

## UNIQUENESS PROPERTY FOR QUASIHARMONIC FUNCTIONS

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ABSTRACT. In this paper we consider a class of continuous functions, called quasiharmonic functions, admitting best approximations by harmonic polynomials. In this class we prove a uniqueness theorem by analogy with the analytic functions.

Let  $K \subset \mathbb{R}^n$  be a compact set and  $f(x) \in C(K)$ . We denote by

$$l_m(f, K) = \inf_{\{q_m\}} \|f(x) - q_m(x)\|_\infty$$

the least deviation of the function  $f$  on  $K$  from harmonic polynomials of degree  $\leq m$ . Zakhariuta (2001) proved the analogue of Bernštein theorem in the class of harmonic functions. That is, if

$$\overline{\lim}_{m \rightarrow \infty} l_m^{1/m}(f, K) < 1 \quad (1)$$

then the function  $f$  can be harmonically extended to some neighborhood of the compact  $K$ . Conversely, if the function  $f$  harmonically extends to some neighbourhood of the compact  $K$ , then inequality (1) holds. We denote by  $qh(K)$  the class of functions  $f(x)$  such that

$$\underline{\lim}_{m \rightarrow \infty} l_m^{1/m}(f, K) < 1.$$

The class  $qh(K)$  is called the class of quasiharmonic functions.

**Main Theorem.** Let  $K \subset \mathbb{R}^n$  be a  $H$ -regular compact and  $f \in qh(K)$ . If the zero set  $E = \{x \in K : f(x) = 0\}$  of the function  $f$  is not an  $N$ -set, then  $f(x) \equiv 0$  on  $K$ .

We note that in the case of quasianalytic functions theorems analogous to our main theorem were proved by Bernštein (1952) and Plesniak (1971).

**The class of functions  $Lh_0(D)$**  (see Imomkulov and Saidov 2008). Let  $D$  be a domain from  $\mathbb{R}^n$  and  $h(D)$  be the space of harmonic functions in  $D$ . We denote by  $Lh_\varepsilon(D)$  - the minimal class of functions, which contains all the functions of the form  $\alpha \ln |u(x)|$ ,  $u(x) \in h(D)$ ,  $0 < \alpha < \varepsilon$ , and closed under the operation of "upper regularization", i.e., for any family of functions  $u_\lambda(x) \in Lh_\varepsilon(D)$ ,  $\lambda \in \Lambda$ , the function

$$\overline{\lim}_{y \rightarrow x} (\sup \{u_\lambda(y) : \lambda \in \Lambda\})$$

also belongs to class  $Lh_\varepsilon(D)$ . The union  $Lh_0(D) = \bigcup_{\varepsilon > 0} Lh_\varepsilon(D)$  is called the class of  $Lh_0$ -functions.

Zahariuta (2001) defined the following extremal function (see also Hecart 1999, 2000; Imomkulov and Saidov 2008):

Let  $E \subset\subset D \subset \mathbb{R}^n$  be a compact set; we fix  $\varepsilon > 0$  and set

$$\chi_\varepsilon(x, E, D) = \overline{\lim}_{y \rightarrow x} \sup \{ \alpha \ln |u(y)| : 0 < \alpha < \varepsilon, u \in h(D), \|u\|_E \leq 1, \|u\|_D^\alpha \leq e \}.$$

It is clear that  $\chi_\varepsilon$  is monotonically decreasing as  $\varepsilon \rightarrow 0$  and that the following limit exists

$$\chi_0(x, E, D) = \lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(x, E, D).$$

Here  $\chi_0(x, E, D)$  is called  $\chi_0$ -measure of compact  $E$  relative to the domain  $D$ . As in the case of  $P$ -measure (see Sadullaev 1981, 1985), we have either  $\chi_0(x, E, D) \equiv 1$  or  $\chi_0(x, E, D) \not\equiv 1$  in the domain  $D$ . In the first case the set  $E \subset D$  for which  $\chi_0(x, E, D) \equiv 1$  is called the set of zero  $\chi_0$ -measure, and in the second case, the set  $E \subset D$  is called the set of nonzero  $\chi_0$ -measure.

