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MULTITIME FLOQUET THEORY

CONSTANTIN UDRIŞTE *

ABSTRACT. This paper is a contribution to the Floquet ideas applied for multitime overdetermined linear first order PDE dynamical systems with multi-periodic coefficients. The main results include: (1) a formula for the fundamental matrix; (2) the equivalence between a T-multi-periodic PDE system and a constant coefficient PDE system; (3) a new criteria for the controllability of overdetermined linear PDE systems.

1. Introduction

The classical Floquet theory was developed in two directions: (1) as a device of reducing single-time dynamical ODE systems involving periodic coefficients to a constant linear ODE system which is much simpler to solve (see Eastham 1974; Floquet 1883; Krasnosel'skii 1968; Teschl 2012; Tu 1994); (2) as an important tool in specific single-time PDEs, as for example hypoelliptic, parabolic, elliptic and Schrödinger PDEs, and boundary value problems arising in applications (see Kuchment 1993).

In this paper we extend the Floquet ideas to the multitime case, *i.e.*, multitime linear first order PDE dynamical systems with multi-periodic coefficients. The results can be applied to simplify the controllability of multitime linear PDE systems (for multitime optimal control, (see our papers: Bejenaru and Udrişte 2012; Tu 1994; Udrişte 2011a; Udrişte and Bejenaru 2011a; Udrişte and Ghiu 2012).

2. Multitime PDE dynamical systems with multi-periodic coefficients

To write a multitime first order PDE dynamical system we need the following ingredients: $t = (t^1, ..., t^m) \in R^m$ or $t = (t^\alpha), \alpha = 1, ..., m$, $x = (x^1, ..., x^n) \in R^n$ or $x = (x^i), i = 1, ..., n$, $M(t) \in R^{n^2 m}$ or $M^i_{j\alpha}(t), i, j = 1, ..., n, \alpha = 1, ..., m$. These determine the first order multitime PDE dynamical system

$$\frac{\partial x}{\partial t} = M(t)x, x(0) = x_0$$

or explicitly

$$\frac{\partial x^i}{\partial t^\alpha}(t) = M^i_{j\alpha}(t)x^j(t), x^i(0) = x^i_0. \tag{1}$$

The existence and uniqueness of the solution asks the complete integrability conditions

$$\frac{\partial M_\alpha}{\partial t^\beta}(t)+M_\alpha(t)M_\beta(t)=\frac{\partial M_\beta}{\partial t^\alpha}(t)+M_\beta(t)M_\alpha(t).$$

Now we condider a pair of homogeneous PDE and nonhomogeneous PDE

$$\frac{\partial x}{\partial t^\alpha}(t)=M_\alpha(t)x(t),\ \frac{\partial x}{\partial t^\alpha}(t)=M_\alpha(t)x(t)+F_\alpha(t), t\in R^m.$$

They are simultaneously completely integrable PDEs systems if and only if

$$\frac{\partial M_\alpha}{\partial t^\beta}(t)+M_\alpha(t)M_\beta(t)=\frac{\partial M_\beta}{\partial t^\alpha}(t)+M_\beta(t)M_\alpha(t)$$

$$M_\alpha(t)F_\beta(t)+\frac{\partial F_\alpha}{\partial t^\beta}(t)=M_\beta(t)F_\alpha(t)+\frac{\partial F_\beta}{\partial t^\alpha}(t).$$

If $\Phi(t)$ is the fundamental matrix of the homogeneous PDE system, then the solution of the Cauchy problem

$$\frac{\partial x}{\partial t^\alpha}(t)=M_\alpha(t)x(t)+F_\alpha(t),\quad x(0)=x_0,\ t\in R^m$$

is given by the *variation of parameters formula*

$$x(t)=\Phi(t)x_0+\int_{\gamma_{0t}}\Phi^{-1}(s)F_\alpha(s)ds^\alpha$$

where γ_{0t} is an arbitrary piecewise C^1 curve joining the points $0,t\in R^m$.

The PDE system (1) is called *T-multi-periodic* if there exists a fixed point $T=(T^1,...,T^m)\in R^m$ such that $M(t)=M(t+T)\in R^{n^2m}$ or $M^i_{j\alpha}(t)=M^i_{j\alpha}(t+T), i,j=1,...,n,\alpha=1,...,m$, for any $t\in R^m$.

Theorem *If the PDE system (1) is T-multi-periodic, then its fundamental matrix $\Phi(t)$ can be written in the form*

$$\Phi(t)=P(t)\exp(B_\beta t^\beta),$$

where $B_\beta,\beta=1,...,m$ are $n\times n$ matrices.

Proof If $\Phi(t)$ is the fundamental (nonsingular) matrix, then so is $\Phi(t+T)$. Since there are only n independent solutions of the PDE system(1), there exists a nonsingular matrix C such that

$$\Phi(t+T)=\Phi(t)C.$$

Since T is fixed, there exist constant $n\times n$ matrices $B_\beta,\beta=1,...,m$ such that $C=\exp(B_\beta T^\beta)$. The matrix $P(t)=\Phi(t)\exp(-B_\beta t^\beta)$ is T-multi-periodic. Indeed,

$$P(t+T)=\Phi(t+T)\exp(-B_\beta(t^\beta+T^\beta))$$

$$=\Phi(t)C\exp(-B_\beta T^\beta)\exp(-B_\beta t^\beta)$$

$$=\Phi(t)\exp(B_\beta T^\beta)\exp(-B_\beta T^\beta)\exp(-B_\beta t^\beta)$$

$$=\Phi(t)\exp(-B_\beta t^\beta)=P(t).$$

The matrix C is called the *monodromy matrix* of the PDE system (1).

