DOI: 10.1478/AAPP.942A6

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 94, No. 2, A6 (2016)

LAPLACE TYPE PROBLEMS FOR A LATTICE WITH CELL COMPOSED BY TWO TRIANGLES AND ONE HEXAGON

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ABSTRACT. We solve a Laplace type problem for a lattice of the Euclidean plane with cell composed by two triangles and one hexagon. We compute the probability that a segment of random position and constant length intersects a side of the lattice.

1. Introduction

In some previous papers (Caristi, Puglisi, and Stoka 2011; Caristi and Stoka 2011; Barilla *et al.* 2014; Caristi, Pettineo, and Stoka 2016) the authors study some Laplace problems with different fundamental cells. The aim of this paper is to consider a lattice $\mathfrak{R}(a,b)$, $a>b>0$, of the euclidean plane with fundamental cell composed by two triangles and one hexagon.

In Section 1 we compute the probability that a segment of random position and constant length l intersects a side of the lattice. In Section 2, we find the values of a,b,l, being $a>b$ the lengths of the sides of the cell.

2. Main result

Let $\mathfrak{R}\,(a,b)$, the lattice with fundamental cell C_0 rapresented in figure 1 with $a>b>0$: With the notations of this figure we have

$$|EH| = |FG| = \sqrt{2}\,(a-b)\,, \tag{1}$$

$$areaC_{01} = areaC_{02} = \frac{(a-b)^2}{2}, \qquad areaC_{03} = b\,(2a-b)\,. \tag{2}$$

We want to compute the probability that a segment s with random position and of constant length l intersects a side of lattice $\mathfrak{R}$, i.e. the probability P_{int} that a segment s intersects a side of the fundamental cell C_0.

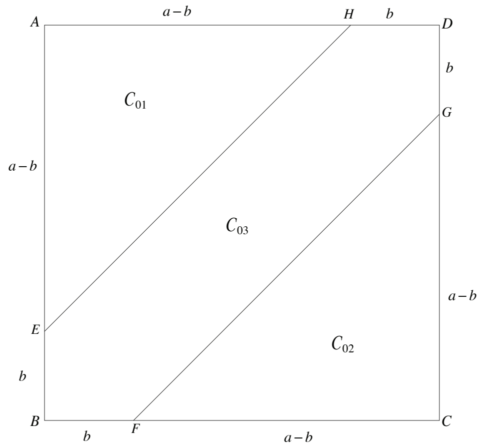

FIGURE 1

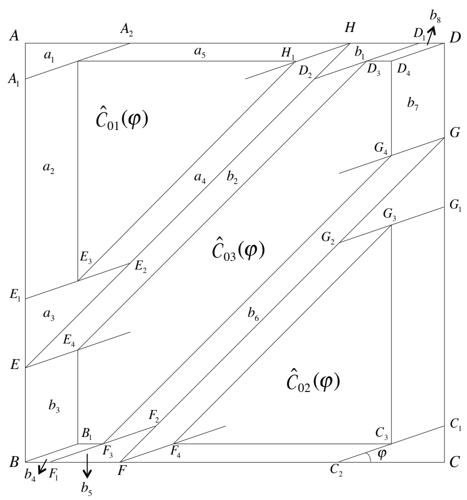

FIGURE 2

Theorem 1. *Let P_{int} be the probability that a random segment s of lenght l intersects a side of the lattice $\Re(a,b)$. Then we have:*

$$P_{int} = \frac{4}{\pi a^2}\left\{l\left[\left(2\sqrt{2}-1\right)a - 2\left(\sqrt{2}-1\right)b\right] - \frac{l^2}{4}\left[3+\left(2\sqrt{2}-3\right)\frac{\pi}{2}\right]\right\}.$$

Proof. The position of the segment s is determinated by its middle point and by the angle φ that s formed with the line AD o BC.

