

Uniqueness property for quasiharmonic functions

S.A.Imomkulov and Z. Sh. Ibragimov

Abstract. In this paper we consider class of continuous functions, called quasiharmonic functions, admitting best approximations by harmonic polynomials. In this class we prove a uniqueness theorem by analogy with the analytic functions.

Key words. Harmonic polynomials, quasiharmonic functions, polynomial approximations, N – sets, H –regular compacts.

Let $K \subset \mathbb{R}^n$ be a compact set and $f(x) \in C(K)$. We denote by

$$l_m(f, K) = \inf_{\{q_m\}} \|f(x) - q_m(x)\|_\infty$$

the least deviation of the function f on K from harmonic polynomials of degree $\leq m$. In paper [1] Zahariuta proved analogue of Bernshtein theorem in the class of harmonic functions. That is, if

$$\overline{\lim}_{m \rightarrow \infty} l_m^{1/m}(f, K) < 1 \quad (1)$$

then the function f can be harmonically extended to some neighborhood of the compact K . Conversely, if the function f harmonically extends to some neighbourhood of the compact K , then inequality (1) holds.

We denote by $qh(K)$ the class of functions $f(x)$ such that

$$\underline{\lim}_{m \rightarrow \infty} l_m^{1/m}(f, K) < 1.$$

The class $qh(K)$ is called the class of quasiharmonic functions.

Main Theorem. Let $K \subset \mathbb{R}^n$ be a H –regular compact and $f \in qh(K)$. If the zero set $E = \{x \in K : f(x) = 0\}$ of the function f is not an N –set, then $f(x) \equiv 0$ on K .

We note that in the case of quasianalytic functions theorems analogous to our main theorem were proved in papers [2-5].

The class of functions $Lh_0(D)$ (see [8]). Let D be a domain from \mathbb{R}^n and $h(D)$ be the space of harmonic functions in D . We denote by $Lh_\varepsilon(D)$ – the minimal class of functions, which contains all the functions of the form $\alpha \ln |u(x)|$, $u(x) \in h(D)$, $0 < \alpha < \varepsilon$, and closed under the operation of “upper regulariation”, i.e., for any family of functions $u_\lambda(x) \in Lh_\varepsilon(D)$, $\lambda \in \Lambda$, the function

$$\overline{\lim}_{y \rightarrow x} (\sup \{u_\lambda(y) : \lambda \in \Lambda\})$$

also belongs to class $Lh_\varepsilon(D)$.

The union $Lh_0(D) = \bigcup_{\varepsilon > 0} Lh_\varepsilon(D)$ is called the class of Lh_0 -functions.

In [1] (see also [8], [9],[10]) the author defined the following extremal function: let $E \subset\subset D \subset \mathbb{R}^n$ be a compact set. We fix $\varepsilon > 0$ and set

$$\chi_\varepsilon(x, E, D) = \overline{\lim}_{y \rightarrow x} \sup \{ \alpha \ln |u(y)| : 0 < \alpha < \varepsilon, u \in h(D), \|u\|_E \leq 1, \|u\|_D^\alpha \leq e \}.$$

It is clear that χ_ε is monotonically decreasing as $\varepsilon \rightarrow 0$ and that the following limit exists

$$\chi_0(x, E, D) = \lim_{\varepsilon \rightarrow 0} \chi_\varepsilon(x, E, D).$$

Here $\chi_0(x, E, D)$ is called χ_0 -measure of compact E relative to the domain D .

As in the case of P -measure (see [6,7]), we have either $\chi_0(x, E, D) \equiv 1$ or $\chi_0(x, E, D) \not\equiv 1$ in the domain D . In the first case the set $E \subset D$ for which $\chi_0(x, E, D) \equiv 1$ is called the set of zero χ_0 -measure, and in the second case, the set $E \subset D$ is called the set of nonzero χ_0 -measure.

Now we provide a lemma "about two constant" for the class of quasiharmonic functions.

Lemma 1. (see [8],[9]). Let D be a domain from \mathbb{R}^n and $E \subset\subset D$ be a compact set. Then for any $\alpha \in (0, 1)$, $\varepsilon \in (0, 1 - \alpha)$ and for any compact $K \subset\subset D_\alpha$ there exists a positive constant $C = C(\alpha, \varepsilon, K, D)$ such that for all harmonic functions $u(x)$ in D the following inequality holds:

$$\|u\|_K \leq C \|u\|_E^{1-\alpha-\varepsilon} \|u\|_D^{\alpha+\varepsilon}, \quad (2)$$

where $D_\alpha = \{x \in D : \chi_0(x, E, D) < \alpha\}$.

This lemma is an analogue of the theorem "about two constants" for the class of holomorphic functions (see for example [6,7]) and plays an important role in the theory of harmonic functions.

We note that from inequality (2) it follows that if $\chi_0(x, E, D) \not\equiv 1$, then E is the uniqueness set for the class of harmonic functions in D .

