

RELATIVISTIC PHENOMENOLOGICAL EQUATIONS AND TRANSFORMATION LAWS OF RELATIVE COEFFICIENTS

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ABSTRACT. The aim of this paper is to derive the phenomenological equations in the context of special relativistic non-equilibrium thermodynamics with internal variables. In particular, after introducing some results developed in our previous paper, by means of classical non-equilibrium thermodynamic procedure and under suitable assumptions on the entropy density production, the phenomenological equations and transformation laws of phenomenological coefficients are derived. Finally, some symmetries of aforementioned coefficients are obtained.

1. Introduction

It is well known that in non-equilibrium thermodynamics models the classical variables are not sufficient to describe some internal phenomena which occur inside the medium (Kluitenberg 1962, 1968; De Groot and Mazur 1984). Whenever a continuum is composed as two or more “internal phases”, the properties are in general known not as single deterministic components but rather as suitable averages of microlocal phenomena, a convenient modelization technique may pass through the use of so-called “*internal variables*” (Maugin and Muschik 1994a,b; Ván 2003). These represent phenomena occurring at the microscopic or at the mesoscopic level that cannot be controlled in their full detail but only in average over small portions of the continuum. These microlocal phenomena (which are typical of non-equilibrium and dissipation processes) are thus summarized into a set of extra variables, the so-called “*internal variables*” (see Kluitenberg 1968; Verhas 1983; Maugin and Muschik 1994a,b; Francaviglia *et al.* 2006), which depend on the model chosen and obey a number of phenomenological equations. Usually, the internal variables enter directly in the state functions and the constitutive laws, but it is sometimes convenient to adopt more general models in which these averages are non-uniform in space so that the internal variables appear together with their space gradients.

In the framework of thermodynamics of continua with internal variables one is often able to better understand the overall behaviour of the continuum in situations far from equilibrium and in the presence of dissipative phenomena (Francaviglia *et al.* 2008, 2009, 2014; Oliveri *et al.* 2016). Two different types of internal variables are distinguished:

one is the internal degrees of freedom and the other one is internal variables of state that, differently from the first one, has not inertia and does not produce external work (Ván *et al.* 2008; Berezovski *et al.* 2009). In this paper we will follow the terminology introduced in the Kluitenberg's theory that considers the internal variables such that its substantial time derivatives does not occur in the first law of thermodynamics. Sometimes internal variables or internal degrees of freedom are introduced without thermodynamic basis under different denomination.

In the last years, in the Kluitenberg's framework of non equilibrium thermodynamic with internal variable, Ciancio *et al.* (2009), Farsaci *et al.* (2010), and Farsaci and Rogolino (2012) have further obtained some correlations between fundamental entities of the theory and directly experimental measurable functions. This leads to the experimental evaluation of some predicted results and some physical phenomena described by phenomenological and state coefficients appearing in the theory. Let us observe that the strain tensors related to irreversible processes respectively, $\gamma_{ik}^{(0)}$ and $\gamma_{ik}^{(i)}$, are quantity experimentally measurable even if, in a particular simple case, it can be shown that it is possible to obtain experimental measurements in relativistic case. Moreover, in a previous paper of us (Farsaci and Rogolino 2016) the extension of the Kluitenberg's theory to the special relativistic case for thermo-mechanical model with internal variables was presented. Our approach is based entirely on the principle of relativity and on the transformation law of the strain tensor as obtained by us (Farsaci and Rogolino 2016). In particular, following the classical non equilibrium thermodynamics approach, we introduce a relativistic entropy function, depending on the corresponding relativistic variables (Von Borzeszkowski and Chrobok 2008). In more details, as in the classical scheme, it is assumed that the relativistic entropy density in an inertial reference frame is dependent on the relativistic internal energy density, the relativistic total strain tensor and some relativistic internal tensorial variables. After having introduced the relativistic equilibrium stress tensor, the relativistic viscous stress tensor and the relativistic memory stress tensor, the expression for the entropy density production is obtained.

