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A MATHEMATICAL-PHYSICAL FORMULATION OF A MESOSCALE LIMITED AREA MODEL IN METEOROLOGY

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ABSTRACT. In this paper we give a detailed mathematical-physical derivation of a limited area meteorological model, used in several applications, describing the motions and the behaviour of the atmosphere, in the *troposphere layer*, at *mesoscale range*, modeled as a mixture of compressible, heat conducting, non viscous components. The mixing ratios of water vapor, cloud water, rainwater, ice, ..., referred to the dry air density are introduced. We work in a non inertial reference frame, solidal to the rotating Earth, where Coriolis force is neglected, being *Rossby number* at the considered mesoscale much bigger than 1. We derive the momentum balance equation, the conservation law for the dry air density, the balance equation for the temperature and the balance equations for the densities of the fluid mixture components in two cases: when a local rectangular Cartesian reference frame, called z-system, solidal to the rotating Earth is used and when a new generalized nondimensional η-system is introduced, by means of the hydrostatic pressure. Detailed calculations (with particular assumptions) are worked out, starting from the case where the atmosphere is modeled as a perfect fluid, the dry air.

1. Introduction

In this contribution a mathematical-physical formulation of a mesoscale limited area meteorological model used in several applications (see Skamarock *et al.* 2008; Caccamo *et al.* 2017; Castorina *et al.* 2018); is given in a non-inertial frame of reference solidal to the rotating Earth, where Coriolis force is neglected, being *Rossby number*, ε, at the considered mesoscale range, $\varepsilon \gg 1$. Particular assumptions are done and detailed calculations are performed. Two types of coordinate systems are introduced, the z-system (where a local rectangular Cartesian reference frame solidal to the rotating Earth is used) and a new η-system, that has a different vertical coordinate. The vertical η coordinate, introduced by means of the hydrostatic pressure, permits, in particular, to consider the Earth surface as a pressure coordinate surface. For a better understanding of the complexity of the general mathematical models governing the atmosphere dynamics we refer to the paper by Georgescu (2017b) (see also Temam and Ziane 2004; Georgescu 2009, 2017a), where we read "*Meteorology is a domain of study which must take into account the complex features* *of the atmosphere fluid and its complicated motion in various domains and at various scales. It is mainly based on thermodynamics of fluids and uses several corresponding mathematical models to describe these .... fluid flows.*" In Georgescu (2017b) the different layers of the atmosphere are presented: the *troposphere layer* (from the ground to a height of 8-20 km), situated near the Earth, influenced by its thermal state and that permits, as an engine, the transport of the heat (coming from the Sun) from the Equator to the poles, due to the temperature gradient that is bigger in the winter and depends on the seasons; *stratosphere* (up to about 50 km), *mesosphere* (up to about 80 km), *thermosphere* (up to about 550 km) and *exosphere* (over 550 km).

The troposphere layer, in direct contact with the earth's surface, has a variable thickness depending on the latitude: at the poles it is thick only 8 km, while it reaches 20 km at the equator. In this layer most of the meteorological phenomena occur, caused by the circulation of air masses. Most of the air pollutants emitted remain confined to the troposphere, some are concentrated near the Earth's surface. It is mainly heated by the Earth. It follows that the temperature decreases with altitude and varies from 15 ^oC to -70 ^oC. The temperature stabilizes around -60 ^oC in a very thin layer, between the troposphere and the stratosphere (the *tropopause*). The other layers of the atmosphere are interesting for applications in other different science fields. In the horizontal directions of the atmosphere four scales are distinguishable, with different dimensions of the domains of atmosferic motions, where particular kinds of flows are present. The model treated in this paper is a *limited area model*, describing the motions and the behaviour of the atmosphere, in the *troposphere layer*, at *mesoscale range*, as a mixture of compressible, heat conducting, non viscous components (water vapor, cloud water, rainwater, ice and others). This model is called *Weather Research Forecast* (WRF) (see Skamarock *et al.* 2008, Holton 2004, Klemp, Skamarock, and Dudhia 2007, Pielke 2002, Thunis and Bornstein 1995), whose spatial horizontal scale is from a few kilometers until to 100 kilometers. In this scale the studied motions present fronts, forced flows and also turbulent flows. Here, we omit considerations about the atmospheric Eckman boundary layer (Georgescu 2017b).

The planetary scale has a range from 4.000 km to 40.000 km and in this scale the motions studied regard the tides, the planetary waves and others.

The synoptical scale has a range from 1.000 km to 4000 km and in this scale the studied motions are the cyclons, the anticyclons, the hurricanes and others (Georgescu 2017b).

The mesoscale is the scale of our interest, which actually includes a range 10 Km - 1000 Km.

The local scale is until a few kilometers and here the pollution and urban meteorology is treated.

In this paper in Section 2 we derive the model equations, describing the physical behavior of a compressible, heat conducting, non viscous Eulerian fluid (in particular the perfect fluid dry air), in a non-inertial local rectangular Cartesian reference frame solidal to the rotating Earth, called z-system, solidal with the Earth rotating around its axis passing through the two South and North poles. The basic assumptions of the model are specified. In Section 3 we introduce a new generalized vertical coordinate s and the two reference frames z-system and s-system (in which only the vertical coordinate is different). Extensive calculations are done to express the partial and time total derivatives, the divergence operator and other operations using the s-coordinate and the z-coordinate. In Section 4 a new η-system is introduced, where the vertical coordinate s is expressed by means of the hydrostatic pressure. A geopotential and a pseudodensity are defined and the momentum balance equation, the conservation law for the fluid density, the balance equation for the temperature and the evolution equation for the geopotential are derived in z-system and η system. Finally, in Section 5 a complete formulation of the limited area model WRF is derived using the η-system, where the whole fluid is a mixture consisting of different compressible, heat conducting, non viscous components, having own density. The mixing ratios of water vapor, cloud water, rainwater, ice, ..., referred to the dry air density are defined and detailed computations are given to derive the model equations. The model under consideration is used in important meteorological applications, see for instance Castorina *et al.* 2018 and Caccamo *et al.* 2017, where simulations of heavy snowfalls and rainfalls events in Sicily were done, respectively.

2. Model equations

This Section deals with a detailed mathematical-physical approach to derive a mesoscale limited area model WRF. We assume that the atmosphere in the *troposphere layer* can be modeled as a compressible, heat conducting Eulerian fluid. In a current Eulerian configuration K_t at the time t, we first give a description referring to an *inertial* rectangular Cartesian levorotatory reference frame $T' \equiv (O', x', y', z')$, with O' coinciding with the center of the Earth and the z' axis coincident with the Earth's rotating axis, then a description referring to a *non inertial* rectangular Cartesian reference frame $T = (O, x, y, z)$, having its origin O coinciding with O', solidal with the rotating Earth with angular velocity $\boldsymbol{\omega}$, passing through the Earth's axis. Finally, we give a description referring to a *local non inertial* rectangular Cartesian reference frame $T^* \equiv (P, x^*, y^*, z^*)$, being T^* a levorotatory triad with respect to Earth SN (South-North), with the origin in a point P of the northern hemisphere of the Earth. In the following, to simplify the formalism we continue to call the reference frame $T^* \equiv (P, x^*, y^*, z^*)$ with the name $T \equiv (P, x, y, z)$.

In the Subsections 2.1-2.3 we treat this approach in full details.

2.1. Model equations respect to an inertial frame T'.

In this Subsection we consider an *inertial* rectangular Cartesian levorotatory reference frame $T' \equiv (O', x', y', z')$, having the origin O' coinciding with the center of the Earth, the z' axis coincident with the Earth's rotating axis and unit vectors of the basis $\mathbf{i}', \mathbf{j}', \mathbf{k}'$, with $\mathbf{k}'$ the unit vector of the $\mathbf{z}'$ axis (see Fig. 1).

The governing equations, in a current Eulerian configuration K_t at the time t, describing the motions and the behavior of the atmosphere, modeled in a first approach as a gas (in particular the perfect gas dry air), are given in the inertial reference frame T', in particular the momentum balance equation, the mass conservation equation and the temperature

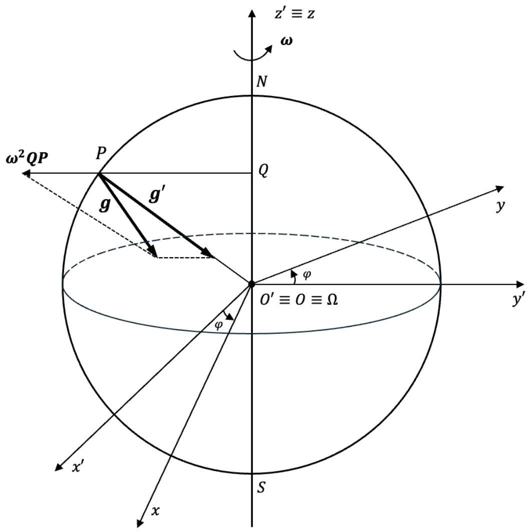

FIGURE 1. "The inertial rectangular Cartesian reference frame $T' \equiv (O', x', y', z')$, considered in Subsection 2.1, and the non-inertial rectangular Cartesian reference frame $T = (O, x, y, z)$, solidal to the rotating Earth, considered in the Subsection 2.2."

balance equation for the considered gas, having the following form:

$$\begin{cases} \rho' \left[\dfrac{\partial \mathbf{v}'}{\partial t} + \mathbf{v}' \cdot \nabla' \mathbf{v}' \right] = -\nabla' p' + \mathbf{F}' + \rho' \mathbf{g}', \\ \dfrac{\partial \rho'}{\partial t} + \nabla' \cdot (\rho' \mathbf{v}') = 0 \\ \dfrac{\partial \theta'}{\partial t} + \mathbf{v}' \cdot \nabla' \theta' = F'_{\theta'} \end{cases} \tag{1}$$

where ρ', $\mathbf{v}'$, θ' and p' are the fields of mass density, velocity, temperature, and pressure, respectively, depending on $(\mathbf{x}', t)$, $\mathbf{x}' \equiv (x', y', z')$ measured in the reference frame T'. The field $\mathbf{F}'$ represents the external force acting on the atmosphere, and $\mathbf{g}'$ is the gravity acceleration, having its application line passing through the center of the earth, assumed constant.

