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ON THE LAGRANGIAN BEING A HOMOGENEOUS FUNCTION OF THE VELOCITY

A METHODOLOGICAL NOTE

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ABSTRACT. In this contribution a careful critical reading of the feasibility to express a Lagrangian function as the sum of several terms each having a different degree of homogeneity with respect to the velocities is presented. Arguments are proposed so to overcome some of the involved difficulties, addressing to visualize a dynamical evolution process of a system of given Lagrangian $\mathscr{L}$ when interacting with an environment as the breaking of the degree of homogeneity with respect to the velocities of the Lagrangian itself due to the dynamical coupling to the environment. Specific attention is devoted to systems the Lagrangians of which are of degree of homogeneity equal to one.

1. Introduction

> *"I would like to begin in an elementary way and I take as my starting point an action principle. That is to say, I assume that there is an action integral which depends on the motion, such that, when one varies the motion, and puts down the conditions for the action integral to be stationary, one gets the equations of motion. ... We start off with an action integral which I denote by $\mathscr{I} = \int \mathscr{L} dt$. It is expressed as a time integral, the integrand $\mathscr{L}$ being the Lagrangian.* ***So with an action principle we have a Lagrangian.*** *. . . You might wonder whether one could not take the Hamiltonian as the starting point and short-circuit this work of beginning with an action integral, getting a Lagrangian from it and passing from the Lagrangian to the Hamiltonian."(Dirac 1964)*

Once more the methodological perspective underlying this contribution can be traced back to the point of view put forward by Giaquinta (2009, 2012) as about the feasibility to obtain "well known, old, familiar, fundamental results" (Giaquinta 2009) from a point of view apparently different with respect to the most widely followed ways as expounded into the current literature (Landau and Lifshitz 1982). To this end, "we investigate what can be deduced from some *a priori* properties of the Lagrangian formalism (as distinct from the Hamiltonian one) as they can be stated by the simple fact that some given basic identities have to be satisfied . Such attitude amounts to underline the eidetic, apodictic character of Lagrangian method instead of its euristic-pragmatic potentialities" particularly when the choice of the new perspective can result best suited to avoid misunderstandings or to be more convenient to visualize and clarify controversial aspects (Giaquinta 2012). We note, in passing, that such a philosophy does not exhaust itself into a mere pedagogical aim: indeed, as we shall see, new originals findings can be obtained from a deeper insight into the basic principles at the very hearth of a theory, even when this last is a well established one. To be more specific our main purpose in this paper is to ascertain to what extend the decomposition of the Lagrangian function as a sum of several terms each having a different degree of homogeneity with respect to the velocity can be thought of in a consistent way. Indeed it will be shown that the feasibility of such occurrence can be viewed as more or less strongly dependent on "when" and "how" the Euler-Lagrange dynamical equations are stated, so to suggest the adoption of proper cautions due to the presence of given *caveat*.

2. The Euler-Lagrange equations "derived"

Well, let us consent to Dirac's definition of the Lagrangian functional $\mathscr{L}$ as the integrand of the Action integral expressed as a time integral and try to deduce some consequences from this statement alone. Firstly, on account to the fact that the spectrum of the Action integral is not negative (so to be bounded from below *naturaliter*) and that a minimum value h, the quantum of Action, does exhist we can state that a variational program can safely be undertook as a well founded procedure provided we impose the restriction $\mathscr{L} \geq 0$ at least almost everywhere due to the fact that $t_f > t_i$, these last being the extremes of the Action integral. Let us note that insofar we are concerned with the "Lagrangian side" of the question, *i.e.*, our dynamics lives in the Space of Configurations, we are not too much involved with the fact that the physical Action may be quantized as it happens to be: for our purpose it is enough that it possesses a minimum value. After these institutional preliminaries have been taken into account we can state that for *any* choice $t : \rightarrow q^i(t)$ [$q^i(t)$ being a piecewise continuos function] the Lagrangian functional becomes a function of time $\mathscr{L}(t)$ and we have *identically*

$$\frac{d\mathscr{L}}{dt} = \dot{q}^i \cdot \frac{\partial \mathscr{L}}{\partial q^i} + \ddot{q}^i \cdot \frac{\partial \mathscr{L}}{\partial \dot{q}^i} + \frac{\partial \mathscr{L}}{\partial t} \tag{1}$$