The purpose of this paper is to derive the relativistic phenomenological equations and the transformation laws of the relativistic phenomenological coefficients for the model introduced by Farsaci and Rogolino (2016), under the assumption that the entropy density depends on the the density of energy (Rindler 1971), the total relativistic strain tensor, and the relativistic internal variables $\gamma_{ik}^{(i)}$.

2. Relativistic thermodynamic model

Let us suppose the medium in motion with respect to an inertial frame of reference Σ , we introduce the Galilean coordinates defined in terms of spatial and time variables (x, y, z, t) as follows:

$$x_0 = ct, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad (1)$$

where c is the scalar velocity of light in the vacuum.

In four dimensional space we will use the following metric:

$$ds^2 = dx_0^2 - dx_i^2 \quad (2)$$

in which Einstein convection is used. It is well known that the coordinate transformation, relating two inertial frames Σ and Σ' in relative general configuration, are the Lorentz transformation which can be written:

$$\begin{cases} x_i = x'_i + \frac{\alpha v_i}{c} x'_0 + (\alpha - 1) \frac{v_i v_k}{v^2} x'_k \\ x_0 = \alpha \left(x'_0 + \frac{v_i x'_i}{c} \right) \end{cases} \quad (3)$$

where $\mathbf{v} \equiv (v_1, v_2, v_3)$ is the uniform velocity of Σ with respect to Σ' and

$$\alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (4)$$

Now we can introduce our second system Σ_0 , so called “*proper reference*”, also moving respect to Σ with the velocity \mathbf{v} , so that the material at the point and time of interest will be at rest in system Σ_0 .

It is well known that the study of the motion of such a medium implies to consider different forms of density of energy flow. Hence, the total density of energy flow L_i is (Landau and Lifshitz 1987):

$$L_i = E_i + \rho c^2 v_i + v^j \phi_{ji}; \quad (5)$$

according to the Einstein relation between mass and energy, it is possible to associate to this quantity the following total momentum density

$$H_i = \frac{L_i}{c^2} = \rho v_i + \frac{E_i}{c^2} + \frac{v^j \phi_{ji}}{c^2}. \quad (6)$$

wherein E_i is the vector representing density of energy flow of not mechanical nature (as the heat), $\rho c^2 v_i$ is the density of energy flow due only to the motion of the medium where ρ is the mass density, the quantity $v^j \phi_{ji}$ is the density of energy flow due to the action of the forces of stress flowing in the positive x_i direction, with ϕ_{ji} the relativistic (no symmetric) stress tensor.

Now we are able to introduce the so called *energy momentum tensor* $\chi_{\alpha\beta}$ (Møller 1952; Landau and Lifshitz 1975; Levi Civita 1982; Landau and Lifshitz 1987; Rindler 2006):

$$\chi_{\alpha\beta} = \begin{cases} \chi_{ik} = H_i v_k + \phi_{ik} \\ \chi_{i0} = T_{0i} = c H_i \\ \chi_{00} = \rho c^2 \end{cases} \quad (7)$$

in which latin index assumes the values 1, 2, 3 and greek index assumes the values 0, 1, 2, 3.

If ρF_i is the unitary volume force, the introduction of a four vector W_α defines as

$$W_\alpha \equiv \left(\frac{\rho v_i F_i}{c}, \rho F_i \right) \quad (8)$$

allows us to write the tensorial equation

$$\frac{\partial \chi_{\alpha\beta}}{\partial x^\beta} = W_\alpha \quad (9)$$

in which the “temporal component” (“0”) index) represents the balance equation for energy density and the spatial components (“1,2,3” index) express the balance equation for momentum density.

By taking into account the transformation law (Landau and Lifshitz 1975)

$$\chi_{\beta\delta} = \frac{\partial x^\mu}{\partial x^{\delta'}} \frac{\partial x^{\nu'}}{\partial x^{\beta'}} \chi_{\mu\nu}^* \quad (10)$$

where $\chi_{\mu\nu}^*$ is the energy momentum tensor in a rest frame of reference, the following relation (Jorné and Jorné 1983; Hayward 1998; Przanowski and Tosiek 2011) yields:

$$\chi_{00} = \alpha^2 \rho_0 c^2 + 2 \frac{\alpha^2 v^i}{c^2} E_i + \frac{\alpha^2}{c^2} v^i v^k \phi_{ik}^* = \rho c^2, \quad (11)$$

where ρ_0 and ϕ_{ik}^* are respectively the mass density and the symmetric Cauchy stress tensor in a rest frame of reference. In the following, a function evaluated in the rest frame of reference will be denoted with the symbol “*” as upper index.