Equation $(1)_3$ is equivalent to the internal energy balance equation. In $(1)_3$ $F'_{\theta'}$ is an heat source. In (1) the symbol " ∇' " is the gradient operator calculated in T', for instance $\nabla' A' \equiv (\frac{\partial A'}{\partial x'}, \frac{\partial A'}{\partial y'}, \frac{\partial A'}{\partial z'})$, where A' is an arbitrary scalar field, measured in T' and the symbol "$\nabla' \cdot$ " is the divergence operator computed in T', for instance $\nabla' \cdot \mathbf{A}' \equiv \frac{\partial A'_{x'}}{\partial x'} + \frac{\partial A'_{y'}}{\partial y'} + \frac{\partial A'_{z'}}{\partial z'}$, with $\mathbf{A}' \equiv (A'_{x'}, A'_{y'}, A'_{z'})$.

The inertial reference frame T' moves with a rectilinear and uniform translational motion with respect to an inertial reference frame solidal with the Sun, which moves with a uniform rectilinear translational motion towards the constellation of Hercules. Furthermore, it can be assumed that the Solar system moves with a uniform rectilinear translational motion within our Galaxy (the Milky Way) following an almost circular orbit around the center of the Galaxy. The universe is also expanding and our galaxy is also moving.

The movement of revolution of the Earth around the Sun for small time intervals with respect to the duration of the solar year can be considered uniform rectilinear translational in good approximation.

Furthermore, it is assumed that the the regular precession motion of the Earth can be neglected with respect to its diurnal rotation.

This is due to the smallness of the precession speed compared to the rotation speed, since the *figure axis*, coinciding with the Earth's rotation axis, takes about twenty-six thousand years (Platonic year) to complete a full turn around the *precession axis*. Also, the Earth is assumed to be spherical with density depending only on its center.

2.2. Model equations respect to an non inertial frame T. In this subsection we consider a non inertial rectangular Cartesian reference frame T solidal with the rotating Earth (see Holton 2004).

In a first step of our modelization we refer to a current Eulerian configuration K_t at the time t, in a *non inertial levorotatory reference frame* T, $T = (O,x,y,z)$, having its origin O coinciding with O', $O \equiv O'$, and the z axis coincident with the z' axis of T', $z \equiv z'$, representing the Earth's axis around which the earth rotates, with unit vectors of basis $\mathbf{i},\mathbf{j},\mathbf{k}$ (see Fig. 1). The Earth's axis is an imaginary line passing through the two Earth's poles, South, S, and North, N, around which the Earth rotates. This rotation is counterclockwise for an observer located in the northern hemisphere and has angular velocity equal to 360^0 in 24 solar hours is $\omega = 7.292 \cdot 10^{-5}$ radians/sec. In Fig. 1 the pseudovector $\boldsymbol{\omega}$ has the direction and towards of the Earth axis and intensity ω. The plane perpendicular to the Earth's axis passing through the center of the Earth intersects the surface of the Earth, determining a maximum circle, whose circumference is called Equator. It divides the Earth into two hemispheres: northern or boreal on the north pole side, southern or austral on the south pole side. In the non inertial rotational motion of the Earth a point P of the continuum medium under consideration (the atmosphere) has an acceleration vector $\mathbf{a}'$ calculated with respect to the inertial reference frame T' given by

$$\mathbf{a}' = \frac{d\mathbf{v}'}{dt} = \frac{d\mathbf{v}}{dt} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times OP) + \dot{\boldsymbol{\omega}} \times OP + 2\boldsymbol{\omega} \times \mathbf{v}, \tag{2}$$

where the velocity vector field $\mathbf{v}'$ is calculated in the inertial reference frame T' and the velocity vector field $\mathbf{v}$ and its time derivative $\frac{d\mathbf{v}}{dt}$ are measured in T. Furthermore, we have taken into account that the translational acceleration is null because the motion is rotational. Considered the projection point Q of P on the rotation axis, being $OP = OQ + QP$, equation (2) takes the form (see Fig. 1)

$$\mathbf{a}' = \frac{d\mathbf{v}}{dt} - \omega^2 QP + \dot{\boldsymbol{\omega}} \times OP + 2\boldsymbol{\omega} \times \mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} - \omega^2 QP + \dot{\boldsymbol{\omega}} \times OP + 2\boldsymbol{\omega} \times \mathbf{v}, \tag{3}$$

where we have taken into consideration that $\boldsymbol{\omega}$ is orthogonal to QP, $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$, with the gradient operator ∇ calculated in T. In (3) $-\omega^2 QP$ is the centripetal acceleration and $2\boldsymbol{\omega} \times \mathbf{v}$ is Coriolis acceleration. Using expression (3) the governing equations (1), referred to the non inertial reference frame T, take the following form (see also Holton 2004, Georgescu 2009, Skamarock *et al.* 2008, Temam and Ziane 2004),

$$\begin{cases} \rho\left[\dfrac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+2\boldsymbol{\omega}\times\mathbf{v}\right]=-\nabla p+\mathbf{F}+\rho\mathbf{g}, \\ \dfrac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \\ \dfrac{\partial \theta}{\partial t}+\mathbf{v}\cdot\nabla\theta=F_\theta \end{cases} \tag{4}$$

where ρ, $\mathbf{v}$, θ and p are the fields of mass density, velocity, temperature and pressure, respectively, depending on $(\mathbf{x},t)$, $\mathbf{x}\equiv(x,y,z)$ measured in the reference frame T, the field $\mathbf{F}$ represents the external force acting on the atmosphere and $2\boldsymbol{\omega}\times\mathbf{v}$ is Coriolis force, measured in T,

$$\mathbf{g}=\mathbf{g}'+\omega^2 QP, \tag{5}$$

having its application line, that does not pass through the center of the earth, except at the equator, and at the poles where QP vanishes. In this model $\mathbf{g}$ is assumed costant, being the term $\boldsymbol{\omega}^2$ very small, so *it is assumed*

$$\mathbf{g}=\mathbf{g}'. \tag{6}$$

The quantity $\dot{\boldsymbol{\omega}}\times OP$ (see (3)) is not present in $(4)_1$, because $\dot{\boldsymbol{\omega}}$ is null, being $\boldsymbol{\omega}$ supposed constant and $-\omega^2 OP$ is neglected. Moreover, the symbol "$\nabla\cdot$" is the divergence operator calculated in T. Furthermore, in equation $(4)_1$ on the right hand side of the momentum balance the Coriolis force can be neglected at the mesoscale range, considered in the limited area meteorological model taken into account in this paper.

Indeed, in equation $(4)_1$ the order of magnitude of the acceleration due to the instantaneous variation of the relative velocity is

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\equiv O(\frac{{V_c}^2}{L}), \tag{7}$$

with V_c and L characteristic velocity and length, respectively.

Thus, the magnitude of the term $\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}$ decreases for increasing scales and depends on V_c^2, while the order of magnitude of the Coriolis acceleration is independent of the length scale L and depends only on the characteristic velocity, (Pielke 2002; Holton 2004), *i.e.*

$$2\boldsymbol{\omega}\times\mathbf{v}\equiv O(2\Omega_c V_c).$$

Therefore the ratio

$$\varepsilon=\frac{\mid\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\mid}{2\mid\boldsymbol{\omega}\times\mathbf{v}\mid}\equiv O(\frac{V_c}{2\Omega_c L}), \tag{8}$$

that defines te *Rossby number,* verifies the inequality $\varepsilon\gg 1$, and therefore, the Coriolis contribution can be neglected at the mesoscale scale, considered in this model. Instead, the *Rossby number* is small for large scale flows, *i.e.* for synoptic or planetary scales, defined for L sufficiently large, consequently the Earth's rotation becomes important. Indeed, in this case the applied forces determine mainly a Coriolis acceleration (Pielke 2002; Holton 2004). As an example, for $L\equiv 10^5 km$, being $\boldsymbol{\omega}=7.292\cdot 10^{-5} radians/sec$, we obtain $\varepsilon=0.137$.

2.3. Model equations respect to an non inertial *local* rectangular Cartesian reference frame T^*. In the second step of our formulation (see Fig.2), assuming that the earth is spherical, we choose to refer the description of the motions of the atmosphere (modeled as a non compressible, heat conducting Eulerian fluid), in its current configuration K_t, to a *local rectangular Cartesian reference frame* $T^* \equiv (P, x^*, y^*, z^*)$, being T^* a levorotatory triad with respect to Earth SN (South-North), with the origin in a point P of the northern hemisphere of the Earth, having the z^* axis tangent to the meridian passing through P and oriented towards north, the y^* axis parallel to the line joining P with the center of the Earth and oriented downwards and consequently the x^* axis tangent to the parallel passing through P and oriented towards east (see Georgescu 2009, Temam and Ziane 2004).