Definition 1 (see [10], [11]). A compact set $E \subset R^n$ is called H -regular at a point x^0 , if for any number $b > 1$ there exist numbers $M > 0$ and $r > 0$ such that for any harmonic polynomial $P(x)$ the following inequality holds:

$$\|P\|_{B(x^0, r)} \leq Mb^{\deg P} \|P(x)\|_{E \cap B(x^0, r)},$$

where $B(x^0, r) = \{x \in R^n : |x - x^0| < r\}$.

If the compact H is regular at each of its point, then it is called H -regular compact.

In [10] it is proved that if a compact E is H -regular at a point $x^0 \in E$, then for any neighbourhood $\Omega \supset \supset E$ we have

$$\chi(x^0, E, \Omega) = 0.$$

The N -sets in R^n (see [8]). Let $\vartheta_k(x) \in Lh_0(D)$ be a monotonically increasing sequence of functions that are locally uniformly bounded from above. Consider the limit

$$\overline{\lim}_{y \rightarrow x} \lim_{k \rightarrow \infty} \vartheta_k(y) = \vartheta(x), \quad x \in D.$$

Then everywhere in D we have the inequality

$$\lim_{k \rightarrow \infty} \vartheta_k(x) \leq \vartheta(x).$$

Definition 2. A set $E \subset R^n$ is called an N -set if for some open set $D \supset E$ there exists a monotonically increasing sequence of functions $\vartheta_k(x) \in Lh_0(D)$ locally uniformly bounded from above and such that the set E is a subset of a set of type

$$\{x \in D : \lim_{k \rightarrow \infty} \vartheta_k(x) < \vartheta(x)\},$$

where $\vartheta(x) = \overline{\lim}_{y \rightarrow x} \lim_{k \rightarrow \infty} \vartheta_k(y)$, $x \in D$.

Proposition 1 [8]. If $\vartheta_k(x) \in Lh_0(D)$ is sequence of functions locally uniformly bounded from above and

$$\overline{\lim}_{y \rightarrow x} \overline{\lim}_{k \rightarrow \infty} \vartheta_k(y) = \vartheta(x), \quad x \in D,$$

then the set

$$E = \{x \in D : \overline{\lim}_{k \rightarrow \infty} \vartheta_k(x) < \vartheta(x)\}$$

consists of a countable union of N -sets.

Indeed, consider the sequence of functions

$$w_{l,j}(x) = \max_{l \leq k \leq j} \vartheta_k(x).$$

Clearly, $\overline{\lim}_{k \rightarrow \infty} \vartheta_k(x) = \lim_{l \rightarrow \infty} \lim_{j \rightarrow \infty} w_{l,j}(x)$. Since the sequence is monotonically increasing in j , we have $\lim_{j \rightarrow \infty} w_{l,j}(x) \leq \overline{\lim}_{y \rightarrow x} \lim_{j \rightarrow \infty} w_{l,j}(y)$, $x \in D$ and the sets

$$E_l = \left\{ x \in D : \lim_{j \rightarrow \infty} w_{l,j}(x) < \overline{\lim}_{y \rightarrow x} \lim_{j \rightarrow \infty} w_{l,j}(y) \right\}, \quad l = 1, 2, \dots,$$

are N -sets. On the other hand, the sequences

$$\lim_{j \rightarrow \infty} w_{l,j}(x) \quad \overline{\lim}_{y \rightarrow x} \lim_{j \rightarrow \infty} w_{l,j}(y), \quad l = 1, 2, \dots,$$

are monotonically decreasing and

$$\overline{\lim}_{k \rightarrow \infty} \vartheta_k(x) = \lim_{l \rightarrow \infty} \lim_{j \rightarrow \infty} w_{l,j}(x) = \vartheta(x) = \lim_{l \rightarrow \infty} \overline{\lim}_{y \rightarrow x} \lim_{j \rightarrow \infty} w_{l,j}(y),$$

$$x \in D \setminus \bigcup_{l=1}^{\infty} E_l.$$

It follows that

$$E \subset \bigcup_{l=1}^{\infty} E_l, \text{ i.e., } E = \bigcup_{l=1}^{\infty} (E_l \cap E).$$

Definition 3. A set $E \subset D$ is called Lh_0 -polar relative to the domain D if there exists a function $\vartheta(x) \in Lh_0(D)$ such that $\vartheta(x) \not\equiv -\infty$ and $\vartheta(x)|_E = -\infty$.

We note that if $u(x) \in h(D)$, $u(x) \not\equiv 0$ and $E \subset \{u(x) = 0\}$, then E is Lh_0 -polar relative to domain D .

Proposition 2 (see [8]). Every Lh_0 -polar set relative to domain D is contained in a countable union of N -sets.