As it is well known, the basic function of the thermodynamics of irreversible processes is the entropy production or the dissipation function which represents the deterioration of free energy due dissipative flows. The rate of entropy production equals the sum of the products of irreversible flows J^i and their forces X_i .

Let we assume, in analogy to nonrelativistic physics, that there always exists a four vector, the entropy vector S_μ , whose spatial part S_i represents the entropy flux and the fourth component characterizes the entropy density. The following entropy inequality holds:

$$\frac{dS_\mu}{dx_\mu} \geq 0 \quad (\mu = 0, 1, 2, 3) \quad (12)$$

that generalizes the Clausius-Duhem inequality to relativity in four-dimensional form which will be valid for any space time coordinate of the type (1). On the other hand, remembering that the entropy is a Lorentz invariant, i. e. $S = S^*$, and the volume $V = \alpha V^*$, from the definition of the entropy density $\phi = \frac{dS}{dV}$, the relation

$$\phi = \alpha \phi^* \quad (13)$$

is obtained (Jorné and Jorné 1983; Farsaci and Rogolino 2016). As a consequence, the following identification for the relativistic temperature in a generic inertial frame Σ (Farsaci and Rogolino 2016)

$$T = \alpha T^* \quad (14)$$

holds, as obtained by Ott (1963).

This transformation for temperature is rather controversial; Ott proposed it as the appropriate transformation, suggesting that a moving body appears relatively hot. The most notorious, the so-called Planck-Ott controversy, generated intensive discussion over the correctness of the transformation of the temperature (Costa and Matsas 1995; Przanowski and Tosiek 2011).

The idea that the temperature might be invariant was also suggested, for example by Landsberg and Matsas (1996). This problem seems to be mathematically simple and therefore there have been many publications about it and is not yet quite dead. Moreover, in other papers (Costa and Matsas 1995; Landsberg and Matsas 1996) the authors claim that there does not exist any universal relativistic transformation of temperature.

Nevertheless, in our opinion, there exist preferred definition of temperature and the most natural transformation formula for absolute temperature in relativistic thermodynamics indeed is given by (14) whose advantage is that it deals relativistic thermodynamics as a direct extension of the nonrelativistic one.

Let us suppose that the entropy density ϕ in an arbitrary inertial frame of reference Σ , depends on the density of energy χ_{00} (Rindler 1971), on the total relativistic strain tensor γ_{ik} , and on the relativistic internal variables $\gamma_{ik}^{(i)}$,

$$\phi = \phi(\chi_{00}\gamma_{ik}, \gamma_{ik}^{(i)}) \quad (15)$$

and in Σ_0 :

$$\phi^* = \phi^*(\chi_{00}^*, \gamma_{ik}^*, \gamma_{ik}^{*(i)}). \quad (16)$$

To satisfy the relativity principle, the mathematical statement of each law must then transform into itself under Lorentz transformation and the constitutive functions are be invariant under a change of frame of type (1) (Rindler 2006). It should be remarked that the set of Lorentz transformations between all systems in unaccelerated uniform motion form a group, such that the combined result of successive transformations is equivalent to a single transformation from the original to the final system of coordinates. The Lorentz group, however, has physical meaning only as a transformation group between inertial frames (Kempers 1989). Moreover it is made the assumption that the constitutive equations of the mechanics of a point mass, i.e. the constitutive equations for momentum and energy of a point mass, should also be Lorentz-invariant. Various attempts (Kempers 1989) have been made in the recent paste to formulate a relativistic objectivity principle on physical grounds especially in general relativity (Cheng 2010).