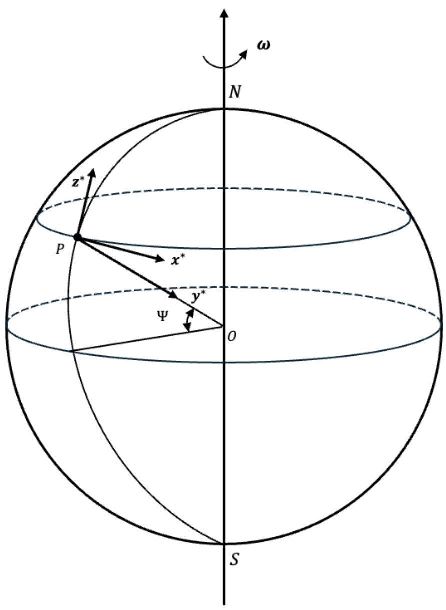

FIGURE 2. The non-inertial rectungalar Cartesian reference frame $T^* \equiv (P, x^*, y^*, z^*)$, solidal to the rotating Earth, that we continue to call $T \equiv (P, x, y, z)$.

Being the deviation of the vertical line from the direction of the Earth radius very small, the gravity acceleration $\mathbf{g}^* = \mathbf{g}$ can be considered parallel to y^* (see equation (6)). We call ψ the latitude of P. Since T^* is also solidal to the rotating Earth, as the reference system T, equations (4) have the same form in both T^* and T, since T^* and T differ only in the origin of the axes and the direction of their axes having constant angles between them. Thus, in the following to simplify the formalism we continue to call the triad T^* with the name T, with $T \equiv (P, x, y, z)$ having the origin in P, so that the same model equations (4) with the same symbols are valid, since nothing changes from the point of view of the description of the various physical quantities present in them measured in T^* or T. Thus, in $T^* \equiv (P, x^*, y^*, z^*)$, called $T \equiv (P, x, y, z)$ equations (4) hold the following form (where Coriolis force is neglected)

$$\begin{cases} \rho\left[\dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\nabla p + \mathbf{F} + \rho\mathbf{g}, \\ \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \\ \dfrac{\partial \theta}{\partial t} + \mathbf{v}\cdot\nabla\theta = F_\theta. \end{cases} \tag{9}$$

In the following this local non-inertial reference frame will be called *z-system* (see (Ferrari 2012), where this non inertial frame is used). Then, using equations (9) we consider a *mesoscale limited area model*, defined in domains where the temporal scale is ranging from a few hours to about 24 h, the spatial horizontal scale is ranging from a few kilometers to thousand kilometers kilometers (10 Km-1000 Km) and the vertical scale is of the order of magnitude of the depth of the troposphere (0-10 km). In particular, in Pielke (2002) it is seen that for mesoscale models, performing a scale analysis of the momentum equation, *i.e.* estimating the order of magnitude of each term present in this equation, the Coriolis acceleration can be neglected.

In Pielke (2002) the momentum equation $(9)_1$, is formulated in a system of rotating spherical coordinates, to evaluate, separately, in both horizontal (in two-dimensional plane) and vertical components of the motion equations, the order of magnitude of Coriolis contribution $2\boldsymbol{\omega}\sin\psi$ ($\boldsymbol{\omega}$ Earth's angular velocity and ψ latitude).

Indeed, a good model for the atmosphere motions should be formulate in spherical coordinates. The use of a rectilinear Cartesian local non-inertial reference system is a good approximation.

Equations (9), that represent one of the simplest models to describe the atmospheric motions are nonlinear, consequently analytical methods to determine the solutions satisfying some initial and boundary conditions are extremely difficult.

For this reason numerical methods are taken into account for weather predictions by Ooyama (1990, 2001), Zhang *et al.* (2012), and Chen *et al.* (2013). To this end different coordinate systems are used to describe the vertical structure of the atmosphere, and the basic equations (9) are formulated in terms of the news introduced coordinates.

Some mathematical aspects of the equations governing geophysical fluid motions, such as weak formulation, existence and uniqueness of solutions, can be found in Temam and Ziane (2004), where, among other, a guide and summary of results "*for the physics oriented reader*" is provided.

3. Introduction of a new generalized coordinate s

In this Section we introduce a new local reference frame solidal with the rotating Earth $T = (P, x, y, s)$, that we continue to call T, having the same origin P, the same x and y axes and as third axis a vertical axis s defined by the function

$$s = s(x, y, z, t), \tag{10}$$

from now on called *new generalized coordinate* (see Kasahara 1974; Pedlosky 1984; Laprise 1992; Eckerman 2008; Ferrari 2012; Grande 2015 and related extensive bibliographies). We assume that (10) is a monotonic single-value function between s and z, for x,y,t fixed. In this case we can consider, inverting (10), the function

$$z = z(x,y,s,t). \tag{11}$$

Any scalar field A can be expressed either in terms of the four independent variables x,y,z,t in z-system, i. e. $A(x,y,z,t)$, or in terms of x,y,s,t in the new introduced s-system $T = (P,x,y,s)$, *i.e.* (Laprise 1992; Ferrari 2012)

$$A(x,y,s,t) \equiv A(x,y,z(x,y,s,t),t), \tag{12}$$

where we have continued to call by the same name A the quantity A in s-system and in z-system. We can perform calculations in both s-system and z-system. For this reason in the following we will indicate by a subscript "s" or "z" the reference frame where we will work.

3.1. Spatial and time partial derivatives of a scalar field. In this Subsection we derive the spatial and time partial derivatives of a scalar field in a s-system and in a z-system. Taking into account (12) we have:

$$\left(\frac{\partial A}{\partial x}\right)_s = \left(\frac{\partial A}{\partial x}\right)_z + \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s, \tag{13}$$

or

$$\left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_s - \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s. \tag{14}$$

In an analogous way we derive:

$$\left(\frac{\partial A}{\partial y}\right)_z = \left(\frac{\partial A}{\partial y}\right)_s - \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial y}\right)_s, \tag{15}$$

$$\left(\frac{\partial A}{\partial t}\right)_z = \left(\frac{\partial A}{\partial t}\right)_s - \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial t}\right)_s. \tag{16}$$

To express $\dfrac{\partial A}{\partial z}$ we can use the following identity:

$$\frac{\partial A}{\partial z} = \frac{\partial A}{\partial s}\frac{\partial s}{\partial z}. \tag{17}$$

Substituting (17) in (14), (15) and (16) we obtain:

$$\left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_s - \frac{\partial A}{\partial s}\frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s, \tag{18}$$

$$\left(\frac{\partial A}{\partial y}\right)_z = \left(\frac{\partial A}{\partial y}\right)_s - \frac{\partial A}{\partial s}\frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial y}\right)_s, \tag{19}$$

$$\left(\frac{\partial A}{\partial t}\right)_z = \left(\frac{\partial A}{\partial t}\right)_s - \frac{\partial A}{\partial s}\frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial t}\right)_s. \tag{20}$$

From (18) and (19) we can introduce the two-dimensional operators

$$\nabla_z^1 \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)_z, \quad \nabla_s^1 \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)_s, \tag{21}$$

satisfying the relation:

$$\nabla_z^1 A = \nabla_s^1 A - \frac{\partial A}{\partial s}\frac{\partial s}{\partial z}\nabla_s^1 z. \tag{22}$$

3.2. Derivation of the time total derivative of a scalar field and relation between $\dot{z} = w$ and $\dot{s}$. In this Subsection we derive the time total derivative of a scalar field in a s-system and in a z-system and a relation between the quantities $w = \dot{z}$ and $\dot{s}$. The total derivative with respect to the time t of a scalar field A must be unchanged either in z-system or in s-system, *i.e.*

$$\left(\frac{dA}{dt}\right)_z = \left(\frac{\partial A}{\partial t}\right)_z + \mathbf{v}_H \cdot \nabla_z^1 A + w\frac{\partial A}{\partial z}, \tag{23}$$

$$\left(\frac{dA}{dt}\right)_s = \left(\frac{\partial A}{\partial t}\right)_s + \mathbf{v}_H \cdot \nabla_s^1 A + \dot{s}\frac{\partial A}{\partial s}, \tag{24}$$

where $w = \dfrac{dz}{dt}$ and $\dot{s} = \dfrac{ds}{dt}$ are the two vertical components of the velocity field in z-system and s-system, respectively,

$$\mathbf{v}_H \equiv (u, v) \tag{25}$$

is *the two-dimensional horizontal velocity*, which remains unchanged in both systems. Thus, in z-system the velocity $\mathbf{v}$ has components $\mathbf{v} \equiv (\dot{x}, \dot{y}, \dot{z}) \equiv (u, v, w)$ and in s-system the velocity has components $(\dot{x}, \dot{y}, \dot{s}) \equiv (u, v, \dot{s})$. Substituting (16), (17), (20) and (22) in (23), from the identity

$$\left(\frac{dA}{dt}\right)_z \equiv \left(\frac{dA}{dt}\right)_s \tag{26}$$

we obtain:

$$\left(\frac{\partial A}{\partial t}\right)_s - \frac{\partial A}{\partial s}\frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial t}\right)_s + \mathbf{v}_H \cdot \left[\nabla_s^1 A - \frac{\partial A}{\partial s}\frac{\partial s}{\partial z}\nabla_s^1 z\right] + w\frac{\partial A}{\partial s}\frac{\partial s}{\partial z} \equiv \tag{27}$$
$$\left(\frac{\partial A}{\partial t}\right)_s + \mathbf{v}_H \cdot \nabla_s^1 A + \dot{s}\frac{\partial A}{\partial s}.$$