After the formal elimination of the accelerations $\ddot{q}^i$ via the Leibnitz's chain rule

$$\ddot{q}^i \left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i} \right) \equiv \frac{d}{dt} \left[\dot{q}^i \left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i} \right) \right] - \dot{q}^i \frac{d}{dt} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i} \right) \tag{2}$$

has been carried out, we obtain the equation:

$$\frac{d}{dt} [\mathscr{E}] \equiv \frac{d}{dt} \left[\dot{q}^i \frac{\partial \mathscr{L}}{\partial \dot{q}^i} - \mathscr{L} \right] = -\dot{q}^i \left[\frac{\partial \mathscr{L}}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i} \right) \right] - \frac{\partial \mathscr{L}}{\partial t} \tag{3}$$

confirming Dirac's hierarchy (Dirac 1964):

Action integral $\Longrightarrow$ Lagrangian $\Longrightarrow$ Energy

Simply by a careful inspection we can appreciate that we obtained a *balance equation* where the total time rate of change of the newly introduced Energy functional

$$\mathscr{E}\left[q^i(t),\dot{q}^i(t),t\right] \equiv \dot{q}^i \frac{\partial \mathscr{L}}{\partial \dot{q}^i} - \mathscr{L}(q^i(t),\dot{q}^i(t),t)$$

equates the negative of the sum of two terms each representing a different kind of contribution, namely a path-dependent term, the other being explicitly time dependent, respectively. It is worth noticing that *the sign minus appears intrinsically and it is not fruit of any convention*. Energy conservation is thus restrict to the Lgrangian being a function not explicitly time dependent ($\frac{\partial \mathscr{L}}{\partial t} = 0$) jointly to the first term of the r.h.s. of Eq. (3) being zero. This last occurrence can be verified under two distinct circumstances:

(a) the chosen curve is the "classical path", *i.e.*, that unique curve that satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}^i} = \frac{\partial \mathscr{L}}{\partial q^i} \tag{4}$$

(b) the chosen curve is such that the velocities and accelerations are orthogonal. Indeed we would claim that

$$\alpha\left(q^i\right)\dot{q}^i\ddot{q}^i = 0 \rightarrow \alpha\left(q^i\right)\frac{d}{dt}\left(\frac{\left(\dot{q}^i\right)^2}{2}\right) = cost \tag{5}$$

$\alpha\left(q^i\right)\ddot{q}^i$ being the factor on front of $\dot{q}^i$ ($\alpha\left(q^i\right) > 0$) in Eq. (3).

Inasmuch as the so defined Energy function is a physical quantity, *i.e.*, it *possesses physical dimensions*, the two composing terms $\dot{q}^i\frac{\partial \mathscr{L}}{\partial \dot{q}^i}$ and $\mathscr{L}$, respectively, have to be both physically and mathematically homogeneous so that we can state under quite general circumstances that *the Lagrangian function has to be a homogeneous function of the velocities*. Indeed let us remind the reader that the theorem of Euler on homogeneous functions

$$x^i\frac{\partial f}{\partial x^i} \equiv kf \qquad \text{for} \qquad f\left(\alpha x^i\right) = \alpha^k f\left(x^i\right) \tag{6}$$

realizes a condition both necessary than sufficient so that each of the two quoted equalities can be looked at as a proper definition of homogeneous function. From this fact we can assess that the Lagrangian has the *same physical dimension of an Energy* as can be realized by the simple fact that under appropriate conditions