Similar to the classic case, by comparing the differential of the entropy density and the generalized Gibbs relation we obtain the equilibrium relativistic stress (Farsaci and Rogolino 2016),

$$\phi_{rs}^{(eq)} = T \frac{\partial \phi}{\partial \gamma_{rs}} = \alpha^2 \phi_{ik}^{*(eq)} \frac{\partial \gamma_{ik}^*}{\partial \gamma_{rs}} \quad (17)$$

where $\phi_{rs}^{*(eq)}$ is the equilibrium stress tensor in Σ_0 ; and the relativistic affinity stress tensor.

$$\phi_{rs}^{(i)} = T \frac{\partial \phi}{\partial \gamma_{rs}^{(i)}} = \alpha^2 \phi_{ik}^{*(i)} \frac{\partial \gamma_{ik}^*}{\partial \gamma_{rs}^{(i)}}. \quad (18)$$

Let us remark that the quantities $\phi_{ik}^{(eq)}$, $\phi_{ik}^{(i)}$ have the same transformation law because of the intrinsic meaning of the stress as ratio between the force and the surface (Landau and Lifshitz 1987) and by indicating with S_{ik} a generic stress tensor (which can be $\phi_{ik}^{(i)}$, ϕ_{ik} , $\phi_{ik}^{(eq)}$) the transformation laws of the stress tensor are well known, i.e. the following

relations yield (Møller 1952; Tolman 1987):

$$\begin{aligned} S_{11} &= S_{11}^* & S_{12} &= \alpha S_{12}^* & S_{13} &= \alpha S_{13}^*, \\ S_{21} &= \frac{1}{\alpha} S_{21}^* & S_{22} &= S_{22}^* & S_{23} &= S_{23}^*, \\ S_{31} &= \frac{1}{\alpha} S_{31}^* & S_{32} &= S_{32}^* & S_{33} &= S_{33}^*. \end{aligned} \quad (19)$$

From these relations follow two properties:

- relativistic stress tensor is not symmetric,
- every component in Σ will depend only on component with same index in Σ_0 .

The transformation laws of stress tensor allows also to get the following transformation law for relativistic strain tensor (see Farsaci and Rogolino 2016):

$$\begin{aligned} \gamma_{11} &= \alpha^2 \gamma_{11}^* & \gamma_{12} &= \alpha \gamma_{12}^* & \gamma_{13} &= \alpha \gamma_{13}^*, \\ \gamma_{21} &= \alpha^3 \gamma_{21}^* & \gamma_{22} &= \alpha^2 \gamma_{22}^* & \gamma_{23} &= \alpha^2 \gamma_{23}^*, \\ \gamma_{31} &= \alpha^3 \gamma_{31}^* & \gamma_{32} &= \alpha^2 \gamma_{32}^* & \gamma_{33} &= \alpha^2 \gamma_{33}^*. \end{aligned}$$

Let us observe that the non-symmetry of the stress tensor implies that the strain tensor is not symmetric.

3. Relativistic phenomenological equations and transformation laws of the transport coefficients

Following way drawn in Classical Irreversible Thermodynamics and, in particular, according to the classical Onsager approach (Gyarmati 1970; Verhas 1983; De Groot and Mazur 1984), in this section the aim is to analyze the expression of relativistic entropy density production, in order to get the relativistic phenomenological equation and the transformation laws of the coefficients involved in them. In particular, as it is usual to proceed in classical irreversible thermodynamics, the evolution of entropy density is given by means of a generalized Gibbs relation; after considering the entropy balance equation, we obtained the following expression for the relativistic entropy density production (Farsaci and Rogolino 2016):

$$\Omega^{(\sigma)} = \frac{1}{T} \left[-T^{-1} \chi_{0i} \frac{\partial T}{\partial x_\gamma} + \Phi_{ik}^{(eq)} \frac{\partial \gamma_{ik}}{\partial x_0} + \phi_{ik}^{(i)} \frac{\partial \gamma_{ik}^{(i)}}{\partial x_0} \right]. \quad (20)$$