From (27), replacing A with z, using expression (22), we derive the relation between w and $\dot{s}$, *i.e.*

$$\dot{s} = \frac{\partial s}{\partial z}\left[w - \left(\frac{\partial z}{\partial t}\right)_s - \mathbf{v}_H \cdot \nabla_s^1 z\right]. \tag{28}$$

3.3. Derivation in s-system of $\left(\frac{\partial w}{\partial z}\right)_z$. To express $\left(\frac{\partial w}{\partial z}\right)_z$ in s-system we take into account (17). Thus, we have:

$$\left(\frac{\partial w}{\partial z}\right)_z = \frac{\partial w}{\partial s}\frac{\partial s}{\partial z} = \frac{\partial}{\partial s}\left(\frac{dz}{dt}\right)_z\frac{\partial s}{\partial z} = \frac{\partial}{\partial s}\left(\frac{dz}{dt}\right)_s\frac{\partial s}{\partial z}. \tag{29}$$

Substituting (16) and (22) in (23) we can verify that the total derivative with respect to time in s-system is given by:

$$\left(\frac{d}{dt}\right)_z \equiv \left(\frac{d}{dt}\right)_s \equiv \left(\frac{\partial}{\partial t}\right)_s + \mathbf{v}_H \cdot \nabla_s^1 + \dot{s}\frac{\partial}{\partial s}. \tag{30}$$

Using (30) we have:

$$w = \left(\frac{dz}{dt}\right)_s = \dot{s}\frac{\partial z}{\partial s} + \left(\frac{\partial z}{\partial t}\right)_s + \mathbf{v}_H \cdot \nabla_s^1 z. \tag{31}$$

Substituting (31) in (29) we obtain

$$\left(\frac{\partial w}{\partial z}\right)_z = \frac{\partial s}{\partial z}\left\{\left[\frac{d}{dt}\left(\frac{\partial z}{\partial s}\right)\right]_s + \frac{\partial \mathbf{v}_H}{\partial s}\cdot\nabla_s^1 z\right\} + \frac{\partial \dot{s}}{\partial s}, \tag{32}$$

where

$$\left[\frac{d}{dt}\left(\frac{\partial z}{\partial s}\right)\right]_s = \dot{s}\frac{\partial^2 z}{\partial s^2} + \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial t}\right)_s + \mathbf{v}_H\cdot\frac{\partial(\nabla_s^1 z)}{\partial s} \tag{33}$$

.

3.4. Derivation of the divergence of v and $\mathbf{v}_H$. Using (14), (16), (17), (21) and (32) we can calculate the expression of the divergence of a three-dimensional vector field $\mathbf{v} \equiv (u,v,w) = (\dot{x},\dot{y},\dot{z})$ in s-system, using the expression:

$$\nabla_z\cdot\mathbf{v} = \nabla_s\cdot\mathbf{v} + \frac{\partial s}{\partial z}\frac{d}{dt}\left(\frac{\partial z}{\partial s}\right), \tag{34}$$

where $\nabla_z\cdot\mathbf{v} = (\frac{\partial u}{\partial x})_z + (\frac{\partial v}{\partial y})_z + (\frac{\partial w}{\partial z})_z$ and $\nabla_s\cdot\mathbf{v} = (\frac{\partial u}{\partial x})_s + (\frac{\partial v}{\partial y})_s + (\frac{\partial \dot{s}}{\partial s})_s$.

For a two-dimensional (horizontal) vector field $\mathbf{v}_H$, given by the expression (24), we have:

$$\nabla_z\cdot\mathbf{v}_H = \nabla_s\cdot\mathbf{v}_H - \frac{\partial s}{\partial z}\frac{\partial \mathbf{v}_H}{\partial s}\cdot\nabla_s^1 z. \tag{35}$$

4. Model equations in s-system

In this Section we transform the prognostic (balance) equations (9) of a mesoscale limited area model in the new introduced *s*-system, with *s* as an arbitrary single-value monotonic function of the geometrical altitude, obviously preserving the invariance of physical representation in the new *s*-system.

In this mesoscale model we neglect the Coriolis contribution (Pielke 2002), obtaining a simplified form of the momentum equation.

Thus, equations (9) in the *z*-system keep the form

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\oplus\mathbf{v}) = -\nabla p + \mathbf{F} + \rho\mathbf{g}, \tag{36}$$

$$\frac{\partial}{\partial t}(\rho\theta) + \nabla\cdot(\rho\mathbf{v}\theta) = \rho\mathbf{F}_\theta, \tag{37}$$

jointly with $(9)_2$. In (36) $\mathbf{F}$ is related to the external force.

To derive (36) we have added $(9)_2$, multiplied by $\mathbf{v}$, to $(9)_1$, multiplied by ρ, and we have taken into account the identity

$$\nabla\cdot(\rho\mathbf{v}\oplus\mathbf{v}) = \mathbf{v}\nabla\cdot(\rho\mathbf{v}) + \rho\mathbf{v}\cdot\nabla\mathbf{v}, \tag{38}$$

where the symbol "$\oplus$" denotes the tensor product. In an analogous way from $(9)_2$, multiplied by θ, and $(9)_3$, multiplied by ρ, we obtain equation (37).

In the following Subsections we derive some quantities in z-system and s-system and, to simplify the use of the tensor calculus methods applied to the vector equation (36), we will consider their scalar components on the horizontal and vertical directions.

4.1. Derivation in z-system and s-system of the first component of the momentum balance equation (36)**.** In the z-system the component of equation (36) in x-direction is the following:

$$\Big[\frac{\partial(\rho u)}{\partial t}\Big]_z + \nabla_z\cdot(\rho \mathbf{v} u) = -\Big(\frac{\partial p}{\partial x}\Big)_z + F_x, \tag{39}$$

where F_x denotes the component of the force $\mathbf{F}$ in x-direction.

Using (20), (22), (31) and (34) we can transform (39) in s-system as follows:

$$\Big[\frac{\partial(\rho u)}{\partial t}\Big]_s - \frac{\partial(\rho u)}{\partial s}\frac{\partial s}{\partial z}\Big(\frac{\partial z}{\partial t}\Big)_s + \mathbf{v}_H\cdot\Big[\nabla_s^1(\rho u) - \frac{\partial(\rho u)}{\partial s}\frac{\partial s}{\partial z}\nabla_s^1 z\Big] + \Big\{\dot{s} + \frac{\partial s}{\partial z}\Big[\Big(\frac{\partial z}{\partial t}\Big)_s + \mathbf{v}_H\cdot\nabla_s^1 z\Big]\Big\}\frac{\partial(\rho u)}{\partial s} + (\rho u)\Big\{\nabla_s\cdot\mathbf{v} + \frac{\partial s}{\partial z}\Big[\frac{\partial}{\partial t}\Big(\frac{\partial z}{\partial s}\Big)_s + \mathbf{v}_H\cdot\nabla_s^1\frac{\partial z}{\partial s} + \dot{s}\frac{\partial}{\partial s}\Big(\frac{\partial z}{\partial s}\Big)\Big]\Big\} = -\Big[\Big(\frac{\partial p}{\partial x}\Big)_s - \frac{\partial p}{\partial s}\frac{\partial s}{\partial z}\Big(\frac{\partial z}{\partial x}\Big)_s\Big] + F_x, \tag{40}$$

or, equivalently as:

$$\Big[\frac{\partial(\rho u)}{\partial t}\Big]_s + \mathbf{v}\cdot\nabla_s(\rho u) + (\rho u)\nabla_s\cdot\mathbf{v} + (\rho u)\frac{\partial s}{\partial z}\frac{d}{dt}\Big(\frac{\partial z}{\partial s}\Big) = -\Big[\Big(\frac{\partial p}{\partial x}\Big)_s - \frac{\partial p}{\partial s}\frac{\partial s}{\partial z}\Big(\frac{\partial z}{\partial x}\Big)_s\Big] + F_x. \tag{41}$$

The previous equation, multiplied by $\dfrac{\partial z}{\partial s}$, becomes:

$$\frac{\partial z}{\partial s}\Big[\Big(\frac{d(\rho u)}{dt}\Big)_s + (\rho u)\nabla_s\cdot\mathbf{v}\Big] + (\rho u)\frac{d}{dt}\Big(\frac{\partial z}{\partial s}\Big) = -\Big[\Big(\frac{\partial p}{\partial x}\Big)_s - \frac{\partial p}{\partial s}\frac{\partial s}{\partial z}\Big(\frac{\partial z}{\partial x}\Big)_s\Big]\frac{\partial z}{\partial s} + F_x\frac{\partial z}{\partial s}. \tag{42}$$

or:

$$\frac{d}{dt}\Big[(\rho u)\frac{\partial z}{\partial s}\Big]_s + (\rho u)\frac{\partial z}{\partial s}\nabla_s\cdot\mathbf{v} = -\Big[\Big(\frac{\partial p}{\partial x}\Big)_s - \frac{\partial p}{\partial s}\frac{\partial s}{\partial z}\Big(\frac{\partial z}{\partial x}\Big)_s\Big]\frac{\partial z}{\partial s} + F_x\frac{\partial z}{\partial s}. \tag{43}$$

Expression (43) takes the following form of balance equation:

$$\frac{\partial}{\partial t}\Big[(\rho u)\frac{\partial z}{\partial s}\Big]_s + \nabla_s\cdot\Big[(\rho u)\frac{\partial z}{\partial s}\mathbf{v}\Big] = -\Big[\Big(\frac{\partial p}{\partial x}\Big)_s - \frac{\partial p}{\partial s}\frac{\partial s}{\partial z}\Big(\frac{\partial z}{\partial x}\Big)_s\Big]\frac{\partial z}{\partial s} + F_x\frac{\partial z}{\partial s}. \tag{44}$$

Let us introduce *the geopotential* $\Phi = \Phi(x,y,s,t)$ (see also Skamarock *et al.* 2008), defined by

$$\Phi = gz(x,y,s,t), \tag{45}$$

having assumed that the surface of constant gravity potential is a sphere, *i.e.* the geopotential is a function of z, and the Earth's gravity acceleration $\mathbf{g}$ constant. Φ is the necessary work to overcome the gravity force and move upwards (at a given height) a unitary mass of atmospheric air.

