x = \frac{a^2}{\pi^2}$:

$$\mathcal{R}_a^*(M^2, \beta_H, \mathcal{P}_r, \mathcal{P}_m, x) = \frac{1}{\mathcal{R}^2} \frac{(1+x)}{x} \left((1+x)^2 + \frac{M^2}{\pi^2} \frac{\mathcal{P}_r}{\mathcal{P}_m} \frac{(1+x)^2 + \frac{M^2}{\pi^2} \frac{\mathcal{P}_r}{\mathcal{P}_m}}{(1+x)^2 + \frac{M^2}{\pi^2} \frac{\mathcal{P}_r}{\mathcal{P}_m} + (\beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r})^2 (1+x)} \right) \tag{67}$$

If $\beta_H = 0$, from Eq. (67) we derive

$$\mathcal{R}_a^*(M^2, \mathcal{P}_r, \mathcal{P}_m, x) = \frac{\pi^4}{\mathcal{R}^2} \frac{1+x}{x} \left[(1+x)^2 + \frac{M^2}{\pi^2} \frac{\mathcal{P}_r}{\mathcal{P}_m} \right], \tag{68}$$

whenever $d > 0$. Equation (68) represents exactly the critical function of linear instability if the principle of exchange of stabilities holds ($\sigma = 0$) (Maiellaro and Palese 1984).

By using the energy employed by Labianca and Palese (n.d.) and, in particular, if d coincides with the value obtained by differentiating the energy, *i.e.*:

$$d = -\frac{2}{\sqrt{\mathcal{R}_a^*}} \frac{(n^2 \pi^2 + a^2)}{\alpha + \frac{1}{\sqrt{\mathcal{R}_a^*}} (1 - \alpha)}, \quad \forall n \in \mathbb{N}, \tag{69}$$

we obtain

$$\alpha = -\frac{\sqrt{\mathcal{R}_a^* + 1}}{\sqrt{\mathcal{R}_a^* - 1}}, \tag{70}$$

as found by Labianca and Palese (n.d.). Hence, we have proved the following theorem:

Theorem 1. *If the principle of exchange of stabilities holds, the inequality*

$$1 \leq \mathcal{R}_a^*,$$

with \mathcal{R}_a^ given by (68), is a sufficient condition of linear and non linear Lyapunov stability, that is, the linear and non linear stability bounds coincide if instability occurs as stationary convection.*

Indeed

$$\mathcal{R}_a^* \geq 1 \iff \mathcal{R}^2 \leq \frac{(1+x)}{x} \left((1+x)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^4 \pi^4 (1+x) \cdot \frac{(1+x)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^4 \pi^4}{(1+x)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^4 \pi^4 + (\beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r})^2 (1+x) n^2 \pi^2} \right) \tag{71}$$

where

$$\frac{(1+x)}{x} \left((1+x)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^4 \pi^4 (1+x) \cdot \frac{(1+x)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^4 \pi^4}{(1+x)^2 + M^2 \frac{\mathcal{P}_r}{\mathcal{P}_m} n^4 \pi^4 + (\beta_H \frac{\mathcal{P}_m}{\mathcal{P}_r})^2 (1+x) n^2 \pi^2} \right) \tag{72}$$

is exactly the Rayleigh function of the linear instability theory, if the principle of exchange of stabilities holds (Maiellaro and Palese 1984). We observe that, owing to Eq. (69), d is a function of n ; hence, it follows that Eq. (48) really defines a sequence of Lyapunov functions. However, *a posteriori* we can consider only the Lyapunov function of that sequence corresponding to $n = 1$, obtaining the same result.

Conclusions

In this paper we have dealt with the point of loss of nonlinear stability of the conduction state of the anisotropic magnetic Bénard problem, when the thermal convection appears. The classical L^2 norm of the perturbation fields, as is well known, is *too weak* to highlight the stabilizing or unstabilizing effects of some varying-sign terms in the perturbation equations. For this reason in the literature there are many variants based on the introduction of additional equations and suitable constants, on the splitting of the Lyapunov functional, on symmetry properties of the operators involved in the energy relation.

In the book by Georgescu and Palese (2009), in the chapter about variants of the energy method – Section 4.2.1. on symmetry and optimality condition – referring to the perturbation evolution equations, we can read that *the central idea of the present variant is to change just these differential equations, and not those integral deduced from them. Moreover this simplification must be done in such a way that the energy relation assumes the simplest*

form. This is the *the central idea* of this paper where, to study the nonlinear Lyapunov stability of the conduction state, we reformulate the problem governing the perturbation evolution in terms of some *essential variables*, by splitting the perturbation equations in *the solenoidal and potential parts*.

The problem has been formulated in terms of *the essential variables* poloidal and toroidal fields, suitable to represent the solenoidal fields in a plane layer. The resulting perturbation energy is a linear combination of some terms due to the additional equations with the classical L^2 norm which can be used, in any case, to dominate the nonlinear terms. By solving the Euler equation associated to the variational problem arising from the deduction of the non linear stability bound and, later, by differentiating with respect to some parameters involved in the Lyapunov function, we have obtained a non linear stability bound that coincides with the Rayleigh function of the linear instability theory, in the subspace of the parameter space where instability occurs as stationary convection.

In the present paper the fluid is electroanisotropic; whence, we have a second-order effect in the evolution equation of the magnetic field. Consequently, we must consider a *more fine* Lyapunov function if compared with the Lyapunov function introduced by Labianca and Palese (n.d.) to study the stability of the thermodiffusive equilibrium in the isotropic case. Namely, the last one can be obtained from the Lyapunov function of the anisotropic case, as a particular case but, on the contrary, the function used in the isotropic case is not suitable in the anisotropic one. In such a way we can also preserve the effect of some skew symmetric terms in the energy relation, *e.g.*, the Coriolis term in the rotating Bénard problem (Palese 2014b), where the absence of subcritical instabilities is shown, in the region of the parameter space in which the principle of exchange of stabilities holds. In this way we were able to avoid the introduction of more complicated Lyapunov functions that can provide the absence of subcritical instabilities, but only under some restrictions on the initial data.

Since Hopf bifurcations and the stabilizing effects of the magnetic field have not been investigated, we are aware that the problem needs a further investigation and, hopefully, a new approach. In this respect we mention a very recent paper of Rionero (2017).

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