According to the Onsager's procedure (Verhas 1983; De Groot and Mazur 1984), we may regard it as a bilinear product of suitable *relativistic generalized thermodynamical forces* $\mathbf{X}^{(\alpha)}$ and their and their *relativistic conjugated thermodynamic fluxes* $\mathbf{J}^{(\alpha)}$ (Pávon *et al.* 1982). In such a case, a sufficient condition to ensure that such a production is always non-negative is to assume that each $\mathbf{J}^{(\alpha)}$ is given by a linear combination of all the vectors $\mathbf{X}^{(\alpha)}$ (Verhas 1983), viz.:

$$\mathbf{J}^{(\alpha)} = \sum_{\beta} L_{\alpha\beta} \mathbf{X}^{(\beta)}, \quad (21)$$

wherein $L_{\alpha\beta}$ are *relativistic phenomenological transport coefficients*. In particular, we can regard the quantities

$$\mathbf{X}^{(j)} = \left\{ -T^{-1} \frac{\partial T}{\partial x_\gamma}, \frac{\partial \gamma_{rs}}{\partial x_0}, \Phi_{rs}^{(i)} \right\}, \quad j = 1, 2, 3$$

as relativistic thermodynamical forces and the quantities

$$\mathbf{J}^{(i)} = \left\{ \chi_{0i}, \Phi_{ik}^{(eq)}, \frac{\partial \gamma_{ik}^{(i)}}{\partial x_0} \right\}, \quad i = 1, 2$$

as their relativistic conjugated thermodynamic fluxes. Therefore, the following relativistic phenomenological equations hold:

$$\chi_{0i} = -T^{-1} L_{i\gamma}^{(11)} \frac{\partial T}{\partial x_\gamma} + L_{irs}^{(12)} \frac{\partial \gamma_{rs}}{\partial x_0} + L_{irs}^{(13)} \Phi_{rs}^{(i)} \quad (22a)$$

$$\Phi_{ik}^{(eq)} = -T^{-1} L_{ik\gamma}^{(21)} \frac{\partial T}{\partial x_\gamma} + L_{ikrs}^{(22)} \frac{\partial \gamma_{rs}}{\partial x_0} + L_{ikrs}^{(23)} \Phi_{rs}^{(i)} \quad (22b)$$

$$\frac{\partial \gamma_{ik}^{(i)}}{\partial x_0} = -T^{-1} L_{ik\gamma}^{(31)} \frac{\partial T}{\partial x_\gamma} + L_{ikrs}^{(32)} \frac{\partial \gamma_{rs}}{\partial x_0} + L_{ikrs}^{(33)} \Phi_{rs}^{(i)} \quad (22c)$$

Moreover, under the hypothesis that $T = cost$, i.e. $\frac{\partial T}{\partial x_\gamma} = 0$, the first term in the expression (20) vanishes and the relativistic phenomenological equations reduce:

$$\Phi_{ik}^{(eq)} = L_{ikrs}^{(11)} \frac{\partial \gamma_{rs}}{\partial x_0} + L_{ikrs}^{(12)} \Phi_{rs}^{(i)} \quad (23a)$$

$$\frac{\partial \gamma_{ik}^{(i)}}{\partial x_0} = L_{ikrs}^{(21)} \frac{\partial \gamma_{rs}}{\partial x_0} + L_{ikrs}^{(22)} \Phi_{rs}^{(i)} \quad (23b)$$

As consequence of Lorentz transformation, the equality $\frac{d\phi}{dt} = \frac{d\phi^*}{dt^*}$ holds, thus the following identity can be written:

$$\frac{1}{T} \left[\Phi_{ik}^{(eq)} \frac{\partial \gamma_{ik}}{\partial x_0} + \Phi_{ik}^{(i)} \frac{\partial \gamma_{ik}^{(i)}}{\partial x_0} \right] = \frac{1}{T_0} \left[\Phi_{ik}^{(eq)*} \frac{\partial \gamma_{ik}^*}{\partial x_0^*} + \Phi_{ik}^{*(i)} \frac{\partial \gamma_{ik}^{*(i)}}{\partial x_0^*} \right] \quad (24)$$

and by substituting the relation (14) it becomes:

$$\Phi_{ik}^{(eq)} \frac{\partial \gamma_{ik}}{\partial x_0} + \Phi_{ik}^{(i)} \frac{\partial \gamma_{ik}^{(i)}}{\partial x_0} = \alpha \left(\Phi_{ik}^{*(eq)} \frac{\partial \gamma_{ik}^*}{\partial x_0^*} + \Phi_{ik}^{*(i)} \frac{\partial \gamma_{ik}^{*(i)}}{\partial x_0^*} \right) \quad (25)$$