Equation (44) becomes:

$$\frac{\partial}{\partial t}\Big[(\rho u)\frac{\partial \Phi}{\partial s}\Big]_s + \nabla_s\cdot\Big[(\rho u)\frac{\partial \Phi}{\partial s}\mathbf{v}\Big] = -\Big[\Big(\frac{\partial p}{\partial x}\Big)_s - \frac{\partial p}{\partial s}\frac{\partial s}{\partial z}\Big(\frac{\partial z}{\partial x}\Big)_s\Big]\frac{\partial \Phi}{\partial s} + F_x\frac{\partial \Phi}{\partial s}. \tag{46}$$

Defining *the pseudodensity* μ as:

$$\mu = -\rho\frac{\partial \Phi}{\partial s}, \tag{47}$$

equation (46) becomes:

$$\frac{\partial}{\partial t}\Big[(\mu u)\Big]_s + \nabla_s \cdot \Big(\mu u \mathbf{v}\Big) = \Big[\Big(\frac{\partial p}{\partial x}\Big)_s - \frac{\partial p}{\partial s}\frac{\partial s}{\partial z}\Big(\frac{\partial z}{\partial x}\Big)_s\Big]\frac{\partial \Phi}{\partial s} - F_x\frac{\partial \Phi}{\partial s}, \tag{48}$$

or:

$$\frac{\partial}{\partial t}\Big[(\mu u)\Big]_s + \nabla_s \cdot \Big(\mu u \mathbf{v}\Big) = \Big[\Big(\frac{\partial p}{\partial x}\Big)_s\frac{\partial \Phi}{\partial s} - \frac{\partial p}{\partial s}\frac{\partial \Phi}{\partial x}\Big] - F_x\frac{\partial \Phi}{\partial s}.$$

An equivalent form of this equation is:

$$\frac{\partial}{\partial t}\Big[(\mu u)\Big]_s + \nabla_s \cdot \Big(\mu u \mathbf{v}\Big) = \frac{\partial}{\partial x}\Big(p\frac{\partial \Phi}{\partial s}\Big)_s - \frac{\partial}{\partial s}\Big(p\frac{\partial \Phi}{\partial x}\Big)_s - F_x\frac{\partial \Phi}{\partial s}. \tag{49}$$

Equation (49) coincides with equations (2.3) in Skamarock *et al.* (2008) and (4) in Castorina *et al.* (2018), when in the mesoscale limited area model Weather Research Forecast (WRF) (see Skamarock *et al.* 2008, Thunis and Bornstein 1995, Pielke 2002 for a useful hierarchy for mesoscale models), a new η-system is introduced, where the vertical coordinate s is chosen as a coordinate η, defined by means of the hydrostatic pressure (Kasahara 1974; Laprise 1992; Holton 2004; Grande 2015), see the following Subsection 4.2. Indeed, by using the vertical coordinate η equation (49) takes the following form:

$$\frac{\partial}{\partial t}\Big[(\mu u)\Big]_\eta + \nabla_\eta \cdot \Big(\mu u \mathbf{v}\Big) - \frac{\partial}{\partial x}\Big(p\frac{\partial \Phi}{\partial \eta}\Big)_\eta + \frac{\partial}{\partial \eta}\Big(p\frac{\partial \Phi}{\partial x}\Big)_\eta = -\frac{\Phi}{\eta}F_x, \tag{50}$$

or

$$\frac{\partial}{\partial t}\Big[(\mu u)\Big]_\eta + \nabla_\eta \cdot \Big(\mu u \mathbf{v}\Big) - \frac{\partial}{\partial x}\Big(p\frac{\partial \Phi}{\partial \eta}\Big)_\eta + \frac{\partial}{\partial \eta}\Big(p\frac{\partial \Phi}{\partial x}\Big)_\eta = \frac{\mu}{\rho}F_x, \tag{51}$$

being, from (47), $\frac{\partial \Phi}{\partial \eta} = -\frac{\mu}{\rho}$.

In meteorology, sometimes, the pressure is measured by radiosondes, so it is used to replace the vertical coordinate z (Pielke 2002) by an other more useful coordinate.

4.2. The η-system as the mass coordinate system. In this Section we introduce a new local reference frame solidal with the rotating Earth $T = (P, x, y, \eta)$, that we continue to call T, having the same origin P, the same coordinates x and y and the vertical coordinate s is chosen as a coordinate η, defined by means of the hydrostatic pressure.

The coordinate η is one nondimensional coordinate that decreases from $\eta = 1$ (at the lower noundary of the atmosphere) to $\eta = 0$ at the top of the atmosphere (in the considered *troposhere layer*). We can defined it by means of the hydrostatic pressure, calling it η, as follows (see equations (2.1) in Skamarock *et al.* (2008) and (2) in Castorina *et al.* (2018):

$$\eta = \frac{p_h - p_{ht}}{p_{h\eta} - p_{ht}} \equiv \frac{p_h - p_{ht}}{\mu}, \tag{52}$$

or, equivalently,

$$p_h - p_{ht} = \mu\eta,$$

where p_h is the hydrostatic component of the pressure (it is the weight of an atmospheric column of unit area standing above a point (x, y) at eight z), p_{ht} is a constant pressure of the atmosphere, related to the weight of an atmospheric column of unit area corresponding to the upper (fictitious) boundary of the troposphere (indeed, we cannot adopt a vertical coordinate which extends to infinity, predicting atmospheric motions using numerical methods), $p_{h\eta}$ denotes the pressure at the surface $\eta = 1$, and

$$\mu = p_{h\eta} - p_{ht}. \tag{53}$$

In equation (53) μ is defined as the difference between the pressures $p_{h\eta}$ and p_{ht}. Thus, it has the physical dimension of a pressure. This is the reason why this coordinate system is sometimes defined as *mass coordinate system*.

In this way, from (52) it follows that the lower boundary coincides with a coordinate pressure surface, *i.e.* $\eta = 1$ corresponding to $p_h = p_{h\eta}$.

By choosing different vertical coordinates we would have a problem with boundary conditions applied to a surface depending on the horizontal variables (e.g. isobaric or the log-pressure coordinate systems), see Kasahara 1974; Pedlosky 1984; Laprise 1992; Eckerman 2008; Grande 2015 and related bibliographies.

In η-coordinate, the lower boundary conditions will be applied to $\eta = 1$.

In the nonhydrostatic case, considered in this paper, the vertical total hydrodynamic pressure p, present in $(9)_1$ represents an unknown of the system, that we can determine from the following *diagnostic* (called also *constitutive*) equation, when we assume the atmosphere is a perfect fluid, in particular *dry air* (see (2.10) in Skamarock *et al.* (2008) and (2) in Castorina *et al.* (2018)):

$$p = p_0\left(\frac{R_d\theta}{p_0\alpha}\right)^{\gamma}. \tag{54}$$

Here R_d is the gas constant for the dry air, $\gamma = c_p/c_v = 1.4$ is the ratio of the heat capacities, $\alpha = 1/\rho$ is the inverse of the dry air density, p_0 is a reference pressure (tipically $10^5 Pa$), $p = p_h + p_n$, with p_n non-hydrostatic pressure. The pressure p_n is defined by $p_n = p - p_h$. The hydrostatic pressure p_h is defined by the equation

$$\frac{\partial p_h}{\partial z} = -\rho g, \tag{55}$$

whose solution gives p_h. From (55) we see that the hydrostatic pressure p_h is a monotonic single value function of z.

Then, we can select p_h as independent variable in the limited area models and therefore by virtue of (52) to choose the vertical coordinate η in these models, defined by means of p_h, in order to describe non-hydrostatic situations.

When we use the vertical coordinate η, defined by p_h, the earth's surface can be considered as a pressure coordinate surface, even if the coordinate η-system has some computational disadvantages near the mountains, as in this case the lower boundary of the atmosphere is not a surface coordinate.

To overcome this problem in Laprise (1992) a further transformation was introduced to incorporate the Earth's orography, the terrain-following coordinates (Laprise 1992; Klemp, Skamarock, and Dudhia 2007).

Indeed, to take into account the Earth's orography, a special procedure must be used in numerical calculus, to solve the problem of the boundary conditions in proximity of the mountains (Laprise 1992; Pielke 2002; Holton 2004). In Ooyama (1990), Laprise (1992), Ooyama (2001), and Chen *et al.* (2013) it is seen that in the hydrostatic case the total vertical pressure p, coinciding with p_h, is chosen as independent variable, and that the choice of the coordinate η, in the isobaric case, allows to obtain an equations set very close to the hydrostatic equations.