$$\frac{\partial \mathscr{L}}{\partial t} = -\frac{d\mathscr{E}}{dt} \tag{7}$$

whereas generally speaking *it is not an Energy*: indeed when time can be considered as homogeneous ($\frac{\partial \mathscr{L}}{\partial t} = 0$) and along the "classical path" *energy is preserved but not the Lagrangian*. This is the most relevance when we compare with what happens along the "Hamiltonian side" of the question. Indeed in that case if $\frac{\partial \mathscr{L}}{\partial t} = 0$ then $\frac{\partial \mathscr{H}}{\partial t} = 0$ so that even $\frac{d\mathscr{H}}{dt} = 0$ (in force of the fact that the Poisson bracket $\{\mathscr{H}, \mathscr{H}\}_{PB} \equiv 0$ and that $\frac{d\mathscr{H}}{dt} = \frac{\partial \mathscr{H}}{\partial t} + \{\mathscr{H}, \mathscr{H}\}_{PB}$): *this means that the Hamiltonian is preserved, but the Hamiltonian is the Energy*. Note that this last statement holds when the involved functions depend on generalized coordinates and momenta as linearly independent state variables, *i.e.*, we live in Phase Space and not in the Configuration Space. As a consequence the concept of Energy as introduced by the Lagrangian approach could be looked at of much larger meaning than that of the Hamiltonian function. At this stage the main objection that could be raised against what we have exposed until now is that we have restated well known results and nothing else, particularly if we take into account Section 2.7 of the treatise by Goldstein, Poole, and Safko (2002). However we still vindicate our perspective as not trivial at all, essentially from the methodological point of view but not uniquely. Indeed the occurrence of the feasibility to decompose the Lagrangian of a system into the sum of three terms with different degrees of homogeneity with respect to the generalized velocities $\dot{q}^i$ (two, one and zero respctively, *i.e.*, a "kinetic energy" term, a "coupling energy" term and a "potential energy" term as they are generally named) is one of the main consequences of that sort of approach to the dynamics of the system which is technically envisaged as the "inverse problem" approach when the involved constrains, describing the embending environment, are non holonomic, *i.e.*, they are explicitly time-dependent. Indeed if one *starts* from second order differential equations with respect to time which are assumed as the "equations of motion" written in terms of the spatial cartesian coordinates of a given subset of $\Re^N$ and looks for the Lagrangian by *partial integration* in most cases one is lead to an expression like

$$\mathscr{L}\left(r^{\alpha}(t); v^{\alpha}(t); t\right) = \mathscr{M}_{\alpha}\left(r^{\alpha}(t); t\right) \frac{v^{\alpha 2}}{2}, \quad \mathscr{M}\left(r^{\alpha}(t); t\right) \geq 0, \ \ \alpha = 1, 2, \ldots, N \qquad (8)$$

Then, if one resorts to generalized coordinates

$$r^{\alpha}(t) = r^{\alpha}\left[\ldots q^i(t) \ldots ; t\right], i = 1, \ldots, (3N - f), \ \ f \equiv numbers\ of\ constrains \qquad (9)$$

the Lagrangian, which originally is a homogeneous function of degree two with respect to the "old" velocities v^{α}, is splitted into the sum of three terms of the "new" generalized velocities $\dot{q}^i$, but no contradiction is encountered insofar the different degrees of homogeneity refer to *distinct* variables and *a priori* we have not any statement about the homogeneity of the Lagrangian itself. What we exposed is different at least under two main points:

(a) we choosed generalized coordinates *ab initio*,
(b) the adopted deductive procedure utilizes identities and nothing else, after starting from the arbitrary choice of any path $t : \rightarrow q^i(t)$.

It is worth noticing that the main finding was the feasibility to state on the same footing (*i.e.*, simultaneously) both the Euler-Lagrange equations and the *status* of the Lagrangians as being a homogeneous function of the velocities.

Before closing this section let us note how the followed procedure could be looked at as a cousin of the family of the variational procedures, particularly those inspired by the formulation of Schwinger (1951), even if we do not follow the most general transformation tecnique including time transformation borrowed from Poincare (Dittrich and Reuter 2001; Ramond 1990).

3. Homogeneity versus additivity of Lagrangians: a few considerations

According to what previously stated in any circumstance where we would be led to take into account of Lagrangian functions $\mathscr{L}$ arising from the sum of terms $\mathscr{L}^i$ each having a different degree of homogeneity with respect to the velocities, we could concile such occurrence with the requirement of $\mathscr{L}$ being a homogeneous function of the velocities *assuming* that the corresponding degree of homogeneity K be the scale-factor dependent quantity

$$K = \frac{1}{(\ln\alpha)} \cdot \ln\left\{ \frac{\mathscr{L}\left(\alpha \dot{q}^i\right)}{\mathscr{L}\left(\dot{q}^i\right)} \right\}, \quad \alpha > 0 \tag{10}$$

each term in the r.h.s. being well defined. This “scaling hypothesis” is remnant of the analogous “scaling hypothesis” referred to “Potential Energy functions”, particularly when, as we shall see shortly, we can introduce an invariant velocity with finite modulus so to be able to consider various space-time cuts and consequently be allotted to visualize different dynamical regimes. For the purposes of this contribution the systematic development of this point and the detailed comparison with the current scenario can be postponed to forthcoming papers, whereas it appears to be more proficous to better ascertain where the additivity of the Lagrangians can be generated and how to interprete these findings so to be consistent with the requirement of homogeneity. In any case let us note how this “scaling hypothesis” could authorize us to speak of Symmetry and consequently of Broken Symmetry when looking at a dynamical process as an interaction process between a system and an environment. Looking at the Euler-Lagrange equations as they stand, *i.e.*, just in force of their “formal status” (independently of the procedure that has been followed to obtain them) some basic facts emerge, namely