Now we provide a lemma “about two constants” for the class of quasiharmonic functions. **Lemma 1.** (see Hecart 2000; Imomkulov and Saidov 2008). *Let  $D$  be a domain from  $\mathbb{R}^n$  and  $E \subset\subset D$  be a compact set. Then for any  $\alpha \in (0, 1)$ ,  $\varepsilon \in (0, 1 - \alpha)$  and for any compact  $K \subset\subset D_\alpha$  there exists a positive constant  $C = C(\alpha, \varepsilon, K, D)$  such that for all harmonic functions  $u(x)$  in  $D$  the following inequality holds:*

$$\|u\|_K \leq C \|u\|_E^{1-\alpha-\varepsilon} \|u\|_D^{\alpha+\varepsilon}, \tag{2}$$

where  $D_\alpha = \{x \in D : \chi_0(x, E, D) < \alpha\}$ .

This lemma is an analogue of the theorem “about two constants” for the class of holomorphic functions (see for example Sadullaev 1981, 1985) and plays an important role in the theory of harmonic functions. We note that from inequality (2) it follows that if  $\chi_0(x, E, D) \not\equiv 1$ , then  $E$  is the uniqueness set for the class of harmonic functions in  $D$ .

**Definition 1** (see Hecart 1999; Nguyen Thanh and Djebbar 1989). *A compact set  $E \subset \mathbb{R}^n$  is called  $H$ -regular at a point  $x^0$ , if for any number  $b > 1$  there exist numbers  $M > 0$  and  $r > 0$  such that for any harmonic polynomial  $P(x)$  the following inequality holds:*

$$\|P\|_{B(x^0, r)} \leq Mb^{\deg P} \|P(x)\|_{E \cap B(x^0, r)},$$

where  $B(x^0, r) = \{x \in \mathbb{R}^n : |x - x^0| < r\}$ .

If the compact  $H$  is regular at each of its point, then it is called  $H$ -regular compact.

Hecart (1999) proved that, if a compact  $E$  is  $H$ -regular at a point  $x^0 \in E$ , then for any neighbourhood  $\Omega \supset\supset E$  we have

$$\chi(x^0, E, \Omega) = 0.$$

**The  $N$ -sets in  $\mathbb{R}^n$**  (see Imomkulov and Saidov 2008). Let  $\vartheta_k(x) \in Lh_0(D)$  be a monotonically increasing sequence of functions that are locally uniformly bounded from above. Consider the limit

$$\overline{\lim}_{y \rightarrow x} \lim_{k \rightarrow \infty} \vartheta_k(y) = \vartheta(x), \quad x \in D.$$

Hence, everywhere in  $D$  we have the inequality

$$\lim_{k \rightarrow \infty} \vartheta_k(x) \leq \vartheta(x).$$

**Definition 2.** A set  $E \subset \mathbb{R}^n$  is called an  $N$ -set if for some open set  $D \supset E$  there exists a monotonically increasing sequence of functions  $\vartheta_k(x) \in Lh_0(D)$  locally uniformly bounded from above and such that the set  $E$  is a subset of a set of type

$$\{x \in D : \lim_{k \rightarrow \infty} \vartheta_k(x) < \vartheta(x)\},$$

where  $\vartheta(x) = \overline{\lim}_{y \rightarrow x, k \rightarrow \infty} \vartheta_k(y)$ ,  $x \in D$ .

**Proposition 1** (Imomkulov and Saidov 2008). If  $\vartheta_k(x) \in Lh_0(D)$  is sequence of functions locally uniformly bounded from above and

$$\overline{\lim}_{y \rightarrow x, k \rightarrow \infty} \vartheta_k(y) = \vartheta(x), \quad x \in D,$$

then the set

$$E = \{x \in D : \overline{\lim}_{k \rightarrow \infty} \vartheta_k(x) < \vartheta(x)\}$$

consists of a countable union of  $N$ -sets.