From relation (25), in absence of relativistic internal variables, one obtains:

$$\Phi_{ik}^{(eq)} \frac{\partial \gamma_{ik}}{\partial x_0} = \alpha \left(\Phi_{ik}^{*(eq)} \frac{\partial \gamma_{ik}^*}{\partial x_0^*} \right) \quad (26)$$

Extending the summation over the indexes i and k from the relation (26) follows:

$$\begin{aligned} & \Phi_{11}^{*(eq)} \frac{\partial \gamma_{11}}{\partial x_0} + \alpha \Phi_{12}^{*(eq)} \frac{\partial \gamma_{12}}{\partial x_0} + \alpha \Phi_{13}^{*(eq)} \frac{\partial \gamma_{13}}{\partial x_0} + \frac{1}{\alpha} \Phi_{21}^{*(eq)} \frac{\partial \gamma_{21}}{\partial x_0} \\ & + \Phi_{22}^{*(eq)} \frac{\partial \gamma_{22}}{\partial x_0} + \Phi_{23}^{*(eq)} \frac{\partial \gamma_{23}}{\partial x_0} + \frac{1}{\alpha} \Phi_{31}^{*(eq)} \frac{\partial \gamma_{31}}{\partial x_0} + \Phi_{32}^{*(eq)} \frac{\partial \gamma_{32}}{\partial x_0} \\ & + \Phi_{33}^{*(eq)} \frac{\partial \gamma_{33}}{\partial x_0} = \alpha \Phi_{ik}^{*(eq)} \frac{\partial \gamma_{ik}^*}{\partial x_0^*} \end{aligned} \quad (27)$$

Thus, by analyzing term by term of the equation (27), the following identities can be emphasized:

$$\begin{aligned} \frac{\partial \gamma_{ik}}{\partial x_0} &= \alpha \frac{\partial \gamma_{ik}^*}{\partial x_0^*} && \text{if } i = k; && \text{if } i \neq k \text{ with } i, k = 2, 3; \\ \frac{\partial \gamma_{ik}}{\partial x_0} &= \frac{\partial \gamma_{ik}^*}{\partial x_0^*} && \text{if } i = 1, k = 2, 3; \\ \frac{\partial \gamma_{ik}}{\partial x_0} &= \alpha^2 \frac{\partial \gamma_{ik}^*}{\partial x_0^*} && \text{if } i = 2, 3, k = 1 \end{aligned} \quad (28)$$

These relations represent the transformation law of $\frac{\partial \gamma_{ik}}{\partial x_0}$, in a similar way, the same laws are also valid for the internal variables $\frac{\partial \gamma_{ik}^{(i)}}{\partial x_0}$.

By considering the equations (23a) and (23b) in a rest frame of reference, Σ_0 , the expressions (28), allow to get the transformation law for the coefficients L_{ikrs} under the assumptions the equations (23a) and (23b) are valid in reference frame Σ . Of course, the non-symmetry of stress and strain tensors does not allow us to use the classical Onsager's symmetry relations; so we have to consider all the components of phenomenological tensors to investigate new symmetry relations.