Indeed, let E be an Lh_0 -polar set relative to domain D . Then by definition there exists a function $\vartheta(x) \in Lh_0(D)$ such that $\vartheta(x) \not\equiv -\infty$,

$\vartheta(x)|_E = -\infty$. Consider a sequence of functions $\vartheta_k(x) = \frac{1}{k}\vartheta(x)$. Clearly, $\vartheta_k(x) \in Lh_0(D)$ and $\vartheta_k(x) \not\equiv -\infty$, $\vartheta_k(x)|_E = -\infty$. Moreover, $\lim_{k \rightarrow \infty} \vartheta_k(x) = 0$ for almost all $x \in D$ and $\lim_{k \rightarrow \infty} \vartheta_k(x) = -\infty$ for all $x \in E$. It then follows that

$$E \subset \left\{ x \in D : \lim_{k \rightarrow \infty} \vartheta_k(x) < \overline{\lim}_{y \rightarrow x, k \rightarrow \infty} \vartheta_k(y) \right\}.$$

On the other hand, as was shown above, the set

$$\left\{ x \in D : \lim_{k \rightarrow \infty} \vartheta_k(x) < \overline{\lim}_{y \rightarrow x, k \rightarrow \infty} \vartheta_k(y) \right\}$$

consists of a countable union of N -sets.

Proof of Main Theorem. Let $f(x) \in qh(K)$ and

$$E = \{x \in K : f(x) = 0\}.$$

By definition of the class $qh(K)$, there is a sequence of harmonic polynomials $p_{m_k}(x)$ such that

$$\lim_{k \rightarrow \infty} \|f - p_{m_k}\|_K^{\frac{1}{m_k}} = d < 1. \quad (3)$$

Since $f|_E = 0$, we have

$$\lim_{k \rightarrow \infty} \|p_{m_k}\|_E^{\frac{1}{m_k}} = d < 1. \quad (4)$$

Inequalities (3) and (4) imply that starting from some number k_0 for all numbers $k \geq k_0$ following two inequalities hold:

$$\|p_{m_k}\|_K \leq 1 + \|f\|, \quad (5)$$

$$\|p_{m_k}\|_E^{\frac{1}{m_k}} < d + \varepsilon < 1, \quad 0 < \varepsilon < 1 - d, \quad (6)$$

Since K is an H -regular compact, by the definition of H -regularity, for any $b : 1 < b < 1/d + \varepsilon$ there are positive numbers M and δ such that for a δ -neighbourhood $U_\delta = \{x : \text{dist}(x, K) < \delta\}$ of the compact K we have following estimate

$$\|P_{m_k}(x)\|_{U_\delta} \leq Mb^{m_k} \|P_{m_k}(x)\|_K \quad (7)$$

On the other hand, since the set E is not an N - set, we have $\chi_0(x, E, U_\delta) / \equiv 1$ and using lemma 1 “about two constants” we obtain that for any $\alpha \in (0, 1)$, $\beta \in (0, 1 - \alpha)$, $\alpha + \beta < 1/2$, and for any open set $U : K \subset\subset U \subset\subset U_{\delta, \alpha}$, where $U_{\delta, \alpha} = \{x \in U_\delta : \chi_0(x, E, U_\delta) < \alpha\}$, there is a positive constant $C = C(\alpha, \beta, K, U_\delta)$ such that

$$\|p_{m_k}(x)\|_U \leq C \|p_{m_k}(x)\|_E^{1-\alpha-\beta} \|p_{m_k}(x)\|_D^{\alpha+\beta}$$

Now using estimations (5), (6) and (7) we obtain

$$\begin{aligned} \|p_{m_k}(x)\|_U &\leq C(d + \varepsilon)^{m_k(1-\alpha-\beta)} \cdot M^{\alpha+\beta} (1 + \|f\|)^{\alpha+\beta} b^{m_k(\alpha+\beta)} \leq \\ &\leq L \cdot (d + \varepsilon)^{m_k(1-\alpha-\beta)} \cdot (d + \varepsilon)^{-m_k(\alpha+\beta)} = L(d + \varepsilon)^{m_k(1-2(\alpha+\beta))}, \end{aligned}$$

where $L = CM^{\alpha+\beta}(1 + \|f\|)^{\alpha+\beta}$.

Here $(d + \varepsilon)^{m_k(1-2(\alpha+\beta))} \rightarrow 0$, $k \rightarrow \infty$, since $d + \varepsilon < 1$ and $\alpha + \beta < 1/2$. Therefore, $\|p_{m_k}(x)\|_U \rightarrow 0$, $k \rightarrow \infty$, i.e., $p_{m_k}(x)$ converges uniformly to zero in a neighbourhood of $U \supset\supset K$. It follows that $f(x) \equiv 0$ on K . The proof is complete.

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Sevdiyov Akramovich Imomkulov
Navoi State Pedagogical Institute (Uzbekistan)
e-mail: sevdiyov_i@mail.ru

Zafar Shavkatovich Ibragimov
Urgench State University (Uzbekistan)
e-mail: z.ibragim@gmail.com