To compute the probability P_{int} we consider the limiting positions of the segment s, for a specified value of φ, in the cells $C_{0i}, (i = 1,2,3)$. Thus we obtain Fig. 2 and the relations:

$$area\widehat{C}_{01}(\varphi) = area\widehat{C}_{02}(\varphi) = areaC_{01} - \sum_{i=1}^{5} areaa_i(\varphi),$$

$$area\widehat{C}_{03}(\varphi) = areaC_{03} - \sum_{i=1}^{8} areab_i(\varphi).$$

By Fig. 2 we have:

$$areaa_1(\varphi) = \frac{l^2}{4}\sin 2\varphi,$$

$$areaa_2(\varphi) = \frac{(a-b)\,l}{2}\cos\varphi - \frac{l^2}{4}(\cos 2\varphi + 1),$$

$$areaa_3(\varphi) = \frac{\sqrt{2}}{4}(\cos 2\varphi - \sin 2\varphi + 1)\,l^2,$$

$$areaa_4(\varphi) = \frac{(a-b)\,l}{2}(\cos\varphi - \sin\varphi) - \frac{1}{4}l^2(\cos 2\varphi - \sin 2\varphi + 1),$$

$$areaa_5(\varphi) = \frac{(a-b)\,l}{2}\sin\varphi - \frac{l^2}{4}\sin 2\varphi.$$

Hence:

$$A_1(\varphi) = \sum_{i=1}^{5} areaa_i(\varphi) = (a-b)\,l\cos\varphi - \frac{\sqrt{2}-1}{4}\left(\sqrt{2}\cos 2\varphi + \sin 2\varphi - \sqrt{2}\right)l^2. \quad (3)$$

In order to compute $area\widehat{C}_{03}(\varphi)$, we consider

$$areab_1(\varphi) = \frac{l^2}{4}(\cos 2\varphi + \sin 2\varphi - 1).$$

$$areab_2(\varphi) = \sqrt{2}(a - b - l\sin\varphi)\frac{l}{2}\sin\left(\frac{\pi}{4} - \varphi\right) =$$

$$\frac{(a-b)\,l}{2}(\cos\varphi - \sin\varphi) - \frac{l^2}{4}(\cos 2\varphi - \sin 2\varphi - 1),$$

$$areab_3(\varphi) = \frac{bl}{2}\cos\varphi,$$

and, as $b_3 = b_7$, we have:

$$areab_7(\varphi) = \frac{bl}{2}\cos\varphi. \quad (4)$$

Then, as $b_5 = b_1$, we have:

$$areab_5(\varphi) = \frac{l^2}{4}(\cos 2\varphi + \sin 2\varphi - 1). \quad (5)$$

$$areab_4(\varphi) = \frac{bl}{2}\sin\varphi - \frac{l^2}{4}(\cos 2\varphi + \sin 2\varphi - 1),$$

$$areab_8(\varphi) = \frac{bl}{2}\sin\varphi - \frac{l^2}{4}(\cos 2\varphi + \sin 2\varphi - 1),$$

$$areab_6(\varphi) = \frac{(a-b)\,l}{2}(\cos\varphi - \sin\varphi) - \frac{l^2}{4}(\cos 2\varphi + \sin 2\varphi - 1).$$

Hence we have:

$$A_3(\varphi) = \sum_{i=1}^{8} areab_i(\varphi) = l\,[a\cos\varphi + (2b-a)\sin\varphi] - \frac{l^2}{2}(\cos 2\varphi + \sin 2\varphi - 1). \quad (6)$$

Replacing in (3) and (4) the relations (5) and (6), respectively, we have:

$$area\widehat{C}_{01}(\varphi) = area\widehat{C}_{02}(\varphi) = areaC_{01} - A_1(\varphi) \quad (7)$$

and

$$area\widehat{C}_{03}(\varphi) = areaC_{03} - A_3(\varphi). \quad (8)$$

Moreover, the relations (3) and (6), with $A_1 = A_2$, give us:

$$\sum_{i=1}^{3} A_i(\varphi) = l\,[(3a-2b)\cos\varphi + (2b-a)\sin\varphi] -$$

$$\frac{l^2}{2}\left[\left(3-\sqrt{2}\right)\cos 2\varphi + \sqrt{2}\sin 2\varphi + \sqrt{2} - 3\right]. \quad (9)$$

Denoting by $M_i\,(i=1,2,3)$ the set of segments s which have their center in C_{0i} and by N_i the set of segments s all contained in the cell C_{0i} , we have (Stoka 1975-76):

$$P_{int} = 1 - \frac{\sum_{i=1}^{3}\mu(N_i)}{\sum_{i=1}^{3}\mu(M_i)}, \quad (10)$$

where μ is the Lesbegue measure in the Euclidean plane. To compute the above measure μ (M_i) and μ (N_i), we use the Poincaré kinematic measure (Poincaré 1912):

$$dk = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinate of middle point of s and φ the fixed angle. We have:

$$\mu(M_i) = \int_0^{\frac{\pi}{4}} d\varphi \int\int_{\{(x,y)\varepsilon C_{0i}\}} dxdy =$$

$$\int_0^{\frac{\pi}{4}} (areaC_{0i})\,d\varphi = \frac{\pi}{4} areaC_{0i},$$

$$(i = 1,2,3).$$

Then

$$\sum_{i=1}^{3}\mu(M_i) = \frac{\pi}{4}\sum_{i=1}^{3} areaC_{01} = \frac{\pi}{4} areaC_0 = \frac{\pi}{4}a^2. \quad (11)$$