(a) the dynamical equivalence of Lagrangians differing by a multiplicative factor

$$\mathscr{L} \to \mathscr{L}' = A\mathscr{L}, \quad A > 0 \tag{11}$$

or additively by the total time derivative of an arbitrary function of coordinates and time

$$\mathscr{L}\left(q(t);\dot{q}(t);t\right) \to \mathscr{L}' = \mathscr{L}\left(q(t);\dot{q}(t);t\right) + \frac{d}{dt} f\left(q(t);t\right) \tag{12}$$

insofar the dynamica equations are unaffected under the aforementioned transformations,

(b) the *momentum* $p_i \equiv \frac{\partial \mathscr{L}}{\partial \dot{q}^i}$ *looks to be invariant under the transformation*

$$\mathscr{L}\left(q(t);\dot{q}(t);t\right) \to \mathscr{L}' = \mathscr{L}\left(q(t);\dot{q}(t);t\right) + \Phi\left(q(t);t\right) \tag{13}$$

$\Phi(q)$ being a purely configurational term, *whereas its time rate of change $\dot{p}_i$ is drastically changed.*

Only the first property under a) looks to be consistent with the requirement of homogeneity insofar the degree of homogeneity is trivially mantained independently of its particular value. For the other occurencies a more detailed inspection is in order. Let us try to investigate some aspects.

Physically speaking the most natural Lagrangians that can be *a priori* conceived to lead to consistent equations of motion are those with degrees of homogeneity with respect to the velocities, two and one respectively. Let us briefly consider them under some of the aspects we are mainly interested in. For *Lagrangian with degree two* we immediately obtain the *fundamental result that the Lagrangian coincides with the corresponding Energy*. Than, the main consequence of the dynamical equivalence stated under Eq. (12) appears to be a symmetry breaking phenomenon insofar the term

$$\frac{d}{dt} f\left(q^i;t\right) \equiv \dot{q}^i \frac{\partial f}{\partial q^i} + \frac{\partial f}{\partial t} \tag{14}$$

introduces *ipso facto* two terms of degree of homogeneity one and zero, respectively, that can be reduced to the unique term of degree one only in the restricted case where the arbitrary function f does not depend on time explicitly. To summarize we obtain a symmetry-breaking phenomenon that modifies both energy and momentum but does not change the equation of motion. Note how this item realizes the converse of what is generally termed a Spontaneously Broken Symmetry phenomenon (Felsager 1998). A quite different scenario opens when we consider Lagrangians with degree of homogeneity equal to one. Firstly let us note that *the corresponding Energy is zero*. In turn this fact implies that, under the effect of the transformation (12), zero energy is retained only when the function f does not depend explicitly on time, whereas, if $\partial_t f \neq 0$, $\mathscr{E}' = -\frac{\partial f}{\partial t}$. In any case *momentum is altered*: $p_i \rightarrow p_i + \frac{\partial f}{\partial q^i}$, *although the original symmetry of the Lagrangian remains unchanged* even when the partial derivative of f with respect to time is different from zero. As it is well known this is at the root of the Gauge-Invariance requirement (Yang and Mills 1954) that plays a heavy role in the fundations of the Electromagnetic Theory as recently underlined by Giaquinta (2012) and Giaquinta and Coco (2013). Among the various consequences of the Lagrangian being a homogeneous function of degree one with respect to the velocity there is the *statement that time cannot be absolute*. Indeed, under a time transformation $t \rightarrow t^*$, the time derivative changes according to $\frac{d}{dt} = \left(\frac{dt^*}{dt}\right)\frac{d}{dt^*}$ so that the scale factor α such that $\dot{q}^* = \alpha\dot{q}$ is given by $\frac{dt}{dt^*}$. If, as assumed, $\mathscr{L}^*\left(\dot{q}^*\right) = \alpha\mathscr{L}\left(\dot{q}\right)$, than the action integral $\int \mathscr{L}^* dt^* = \int \mathscr{L} dt$ is invariant, as it has to be, showing unambigously that time cannot be absolute. Among the consequences, the dependence of the theory on the ratios of the involved velocities so that cut-off maximum velocities have to be introduced up to a maximum invariant velocity with finite modulus. We close this section with a brief comment on the ligth that can be shaded by these findings on the questions posed by the indeterminary under Eq. (13), *i.e.*, the introduction of the potential energy function. To this end consider how starting from any configurational function $\Phi(q)$ and any time t we can construct a corresponding f function via the simple trick