Indeed, consider the sequence of functions

$$w_{l,j}(x) = \max_{l \leq k \leq j} \vartheta_k(x).$$

Clearly,  $\overline{\lim}_{k \rightarrow \infty} \vartheta_k(x) = \lim_{l \rightarrow \infty} \lim_{j \rightarrow \infty} w_{l,j}(x)$ . Since the sequence is monotonically increasing in  $j$ , we have  $\lim_{j \rightarrow \infty} w_{l,j}(x) \leq \overline{\lim}_{y \rightarrow x, j \rightarrow \infty} w_{l,j}(y)$ ,  $x \in D$  and the sets

$$E_l = \left\{ x \in D : \lim_{j \rightarrow \infty} w_{l,j}(x) < \overline{\lim}_{y \rightarrow x, j \rightarrow \infty} w_{l,j}(y) \right\}, \quad l = 1, 2, \dots,$$

are  $N$ -sets. On the other hand, the sequences

$$\lim_{j \rightarrow \infty} w_{l,j}(x) \quad \overline{\lim}_{y \rightarrow x, j \rightarrow \infty} w_{l,j}(y), \quad l = 1, 2, \dots,$$

are monotonically decreasing and

$$\overline{\lim}_{k \rightarrow \infty} \vartheta_k(x) = \lim_{l \rightarrow \infty} \lim_{j \rightarrow \infty} w_{l,j}(x) = \vartheta(x) = \lim_{l \rightarrow \infty} \overline{\lim}_{y \rightarrow x, j \rightarrow \infty} w_{l,j}(y),$$

$$x \in D \setminus \bigcup_{l=1}^{\infty} E_l.$$

It follows that

$$E \subset \bigcup_{l=1}^{\infty} E_l, \quad \text{i.e.} \quad E = \bigcup_{l=1}^{\infty} (E_l \cap E).$$

**Definition 3.** A set  $E \subset D$  is called  $Lh_0$ -polar relative to the domain  $D$  if there exists a function  $\vartheta(x) \in Lh_0(D)$  such that  $\vartheta(x) \not\equiv -\infty$  and  $\vartheta(x)|_E = -\infty$ .

We note that, if  $u(x) \in h(D)$ ,  $u(x) \not\equiv 0$  and  $E \subset \{u(x) = 0\}$ , then  $E$  is  $Lh_0$ -polar relative to domain  $D$ .

**Proposition 2** (see Imomkulov and Saidov 2008). Every  $Lh_0$ -polar set relative to domain  $D$  is contained in a countable union of  $N$ -sets.

Indeed, let  $E$  be an  $Lh_0$ -polar set relative to domain  $D$ . Then by definition there exists a function  $\vartheta(x) \in Lh_0(D)$  such that  $\vartheta(x) \not\equiv -\infty$ ,  $\vartheta(x)|_E = -\infty$ . Consider a sequence of functions  $\vartheta_k(x) = \frac{1}{k} \vartheta(x)$ . Clearly,  $\vartheta_k(x) \in Lh_0(D)$  and  $\vartheta_k(x) \not\equiv -\infty$ ,  $\vartheta_k(x)|_E = -\infty$ . Moreover,  $\lim_{k \rightarrow \infty} \vartheta_k(x) = 0$  for almost all  $x \in D$  and  $\lim_{k \rightarrow \infty} \vartheta_k(x) = -\infty$  for all  $x \in E$ . It then follows that

$$E \subset \left\{ x \in D : \lim_{k \rightarrow \infty} \vartheta_k(x) < \overline{\lim}_{y \rightarrow x, k \rightarrow \infty} \vartheta_k(y) \right\}.$$

On the other hand, as was shown above, the set

$$\left\{ x \in D : \lim_{k \rightarrow \infty} \vartheta_k(x) < \overline{\lim}_{y \rightarrow x, k \rightarrow \infty} \vartheta_k(y) \right\}$$

consists of a countable union of  $N$ -sets.