4.3. Diagnostic equation for the hydrostatic pressure as definition of the pseudodensity. We observe that the definition of pseudodensity given by (47), in the case of the WRF model becomes the diagnostic equation for the hydrostatic pressure (55). Indeed, equation (55) can be written equivalently as follows:

$$\frac{\partial p_h}{\partial \eta}\frac{\partial \eta}{\partial z} = -\rho g, \tag{56}$$

or

$$\frac{\partial p_h}{\partial \eta} = -\rho g\frac{\partial z}{\partial \eta} = -\rho\frac{\partial \Phi}{\partial \eta}. \tag{57}$$

Taking into account that $\mu = p_{hs} - p_{ht}$ does not depend on η, it takes the form

$$\frac{\partial p_h}{\partial \eta} = \frac{\partial p_h}{\partial(\mu\eta)}\frac{\partial(\mu\eta)}{\partial \eta} = \mu. \tag{58}$$

Finally, substituting (58) in (57) we obtain the definition of pseudensity (47) (see equations (2.9) in Skamarock *et al.* (2008) and Castorina *et al.* (2018)):

$$\mu = -\rho\frac{\partial \Phi}{\partial \eta}, \tag{59}$$

where it is seen that the definition of pseudodensity becomes the diagnostic equation for hydrostatic pressure.

4.4. Derivation in z-system and η-system of the second component of the momentum balance equation (36)**.** In this Subsection, in an analogous way as in Subsection 4.1 we can derive the second component of equation (36) in the y-direction.

$$\frac{\partial}{\partial t}\Big[(\mu v)\Big]_\eta + \nabla_\eta\cdot\Big(\mu v\mathbf{v}\Big) - \frac{\partial}{\partial y}\Big(p\frac{\partial\Phi}{\partial\eta}\Big)_\eta + \frac{\partial}{\partial\eta}\Big(p\frac{\partial\Phi}{\partial y}\Big)_\eta = \frac{\mu}{\rho}F_y. \tag{60}$$

with $\frac{\mu}{\rho}F_y = -F_y\frac{\partial\Phi}{\partial\eta}$.

Equation (60) coincides with equations (2.4) in Skamarock *et al.* (2008) and (5) in Castorina *et al.* (2018).

4.5. Derivation in z-system and η-system of the third component of the momentum balance equation (36)**.** In this Subsection we derive in z-system and η-system the third component of the momentum balance equation (36). We recall that the true vertical component of the velocity vector is $w = \dot{z}$. In the z-system the component of (36) in z-direction is the following:

$$\Big[\frac{\partial(\rho w)}{\partial t}\Big]_z + \nabla_z\cdot(\rho\mathbf{v}w) = -\Big(\frac{\partial P}{\partial z}\Big)_z + F_z - \rho g. \tag{61}$$

In the η-system, using (17), (20), (22) and (34), equation (61) takes the form

$$\Big[\frac{\partial(\rho w)}{\partial t}\Big]_\eta - \frac{\partial(\rho w)}{\partial\eta}\frac{\partial\eta}{\partial z}\Big(\frac{\partial z}{\partial t}\Big)_\eta + \mathbf{v}_H\cdot\Big[\nabla^1_\eta(\rho w) - \frac{\partial(\rho w)}{\partial\eta}\frac{\partial\eta}{\partial z}\nabla^1_\eta z\Big] + \Big\{\dot{\eta} + \frac{\partial\eta}{\partial z}\Big[\Big(\frac{\partial z}{\partial t}\Big)_\eta$$

$$+\mathbf{v}_H\cdot\nabla^1_\eta z\Big]\Big\}\frac{\partial(\rho w)}{\partial\eta}+(\rho w)\Big\{\nabla_\eta\cdot\mathbf{v}+\frac{\partial\eta}{\partial z}\Big[\frac{\partial}{\partial t}\Big(\frac{\partial z}{\partial\eta}\Big)_\eta+\mathbf{v}_H\cdot\nabla^1_\eta\frac{\partial z}{\partial\eta}+\dot\eta\frac{\partial}{\partial\eta}\Big(\frac{\partial z}{\partial\eta}\Big)\Big]\Big\}=$$

$$-\Big(\frac{\partial p}{\partial\eta}\Big)_\eta\frac{\partial\eta}{\partial z}+F_z-\rho g, \tag{62}$$

or, equivalently as:

$$\Big[\frac{\partial(\rho w)}{\partial t}\Big]_\eta+\mathbf{v}\cdot\nabla_\eta(\rho w)+(\rho w)\nabla_\eta\cdot\mathbf{v}+(\rho w)\frac{\partial\eta}{\partial z}\frac{d}{dt}\Big(\frac{\partial z}{\partial\eta}\Big)=-\Big(\frac{\partial p}{\partial\eta}\Big)_\eta\frac{\partial\eta}{\partial z}+F_z-\rho g. \tag{63}$$

The previous equation, multiplied by $\dfrac{\partial z}{\partial\eta}$, becomes:

$$\frac{\partial z}{\partial\eta}\Big[\Big(\frac{d(\rho w)}{dt}\Big)_\eta+(\rho w)\nabla_\eta\cdot\mathbf{v}\Big]+(\rho w)\frac{d}{dt}\Big(\frac{\partial z}{\partial\eta}\Big)=-\Big(\frac{\partial p}{\partial\eta}\Big)_\eta+\frac{\partial z}{\partial\eta}F_z-\frac{\partial z}{\partial\eta}\rho g, \tag{64}$$

or, taking into account (45), equation (64) takes the form:

$$\frac{d}{dt}\Big[(\rho w)\frac{\partial\Phi}{\partial\eta}\Big]_\eta+(\rho w)\frac{\partial\Phi}{\partial\eta}\nabla_\eta\cdot\mathbf{v}=-g\Big(\frac{\partial p}{\partial\eta}\Big)_\eta+\frac{\partial\Phi}{\partial\eta}F_z-\frac{\partial\Phi}{\partial\eta}\rho g. \tag{65}$$

Using the diagnostic (constitutive) equation for the hydrostatic pressure (59), equation (65) can be written as follows:

$$\frac{\partial}{\partial t}\Big[(\mu w)\Big]_\eta+\nabla_\eta\cdot(\mu w\mathbf{v})+g\Big[\mu-\frac{\partial p}{\partial\eta}\Big]=\frac{\mu}{\rho}F_z, \tag{66}$$

with $\frac{\mu}{\rho}F_z=-\frac{\partial\Phi}{\partial\eta}F_z$.

Equation (66) coincides with the equations (2.5) in Skamarock *et al.* (2008) and (6) in Castorina *et al.* (2018). The system of equations (9) in η-system is transformed in (28), (49), (60) and (66), because we have two vertical velocity components, *i.e.* $\dot\eta$ and w.

4.6. Pseudodensity μ conservation equation in η-system. In this Subsection we determine the pseudodensity μ conservation equation in η-system, equivalent to the mass conservation equation $(9)_2$ in the z-system.

Thus, we transform $(9)_2$, using (17), (22), (28), as follows:

$$\Big(\frac{\partial\rho}{\partial t}\Big)_\eta-\frac{\partial\rho}{\partial\eta}\frac{\partial\eta}{\partial z}\Big(\frac{\partial z}{\partial t}\Big)_\eta+\mathbf{v}_H\cdot\Big[\nabla^1_\eta\rho-\frac{\partial\rho}{\partial\eta}\frac{\partial\eta}{\partial z}\nabla^1_\eta z\Big]+\Big\{\dot\eta+\frac{\partial\eta}{\partial z}\Big[\Big(\frac{\partial z}{\partial t}\Big)_\eta$$

$$+\mathbf{v}_H\cdot\nabla^1_\eta z\Big]\Big\}\frac{\partial\rho}{\partial\eta}+\rho\Big\{\nabla_\eta\cdot\mathbf{v}+\frac{\partial\eta}{\partial z}\Big[\frac{\partial}{\partial t}\Big(\frac{\partial z}{\partial\eta}\Big)_\eta+\mathbf{v}_H\cdot\nabla^1_\eta\frac{\partial z}{\partial\eta}+\dot\eta\frac{\partial}{\partial\eta}\Big(\frac{\partial z}{\partial\eta}\Big)\Big]\Big\}=0.$$

This last equation is equivalent to the following one:

$$\Big(\frac{\partial\rho}{\partial t}\Big)_\eta+\mathbf{v}\nabla_\eta\rho+\rho\nabla_\eta\cdot\mathbf{v}+\rho\frac{\partial\eta}{\partial z}\frac{d}{dt}\Big(\frac{\partial z}{\partial\eta}\Big)=0. \tag{67}$$

Multiplying (67) by $\dfrac{\partial z}{\partial\eta}$, we have:

$$\frac{d}{dt}\Big(\rho\frac{\partial z}{\partial\eta}\Big)_\eta+\rho\frac{\partial z}{\partial\eta}\nabla_\eta\cdot\mathbf{v}=0.$$

Using (45) and (47) the previous equation becomes:

$$\frac{\partial \mu}{\partial t} + \nabla_\eta \cdot (\mu \mathbf{v}) = 0. \tag{68}$$

Equation (68) coincides with equations (2.7) in Skamarock *et al.* (2008) and (8) in Castorina *et al.* (2018).