For the sake of simplicity, it will be indicated the procedure in order to obtain the transformation law in the case of one component of the phenomenological relation (23a), e.g. $i = 1, k = 2$, thus it is need to repeat the same reasoning for all components of the equations (23a) and (23b).

$$\begin{aligned} \Phi_{12}^{(eq)} &= L_{12rs}^{(11)} \frac{\partial \gamma_{rs}}{\partial x_0} + L_{12rs}^{(12)} \Phi_{rs}^{(i)} = L_{1211}^{(11)} \frac{\partial \gamma_{11}}{\partial x_0} + L_{1212}^{(11)} \frac{\partial \gamma_{12}}{\partial x_0} + L_{1213}^{(11)} \frac{\partial \gamma_{13}}{\partial x_0} + \\ & L_{1221}^{(11)} \frac{\partial \gamma_{21}}{\partial x_0} + L_{1222}^{(11)} \frac{\partial \gamma_{22}}{\partial x_0} + L_{1223}^{(11)} \frac{\partial \gamma_{23}}{\partial x_0} + L_{1231}^{(11)} \frac{\partial \gamma_{31}}{\partial x_0} + L_{1232}^{(11)} \frac{\partial \gamma_{32}}{\partial x_0} + \\ & L_{1233}^{(11)} \frac{\partial \gamma_{33}}{\partial x_0} + \sum_{r,s=1,\dots,3} L_{12rs}^{(12)} \Phi_{rs}^{(i)} \end{aligned} \quad (29)$$

In any addendum replacing suitable component of the relations (19) and (28), assuming that every other component in (29) different from the one that is being considered is zero, i.e.

$\frac{\partial \gamma_{rs}}{\partial x_0} = 0, \forall i$ and j except for $i = 1$ and $j = 2$ the following coefficients are defined:

$$\begin{aligned} L_{1211}^{*(11)} &= \alpha L_{1211}^{(11)}, & L_{1212}^{*(11)} &= L_{1212}^{(11)}, & L_{1221}^{*(11)} &= \alpha^2 L_{1221}^{(11)}, \\ L_{1222}^{*(11)} &= \alpha L_{1222}^{(11)}, & L_{1223}^{*(11)} &= \alpha L_{1223}^{(11)}, & L_{1231}^{*(11)} &= \alpha^2 L_{1231}^{(11)}, \\ L_{1232}^{*(11)} &= \alpha L_{1232}^{(11)}, & L_{1233}^{*(11)} &= \alpha L_{1233}^{(11)} \end{aligned}$$

By considering every component of (23a) and (23b), let us observe that the coefficients of the second addendum of (23a), $L_{ikrs}^{(12)}$, and the first one of equation (23b), $L_{ikrs}^{(21)}$ represent scalar value, while one obtains identifications for the phenomenological coefficient $L_{ikrs}^{(11)}$ that can be summarized in the following table.