**Proof of Main Theorem.** Let  $f(x) \in qh(K)$  and

$$E = \{x \in K : f(x) = 0\}.$$

By definition of the class  $qh(K)$ , there is a sequence of harmonic polynomials  $p_{m_k}(x)$  such that

$$\lim_{k \rightarrow \infty} \|f - p_{m_k}\|_K^{\frac{1}{m_k}} = d < 1. \tag{3}$$

Since  $f|_E = 0$ , we have

$$\lim_{k \rightarrow \infty} \|p_{m_k}\|_E^{\frac{1}{m_k}} = d < 1. \tag{4}$$

Inequalities (3) and (4) imply that, starting from some number  $k_0$  for all numbers  $k \geq k_0$ , the following two inequalities hold:

$$\|p_{m_k}\|_K \leq 1 + \|f\|, \tag{5}$$

$$\|p_{m_k}\|_E^{\frac{1}{m_k}} < d + \varepsilon < 1, \quad 0 < \varepsilon < 1 - d, \tag{6}$$

Since  $K$  is an  $H$ -regular compact, by the definition of  $H$ -regularity, for any  $b : 1 < b < 1/d + \varepsilon$  there are positive numbers  $M$  and  $\delta$  such that for a  $\delta$ -neighbourhood  $U_\delta = \{x : \text{dist}(x, K) < \delta\}$  of the compact  $K$  we have following estimate

$$\|P_{m_k}(x)\|_{U_\delta} \leq Mb^{m_k} \|P_{m_k}(x)\|_K \tag{7}$$

On the other hand, since the set  $E$  is not an  $N$ -set, we have  $\chi_0(x, E, U_\delta) \not\equiv 1$  and using lemma 1 “about two constants” we obtain that for any  $\alpha \in (0, 1)$ ,  $\beta \in (0, 1 - \alpha)$ ,  $\alpha + \beta < 1/2$ , and for any open set  $U : K \subset\subset U \subset\subset U_{\delta, \alpha}$ , where  $U_{\delta, \alpha} = \{x \in U_\delta : \chi_0(x, E, U_\delta) < \alpha\}$ , there is a positive constant  $C = C(\alpha, \beta, K, U_\delta)$  such that

$$\|p_{m_k}(x)\|_U \leq C \|p_{m_k}(x)\|_E^{1-\alpha-\beta} \|p_{m_k}(x)\|_D^{\alpha+\beta}$$

Now using estimations (5), (6) and (7) we obtain

$$\begin{aligned} \|p_{m_k}(x)\|_U &\leq C(d + \varepsilon)^{m_k(1-\alpha-\beta)} \cdot M^{\alpha+\beta} (1 + \|f\|)^{\alpha+\beta} b^{m_k(\alpha+\beta)} \leq \\ &\leq L \cdot (d + \varepsilon)^{m_k(1-\alpha-\beta)} \cdot (d + \varepsilon)^{-m_k(\alpha+\beta)} = L(d + \varepsilon)^{m_k(1-2(\alpha+\beta))}, \end{aligned}$$

where  $L = CM^{\alpha+\beta}(1 + \|f\|)^{\alpha+\beta}$ . Here  $(d + \varepsilon)^{m_k(1-2(\alpha+\beta))} \rightarrow 0$ ,  $k \rightarrow \infty$ , since  $d + \varepsilon < 1$  and  $\alpha + \beta < 1/2$ . Therefore,  $\|p_{m_k}(x)\|_U \rightarrow 0$ ,  $k \rightarrow \infty$ , i.e.,  $p_{m_k}(x)$  converges uniformly to zero in a neighbourhood of  $U \supset \supset K$ . It follows that  $f(x) \equiv 0$  on  $K$ . The proof is complete.

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