4.7. Temperature balance equation in η-system. In this Subsection we transform equation (37) in the η-system.

Thus, we have:

$$\left[\frac{\partial(\rho\theta)}{\partial t}\right]_\eta - \frac{\partial \rho\theta}{\partial \eta}\frac{\partial \eta}{\partial z}\left(\frac{\partial z}{\partial t}\right)_\eta + \mathbf{v}_H \cdot \left[\nabla^1_\eta(\rho\theta) - \frac{\partial(\rho\theta)}{\partial \eta}\frac{\partial \eta}{\partial z}\nabla^1_\eta z\right] + \left\{\dot{\eta} + \frac{\partial \eta}{\partial z}\left[\left(\frac{\partial z}{\partial t}\right)_\eta + \mathbf{v}_H \cdot \nabla^1_\eta z\right]\right\}\frac{\partial(\rho\theta)}{\partial \eta} + \rho\theta\left\{\nabla_\eta \cdot \mathbf{v} + \frac{\partial \eta}{\partial z}\left[\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \eta}\right)_\eta + \mathbf{v}_H \cdot \nabla^1_\eta \frac{\partial z}{\partial \eta} + \dot{\eta}\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \eta}\right)\right]\right\} = \rho F_\theta. \tag{69}$$

Equation (69) is equivalent to the following one:

$$\left[\frac{\partial(\rho\theta)}{\partial t}\right]_\eta + \mathbf{v} \cdot \nabla_\eta(\rho\theta) + \rho\theta\nabla_\eta \cdot \mathbf{v} + \rho\theta\frac{\partial \eta}{\partial z}\frac{d}{dt}\frac{\partial z}{\partial \eta} = \rho F_\theta. \tag{70}$$

Multiplying (70) by $\dfrac{\partial z}{\partial \eta}$ we have:

$$\frac{d}{dt}\left[(\rho\theta)\frac{\partial z}{\partial \eta}\right]_\eta + (\rho\theta)\frac{\partial z}{\partial \eta}\nabla_\eta \cdot \mathbf{v} = \rho\frac{\partial z}{\partial \eta}F_\theta. \tag{71}$$

Finally, taking into account equations (45) and (47) we have

$$\frac{\partial}{\partial t}(\mu\theta) + \nabla_\eta \cdot (\mu \mathbf{v}\theta) = \rho\frac{\partial \Psi}{\partial \eta}F_\theta, \tag{72}$$

or

$$\frac{\partial}{\partial t}(\mu\theta) + \nabla_\eta \cdot (\mu \mathbf{v}\theta) = -\mu F_\theta, \tag{73}$$

being $\frac{\partial \Psi}{\partial \eta} = -\frac{\mu}{\rho}$.

Equation (72) coincides with equations (2.6) in Skamarock *et al.* (2008) and (7) in Castorina *et al.* (2018).

We point out that the governing equations obtained in the η-system, given by (49), (60), (66), (68), (72) and (28), are equivalent to the equations (9) in the z-system, because the evolution equation for the pseudodensity μ in η-system is equivalent to the mass conservation equation in z-system, the Euler and temperature balance equations in η-system are obtained from the corresponding ones in z-system taking into account the mass conservation equation.

4.8. Evolution equation for the geopotential. In this Subsection we see that the additional equation (28) is due to the presence of two vertical components of the velocity field and is equivalent to the evolution equation for the geopotential Φ, defined by equation (45). Namely, from the definition of geopotential it follows:

$$\frac{d}{dt}\Phi = gw. \tag{74}$$

Taking into account (30), from the previous equality we have:

$$\left(\frac{\partial \Phi}{\partial t}\right)_\eta + \mathbf{v}_H \cdot \nabla^1_\eta \Phi + \dot{\eta}\frac{\partial \Phi}{\partial \eta} = gw. \tag{75}$$

Equation (75) is equivalent to the following one:

$$\left(\frac{\partial \Phi}{\partial t}\right)_\eta + \mathbf{v}_H \cdot \nabla_\eta \Phi = gw. \tag{76}$$

In terms of the "flux variables" $\mathbf{V}_H \equiv \mu \mathbf{v_H} \equiv \mu(u,v) = (U,V)$ and $W \equiv \mu w$, introduced in Castorina *et al.* (2018) equation (76) takes the form:

$$\left(\frac{\partial \Phi}{\partial t}\right)_\eta + \mu^{-1}[\mathbf{V}_H \cdot \nabla_\eta \Phi - gW] = 0. \tag{77}$$

Equation (77) coincides with equations (2.8) in Skamarock *et al.* (2008) and (9) in Castorina *et al.* (2018).

We point out that the equation (76) becomes an identity substituting in it w given by (28). In fact, we obtain

$$\left(\frac{\partial \Phi}{\partial t}\right)_\eta + \mathbf{v}_H \cdot \nabla_\eta \Phi \equiv g\left[\dot{\eta}\frac{\partial z}{\partial \eta} + \left(\frac{\partial z}{\partial t}\right)_\eta + \mathbf{v}_H \cdot \nabla^1_\eta z\right]. \tag{78}$$

Therefore, from (78) the evolution equation (76) is equivalent to the relation (28) between $w = \dot{z}$ and $\dot{\eta}$, *i.e.* it represents the additional equation due to the introduction of two vertical components of the velocity field w and $\dot{\eta}$.

5. Governing equations describing the motions and the behaviour of the atmosphere modeled as a mixture

Let us consider equations (9), describing the behaviour and the motions of the atmosphere, in the *troposhere layer*, at mesoscale range, in the case when the atmosphere is modeled as a mixture consisting of different non-viscous, compressible, heat conducting components, having own density. Thus, we introduce the densities ρ_m $(m = v, c, r, i, , \dots)$ of these components (called in Castorina *et al.* (2018) flux variables), as the dry air density ρ_d, the water vapor ρ_v, the cloud water ρ_c, the rainwater density ρ_r, the ice density ρ_i, so that the total mass density ρ of the atmosphere is defined as the sum of these densities (Klemp, Skamarock, and Dudhia 2007), *i.e.*

$$\rho = \rho_d + \rho_v + \rho_c + \rho_r + \rho_i + \dots \tag{79}$$

We define the mixing ratios

$$q_m = \frac{\rho_m}{\rho_d} \qquad (m = v, c, r, i, \dots) \tag{80}$$

of water vapor, cloud water, rainwater, icedensity, that are referred to the dry air density ρ_d, because, among all the various species density, ρ_d is a quantity that is conserved and can be used to derive the evolution equations for the densities of the other components. We know that *the dry air cannot be condensed.*

By virtue of (80), equation (79) keeps the form

$$\rho = \rho_d(1 + q_v + q_c + q_r + q_i + \dots), \tag{81}$$

or, introducing the following definition

$$\alpha = \frac{1}{\rho} \tag{82}$$

equation (81) takes the form

$$\alpha = \alpha_d(1 + q_v + q_c + q_r + q_i + \dots)^{-1}. \tag{83}$$

At mesoscale ranges, neglecting the Coriolis force, the model equations for the considered mixture, in the z-system (introduced in the Subsection 2.3), have the form (see equations (9) of Subsection 2.2 for a comparison):

$$\begin{cases} \rho \dfrac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{F} + \rho \mathbf{g}, \\ \dfrac{\partial \rho_d}{\partial t} + \nabla \cdot (\rho_d \mathbf{v}) = 0 \\ \dfrac{\partial \rho_m}{\partial t} + \nabla \cdot (\mathbf{v} \rho_m) = F_{\rho_m} \qquad (m = v, c, r, i, \dots) \\ \dfrac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta = F_\theta, \end{cases} \tag{84}$$

where the Cariolis force is neglected, ρ, ρ_m, θ, p, $\mathbf{F}$ are the total mass density, the m-contituent density, the temperature, the pressure, the external force fields, respectively, referred to the mixture, depending on $(\mathbf{x}, \mathbf{t})$, $\mathbf{x} \equiv (x, y, z)$ and measured in the local rectangular Cartesian reference frame $T \equiv (P, x, y, z)$ solidal to the rotating Earth and the gravity acceleration $\mathbf{g}$, supposed constant, is defined as in Subsection 2.2. Furthermore from $(84)_1$ it is seen that $\mathbf{v}$ is the velocity field, with components u, v, w with respect to the z-system, $\mathbf{v} = (u, v, w)$, of the whole mixture, having density ρ given by (79).

Furthermore, in (84) the symbols ∇" and "$\nabla \cdot$ " are the gradient and the divergence operators calculated in T and $F_{\rho_m} \quad (m = v, c, r, i, \dots)$ represent the source or subtraction terms for the m components of the mixture.

Equation $(84)_1$ in balance equation form is given by

$$\frac{\partial(\rho_d \mathbf{v})}{\partial t} + \nabla_z \cdot (\rho_d \mathbf{v} \oplus \mathbf{v}) = -\frac{\rho_d}{\rho} \nabla_z p + \frac{\rho_d}{\rho} \mathbf{F} + \rho_d \mathbf{g}, \tag{85}$$

obtained multiplying equation $(84)_1$ by $\frac{\rho_d}{\rho}$, equation $(84)_2$ by $\mathbf{v}$, summing the two obtained results and taking into account the following identity

$$\nabla_z \cdot (\rho_d \mathbf{v} \oplus \mathbf{v}) = \mathbf{v} \nabla_z \cdot (\rho_d \mathbf{v}) + \rho_d \mathbf{v} \cdot \nabla_z \mathbf{v}. \tag{86}$$

In (85) and (86) the symbol ∇ is indicated by ∇_z to specify that the used non-inertial reference frame $T \equiv (P, x, y, z)$ is also called z-system. Equation $(84)_4$ can be written in the following balance equation form

$$\frac{\partial}{\partial t}(\rho_d\theta)+\nabla_z\cdot(\rho_d\mathbf{v}\theta)=\rho_d\mathbf{F}_\theta, \tag{87}$$

that can be obtained adding $(84)_4$ multiplied by ρ_d to $(84)_2$ multiplied by θ.