$\alpha L_{1111}^{(11)}$	$L_{1112}^{(11)}$	$L_{1113}^{(11)}$	$\alpha^2 L_{1121}^{(11)}$	$\alpha L_{1122}^{(11)}$	$\alpha L_{1123}^{(11)}$	$\alpha^2 L_{1131}^{(11)}$	$\alpha L_{1132}^{(11)}$	$\alpha L_{1133}^{(11)}$
$\alpha L_{1211}^{(11)}$	$\frac{1}{\alpha} L_{1212}^{(11)}$	$\frac{1}{\alpha} L_{1213}^{(11)}$	$\alpha L_{1221}^{(11)}$	$L_{1222}^{(11)}$	$L_{1223}^{(11)}$	$\alpha L_{1231}^{(11)}$	$L_{1232}^{(11)}$	$L_{1233}^{(11)}$
$L_{1311}^{(11)}$	$\frac{1}{\alpha} L_{1312}^{(11)}$	$\frac{1}{\alpha} L_{1313}^{(11)}$	$\alpha L_{1321}^{(11)}$	$L_{1322}^{(11)}$	$L_{1323}^{(11)}$	$\alpha L_{1331}^{(11)}$	$L_{1332}^{(11)}$	$L_{1333}^{(11)}$
$\alpha^3 L_{2111}^{(11)}$	$\alpha^2 L_{2112}^{(11)}$	$\alpha^2 L_{2113}^{(11)}$	$\alpha^3 L_{2121}^{(11)}$	$\alpha^2 L_{2122}^{(11)}$	$\alpha^2 L_{2123}^{(11)}$	$\alpha^3 L_{2131}^{(11)}$	$\alpha^2 L_{2132}^{(11)}$	$\alpha^2 L_{2133}^{(11)}$
$\alpha L_{2211}^{(11)}$	$L_{2212}^{(11)}$	$L_{2213}^{(11)}$	$\alpha^2 L_{2221}^{(11)}$	$\alpha L_{2222}^{(11)}$	$\alpha L_{2223}^{(11)}$	$\alpha^2 L_{2231}^{(11)}$	$\alpha L_{2232}^{(11)}$	$\alpha L_{2233}^{(11)}$
$\alpha L_{2311}^{(11)}$	$L_{2312}^{(11)}$	$L_{2313}^{(11)}$	$\alpha^2 L_{2321}^{(11)}$	$\alpha L_{2322}^{(11)}$	$\alpha L_{2323}^{(11)}$	$\alpha^2 L_{2331}^{(11)}$	$\alpha L_{2332}^{(11)}$	$\alpha L_{2333}^{(11)}$
$\alpha^3 L_{3111}^{(11)}$	$\alpha^2 L_{3112}^{(11)}$	$\alpha^2 L_{3113}^{(11)}$	$\alpha^3 L_{3121}^{(11)}$	$\alpha^2 L_{3122}^{(11)}$	$\alpha^2 L_{3123}^{(11)}$	$\alpha^3 L_{3131}^{(11)}$	$\alpha^2 L_{3132}^{(11)}$	$\alpha^2 L_{3133}^{(11)}$
$\alpha L_{3211}^{(11)}$	$L_{3212}^{(11)}$	$L_{3213}^{(11)}$	$\alpha^2 L_{3221}^{(11)}$	$\alpha L_{3222}^{(11)}$	$\alpha L_{3223}^{(11)}$	$\alpha^2 L_{3231}^{(11)}$	$\alpha L_{3232}^{(11)}$	$\alpha L_{3233}^{(11)}$
$\alpha L_{3311}^{(11)}$	$L_{3312}^{(11)}$	$L_{3313}^{(11)}$	$\alpha^2 L_{3321}^{(11)}$	$\alpha L_{3322}^{(11)}$	$\alpha L_{3323}^{(11)}$	$\alpha^2 L_{3331}^{(11)}$	$\alpha L_{3332}^{(11)}$	$\alpha L_{3333}^{(11)}$

Let us remark that these table summarizes the relations of the phenomenological coefficients which are invariant under the Lorentz transformation between the frame reference Σ and the rest frame reference Σ_0 . We can note that there is not a symmetry on all indexes, but only for particular pairs of values of indexes i, k, r, s , of L_{ikrs} .

In a similar way, by repeating the same procedure, from relation (23b) it is possible to deduce the transformation laws of the coefficient $L_{ikrs}^{(22)}$. These laws can be directly deduced from the table with the difference that the coefficients of proportionality are the reciprocal of those which are involved in the $L_{ikrs}^{(11)}$. Obviously, it is possible to obtain the classical case of phenomenological equations and phenomenological transport coefficient if we refer to a rest reference frame.

4. Conclusions

In this paper we have deduced the relativistic phenomenological equations and transformation laws of the relativistic transport coefficients by using a relativistic thermodynamical model formerly introduced by us (Farsaci and Rogolino 2016) and entirely based on the relativity principle and the transformation law of the strain tensor. In particular, following the procedure in the classical thermodynamics, we introduced a relativistic entropy density production depending on the gradient of relativistic temperature, the relativistic flow of the unitary volume force, the relativistic equilibrium strain and the relativistic affinity stress.

As a final remark, we observe that, because of the non-symmetry of the relativistic strain tensor, γ_{ik} , also the coefficients $L_{ikrs}^{(lj)}$ with $(l, j = 1, 2)$ are in general not symmetric. Obviously, it is possible to obtain the classical case if we refer to a rest reference frame. This approach can be useful for the study of astrophysical problems, e.g. in the black holes, in which it needs a more complete description (Weinberg 1972).

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