$$\Phi\left(q(t);t\right) \rightarrow f\left(q(t);t\right) = t \cdot \Phi\left(q(t)\right) \tag{15}$$

so that

$$\frac{d}{dt} f(q(t)) = \Phi(q(t)) + t \cdot \frac{d\Phi}{dt} = \Phi(q(t)) + \frac{d}{dt}\left(\frac{t^2}{2} \cdot \frac{d\Phi}{dt}\right) - \frac{t^2}{2} \cdot \frac{d^2\Phi}{dt^2} = \Phi(q(t)) + \frac{d}{dt}\left(\frac{t^2}{2} \cdot \frac{d\Phi}{dt}\right) - \frac{d}{dt}\left(\frac{t^3}{2 \cdot 3} \cdot \frac{d^2\Phi}{dt^2}\right) + \frac{t^3}{2 \cdot 3} \cdot \frac{d^3\Phi}{dt^3} = \Phi(q(t)) + \cdots + \frac{d}{dt}\left(\frac{t^{(i-1)}}{i!} \cdot \frac{d^{(i-2)}\Phi}{dt^{(i-2)}}\right) + (-1)^i \frac{d}{dt}\left(\frac{t^i}{i!} \cdot \frac{d^{(i-1)}\Phi}{dt^{(i-1)}}\right) + (-1)^{(i+1)} \frac{t^i}{i!} \cdot \frac{d^i\Phi}{dt^i}$$

When any total time derivative is expressed conformly to Eq. (13) we see that each term in the r.h.s., with exception of the first one, contributes weigthed by powers of velocity ratios faster and faster going to zero so to be rapidly negligible: the philosophy of the Gauge Equivalence seems to be restored.

4. Conclusions and perspectives

Throughout this paper we have been concerned with some problems arising, in our opinion, when considering Lagrangian functions built up by various terms each having a specific degree of homogeneity with respect to the involved velocities, insofar according to the methodological procedure adopted by us, the *Lagrangian as a whole* should be a homogeneous function of the velocities. Obviously this does not mean that we restrict ourselves to "kinetic" terms only, nor that we reject *tout-court* the criterion of *additivity* of Lagrangians for complex, interacting systems. For the sake of completeness let us remind how we have been induced to these considerations so to adds some comments and reflections, even if our procedure could be judged mainly as the consequence of an apparently trivial formal fact. Indeed when considering the total time derivative of the Lagrangian function along the path $q^i(t)$, if the coefficient $\frac{\partial \mathscr{L}}{\partial q^i}$ appearing in the r.h.s. of Eq. (1) is substituted by the term $\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i}\right)$, Eq. (7) is immediately obtained. Let us remind that we are authored to utilize this *equality* only if we are along the classical path, *i.e.*, *after* having established the Euler-Lagrange equations in some a way, whatsoever it may be. In this case the formal properties to which the Lagrangian function has to fulfil are dictated *only* by the formal properties deriving from the structure in itself of the equations of motion. This could be considered as consistent insofar the equations by Euler and Lagrange can be looked at as "intrinsic", *i.e.*, due to their being form covariant under any arbitrary transformation of the coordinates insofar the ways under which the velocities are transformed accordingly do not alter the form of the equations altogether (Kleinert 2009). Vice versa adopting the *identity* under Eq. (2) to express the term $\ddot{q}^i \frac{\partial \mathscr{L}}{\partial \dot{q}^i}$ appearing in the r.h.s. of Eq. (1) we have "symmetrized" the equation in the sense that only q^i and $\dot{q}^i$ functions are involved (as it happens into the Lagrangian) and we have stressed the simple fact that Eq. (1) has to be *identically* satisfied for *any* function $q^i(t)$.

To conclude, we can assess that the declared purpose to deserve a future engaging to develop a systematic and detailed examination of what we have offered to the attention of the readers so to get a deeper insight is not a formal, ritual and rethoric promise. To this end the cooperation of the interested reader through observations, criticism and suggestions will be well accepted and we kindly acknowledge in advance. In any case we hope to have been successful to show our methodological attitude to follow the hierarchical chain

$$\textit{a priori}\text{ statement} \Longrightarrow \text{scaling requirement} \Longleftrightarrow \text{symmetry}$$

that was skilfully synthetized by Weyl (1952) under the two statements:

> *"As far as I see, all **a priori** statements in physics have their origin in symmetry."*

> *"Symmetry is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection."*

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