3. Setting the constant coefficients PDE system

The most important result of Floquet type is

Theorem *The multi-periodic PDE system (1) is equivalent to the constant coefficient PDE system*

$$\frac{\partial y}{\partial t^\alpha}(t) = B_\alpha y(t). \tag{2}$$

Proof We start from a multi-periodic PDE system (1) and we make the substitution $x(t) = P(t)y$, where $P(t)$ has the foregoing meaning. Then

$$\frac{\partial x}{\partial t} = \frac{\partial P}{\partial t}y + P\frac{\partial y}{\partial t} = M_\alpha x = M_\alpha P y$$

or

$$\frac{\partial y}{\partial t^\alpha} = P^{-1}(M_\alpha P - \frac{\partial P}{\partial t^\alpha})y.$$

On the other hand, we have

$$\frac{\partial \Phi}{\partial t^\alpha} = M_\alpha(t)\Phi$$

(by definition) and

$$\frac{\partial \Phi}{\partial t^\alpha}\exp(-B_\beta t^\beta) = M_\alpha(t)\Phi(t)\exp(-B_\beta t^\beta) = M_\alpha(t)P(t).$$

Taking the partial derivative of the function

$$P(t) = \Phi(t)\exp(-B_\beta t^\beta),$$

we find

$$\begin{aligned}\frac{\partial P}{\partial t^\alpha} &= \frac{\partial \Phi}{\partial t^\alpha}\exp(-B_\beta t^\beta) - \Phi\exp(-B_\beta t^\beta)B_\alpha \\ &= M_\alpha P - PB_\alpha,\end{aligned}$$

The substitution gives

$$\frac{\partial y}{\partial t^\alpha} = P^{-1}(M_\alpha P - M_\alpha P + PB_\alpha)y = B_\alpha y.$$

In this way, the Floquet point of view brings about an important simplification: the linear PDE system (1) is reduced to the linear constant coefficients PDE system (2).

4. Spectral theory

The eigenvalues ρ of the monodromy matrix C are called characteristic multipliers, and the complex vector $\lambda = (\lambda_\beta)$ defined by $\rho = \exp(\lambda_\beta T^\beta)$ is called *characteristic vector* or *Floquet exponent vector*.

The complete integrability conditions show that the matrices B_α commute. Let λ_α be the generic eingenvalue of the matrix B_α, *i.e.*, a solution of eigen-vector-value equation $(B_\alpha - \lambda_\alpha)u = 0, u \neq 0$, and ρ be the eigenvalue of the matrix $C = \exp(B_\beta T^\beta)$, *i.e.*, a solution of eigen-vector-value equation $(C - \rho)v = 0, v \neq 0$. Assuming that each B_α and C are simple matrices, then using the canonical forms $V^{-1}B_\alpha V = diag(\lambda_\alpha)$, for each α, and $T^{-1}CT = diag(\rho)$, it follows $diag(\rho) = \exp(\lambda_\beta T^\beta)$.

A necessary and sufficient condition for asymptotic stability is that the characteristic vectors λ have negative real part or equivalently ρ have modulus less than 1.

Theorem *The relations*

$$\det C = \Pi\rho = \exp\left(\int_0^T trB_\alpha dt^\alpha\right)$$

$$trB_\alpha T^\alpha = \sum \lambda_\alpha = \int_0^T \left(trM_\beta dt^\beta\right)$$

hold true. The components λ_α of the characteristic vectors $\lambda = (\lambda_\alpha)$ are determined respectively modulo $\frac{2\pi i}{T^\alpha}$.

Proof If we start from $\Phi(t)$, with $\Phi(0) = I$, we find

$$\det\Phi(t) = \det(P(t)\exp(B_\beta t^\beta)) = \det(P(t))\exp(\text{tr }(B_\beta t^\beta)).$$

On the other hand $P(T) = \Phi(0) = I$ and

$$\det\Phi(t) = \text{wronkian} = \exp\left(\int_0^t \text{tr}M_\beta ds^\beta\right).$$

5. Simplifying the controllability of multitime linear PDE systems

To the homogeneous PDE system

$$\frac{\partial x}{\partial t^\alpha}(t) = M_\alpha(t)x(t),$$

we associate a controlled linear PDE system

$$\frac{\partial x}{\partial t^\alpha}(t) = M_\alpha(t)x(t) + N_\alpha(t)u(t). \tag{3}$$

Now we want to underline that the controllability of the linear PDE system (3) can be reduced to the controllability of constant coefficient linear PDE system (for the controllability of linear PDE systems, (see Tu 1994; Udrişte 2008b; Udrişte and Ghiu 2012).

Theorem *The controlled PDE system (3) is equivalent to the constant coefficient controlled PDE system*

$$\frac{\partial y}{\partial t^\alpha}(t) = B_\alpha y(t) + C_\alpha u(t), \tag{4}$$

where $\Phi(t)$ is the fundamental matrix of the PDE system (1) and $C_\alpha(t) = \exp(B_\beta t^\beta)\Phi^{-1}(t)N_\alpha(t)$.

Proof We use the substitution $x(t) = P(t)y$, where $\Phi(t) = P(t)\exp(B_\beta t^\beta)$, and commutation of the matrices M_α and $\exp(B_\beta t^\beta)$.

Some of our results regarding the controllability and observability of multitime linear PDE systems (see Tu 1994; Udrişte 2008b; Udrişte and Ghiu 2012) can be improved via the foregoing Theorem.

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* University Politehnica of Bucharest
Faculty of Applied Sciences, Department of Mathematics-Informatics
Splaiul Independentei 313, 060042 Bucharest, Romania

Email: udriste@mathem.pub.ro

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