In the same way, we can write

$$\mu(N_i) = \int_0^{\frac{\pi}{4}} d\varphi \int\int_{\left\{(x,y)\varepsilon \widehat{C}_{0i}(\varphi)\right\}} dxdy =$$

$$\int_0^{\frac{\pi}{4}} \left[area\widehat{C}_{0i}(\varphi)\right] d\varphi = \int_0^{\frac{\pi}{4}} [areaC_{0i} - A_i(\varphi)]\,d\varphi = \frac{\pi}{4} areaC_{0i} - \int_0^{\frac{\pi}{4}} [A_i(\varphi)]\,d\varphi.$$

Then

$$\sum_{i=1}^{3} \mu(N_i) = \frac{\pi}{4} a^2 - \int_0^{\frac{\pi}{4}} \left[\sum_{i=1}^{3} A_i(\varphi) \right] d\varphi.$$

Formula (9) gives us

$$\int_0^{\frac{\pi}{4}} \left[\sum_{i=1}^{3} A_i(\varphi) \right] d\varphi = l \left[\left(2\sqrt{2}-1\right) a - 2\left(\sqrt{2}-1\right) b \right] - \frac{l^2}{4} \left[3 + \left(2\sqrt{2}-3\right) \frac{\pi}{2} \right].$$

Consequently,

$$\sum_{i=1}^{3} \mu(N_i) = \frac{\pi}{4} a^2 - l \left[\left(2\sqrt{2}-1\right) a - 2\left(\sqrt{2}-1\right) b \right] - \frac{l^2}{4} \left[3 + \left(2\sqrt{2}-3\right) \frac{\pi}{2} \right]. \tag{12}$$

Replacing the relations (11) and (12) in (10), we obtain:

$$P_{int} = \frac{4}{\pi a^2} \left\{ l \left[\left(2\sqrt{2}-1\right) a - 2\left(\sqrt{2}-1\right) b \right] - \frac{l^2}{4} \left[3 + \left(2\sqrt{2}-3\right) \frac{\pi}{2} \right] \right\}. \tag{13}$$

□

3. Limiting positions

Supposing a variable with $a > b$, we want to show that there are the values for a, b, l for which the probability P_{int} is maximum. The result is given by:

Corollary 1. *Let a, b, l such that the following relation is satisfied:*

$$\left(2\sqrt{2}-1\right) a - 4\left(\sqrt{2}-1\right) b - \frac{l}{2} \left[3 + \left(2\sqrt{2}-3\right) \frac{\pi}{2} \right] = 0,$$

then for these values, the probability P_{int} is maximum.

Proof. We indicate:

$$f(a) = \frac{l \left[\left(2\sqrt{2}-1\right) a - 2\left(\sqrt{2}-1\right) b \right] - \frac{l^2}{4} \left[3 + \left(2\sqrt{2}-3\right) \frac{\pi}{2} \right]}{a^2},$$

the probability P_{int} given by (13) is written

$$P_{int} = \frac{4}{\pi} f(a).$$

We have

$$f'(a) = \frac{-l \left(2\sqrt{2}-1\right) a + 4l \left(\sqrt{2}-1\right) b + \frac{l^2}{2} \left[3 + \left(2\sqrt{2}-3\right) \frac{\pi}{2} \right]}{a^3}.$$

Then, if the following relation is satisfied

$$\left(2\sqrt{2}-1\right) a - 4\left(\sqrt{2}-1\right) b - \frac{l}{2} \left[3 + \left(2\sqrt{2}-3\right) \frac{\pi}{2} \right] = 0,$$

we have $f'(a) = 0$ and the probability P_{int} is maximum. □

Acknowledgments

This paper is dedicated to Dr. Maria Filippi (Accademia delle Scienze di Torino).

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Manuscript received 8 January 2016; communicated 23 June 2016; published online 12 September 2016