To the previous two equations (85) and (87) we must add $(84)_{2,3}$ to close the system of equations describing the atmospheric motions.

In the following Subsections we will perform some calculations in z-system and η-system.

5.1. Derivation of the first component of the momentum balance equation (85). In the z-system the component of (85) in x-direction has the following form:

$$\left[\frac{\partial(\rho_d u)}{\partial t}\right]_z+\nabla_z\cdot(\rho_d\mathbf{v}u)=-\frac{\rho_d}{\rho}\left(\frac{\partial p}{\partial x}\right)_z+\frac{\rho_d}{\rho}F_x, \tag{88}$$

where F_x stands for the component of the force in x-direction.

Using (20), (22), (31) and (34) we can transform (88) in η-system as follows:

$$\left[\frac{\partial(\rho_d u)}{\partial t}\right]_\eta-\frac{\partial(\rho_d u)}{\partial\eta}\frac{\partial\eta}{\partial z}\left(\frac{\partial z}{\partial t}\right)_\eta+\mathbf{v}_H\cdot\left[\nabla^1_\eta(\rho_d u)-\frac{\partial(\rho_d u)}{\partial\eta}\frac{\partial\eta}{\partial z}\nabla^1_\eta z\right]+\Big\{\dot\eta+\frac{\partial\eta}{\partial z}\Big[\left(\frac{\partial z}{\partial t}\right)_\eta+ \mathbf{v}_H\cdot\nabla^1_\eta z\Big]\Big\}\frac{\partial(\rho_d u)}{\partial\eta}+(\rho_d u)\Big\{\nabla_\eta\cdot\mathbf{v}+\frac{\partial\eta}{\partial z}\Big[\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial\eta}\right)_\eta+\mathbf{v}_H\cdot\nabla^1_\eta\frac{\partial z}{\partial\eta}+\dot\eta\frac{\partial}{\partial\eta}\left(\frac{\partial z}{\partial\eta}\right)\Big]\Big\}= -\frac{\rho_d}{\rho}\left[\left(\frac{\partial p}{\partial x}\right)_\eta-\frac{\partial p}{\partial\eta}\frac{\partial\eta}{\partial z}\left(\frac{\partial z}{\partial x}\right)_\eta\right]+\frac{\rho_d}{\rho}F_x, \tag{89}$$

or, equivalently as:

$$\left[\frac{\partial(\rho_d u)}{\partial t}\right]_\eta+\mathbf{v}\cdot\nabla_\eta(\rho_d u)+(\rho_d u)\nabla_\eta\cdot\mathbf{v}+(\rho_d u)\frac{\partial\eta}{\partial z}\frac{d}{dt}\left(\frac{\partial z}{\partial\eta}\right)= -\frac{\rho_d}{\rho}\left[\left(\frac{\partial p}{\partial x}\right)_\eta-\frac{\partial p}{\partial\eta}\frac{\partial\eta}{\partial z}\left(\frac{\partial z}{\partial x}\right)_\eta\right]+\frac{\rho_d}{\rho}F_x, \tag{90}$$

with $\mathbf{v}_H\equiv(u,v)$ the horizontal barycentric velocity of the mixture.

Equation (41) multiplied by $\dfrac{\partial z}{\partial\eta}$ keeps the form:

$$\frac{\partial z}{\partial\eta}\left[\left(\frac{d(\rho_d u)}{dt}\right)_\eta+(\rho_d u)\nabla_\eta\cdot\mathbf{v}\right]+(\rho_d u)\frac{d}{dt}\left(\frac{\partial z}{\partial\eta}\right)= -\frac{\rho_d}{\rho}\left[\left(\frac{\partial p}{\partial x}\right)_\eta-\frac{\partial p}{\partial\eta}\frac{\partial\eta}{\partial z}\left(\frac{\partial z}{\partial x}\right)_\eta\right]\frac{\partial z}{\partial\eta}+\frac{\rho_d}{\rho}F_x\frac{\partial z}{\partial\eta}, \tag{91}$$

or equivalently as

$$\frac{d}{dt}\left[(\rho_d u)\frac{\partial z}{\partial\eta}\right]_\eta+(\rho_d u)\frac{\partial z}{\partial\eta}\nabla_\eta\cdot\mathbf{v}=-\frac{\rho_d}{\rho}\left(\frac{\partial p}{\partial x}\right)_\eta\frac{\partial z}{\partial\eta}+\frac{\rho_d}{\rho}\frac{\partial p}{\partial\eta}\left(\frac{\partial z}{\partial x}\right)_\eta+\frac{\rho_d}{\rho}F_x\frac{\partial z}{\partial\eta}. \tag{92}$$

Equation (92) takes the following balance equation form:

$$\frac{\partial}{\partial t}\left[(\rho_d u)\frac{\partial z}{\partial\eta}\right]_\eta+\nabla_\eta\cdot\left[(\rho_d u)\frac{\partial z}{\partial\eta}\mathbf{v}\right]+\frac{\rho_d}{\rho}\left(\frac{\partial p}{\partial x}\right)_\eta\frac{\partial z}{\partial\eta}-\frac{\rho_d}{\rho}\frac{\partial p}{\partial\eta}\left(\frac{\partial z}{\partial x}\right)_\eta=\frac{\rho_d}{\rho}F_x\frac{\partial z}{\partial\eta}. \tag{93}$$

If we introduce now the geopotential $\Phi(x,y,\eta,t)$ for the mixture, that we continue to call Φ as in (45), defined by

$$\Phi = gz(x,y,\eta,t), \tag{94}$$

after having assumed g constant, equation (93) becomes:

$$\frac{\partial}{\partial t}\left[(\rho_d u)\frac{\partial \Phi}{\partial \eta}\right]_\eta + \nabla_\eta \cdot \left[(\rho_d u)\frac{\partial \Phi}{\partial \eta}\mathbf{v}\right] + \frac{\rho_d}{\rho}\left(\frac{\partial p}{\partial x}\right)_\eta \frac{\partial \Phi}{\partial \eta} - \frac{\rho_d}{\rho}\frac{\partial p}{\partial \eta}\left(\frac{\partial \Phi}{\partial x}\right)_\eta = \frac{\rho_d}{\rho}F_x\frac{\partial \Phi}{\partial \eta}, \tag{95}$$

being $\frac{\partial z}{\partial \eta} = \frac{1}{g}\frac{\partial \Phi}{\partial \eta}$. Let us consider the hydrostatic pressure equation for the dry air, in terms of Φ as follows (see equations (2.20) in Skamarock *et al.* (2008) and (21) in Castorina *et al.* (2018)):

$$\mu_d = -\rho_d \frac{\partial \Phi}{\partial \eta}, \tag{96}$$

with (see equations (2.11) in Skamarock *et al.* (2008) and (21) in Castorina *et al.* (2018))

$$\eta = \frac{p_{hd} - p_{hdt}}{p_{hd\eta} - p_{hdt}} \equiv \frac{p_{hd\eta} - p_{hdt\eta}}{\mu_d}, \tag{97}$$

or equivalently as:

$$p_{hd\eta} - p_{hdt} = \mu_d \eta,$$

where p_{hd} is the hydrostatic component of the pressure of dry air (related to the weight of an atmospheric column of unit area above a point (x,y) at eight z), p_{hdt} a constant pressure of the dry air, related to the weight of an atmospheric column of unit area corresponding to the upper (fictitius) boundary of the troposphere (indeed, we cannot adopt a vertical coordinate which extends to infinity, predicting atmospheric motions using numerical methods), $p_{hd\eta}$ stands for the pressure of the dry air at surface $\eta = 1$ and

$$p_{hd\eta} - p_{hdt} = \mu_d. \tag{98}$$

In equation (98) μ_d is defined as the difference between the pressures $p_{hd\eta}$ and p_{hdt}. Thus, it has the physical dimension of a pressure.

Being the coordinate η decreasing from $\eta = 1$, conciding with the lower boundary, to $\eta = 0$ at the top of the atmosphere (in the considered *troposhere layer*), from (97) it follows that the lower boundary coincides with a coordinate pressure surface, *i.e.* $\eta = 1$ corresponding to $p_{hd} = p_{hd\eta}$.

The hydrostatic pressure p_{hd} is defined by the equation

$$\frac{\partial p_{hd}}{\partial z} = -\rho_d g, \tag{99}$$

whose solution gives the p_{hd}. From (99) we see that the hydrostatic pressure p_{hd} is a monotonic single value function of z.

Then, we can select p_{hd} as independent variable in the limited area models and therefore by virtue of (97) to choose in these models the vertical coordinate η, defined by means of p_{hd}, as in Subsection 4.2, in order to describe non-hydrostatic situations.

When we use the vertical coordinate η, defined by p_{hd}, the Earth surface can be considered as a pressure coordinate surface, even if the coordinate η-system has some computational disadvantages near the mountains, as in this case the lower boundary of the atmosphere is not a surface coordinate.

We observe that the diagnostic equation for the hydrostatic pressure of the dry air can be written as follows:

$$\frac{\partial p_{hd}}{\partial \eta}\frac{\partial \eta}{\partial z} = -\rho_d g, \tag{100}$$

or

$$\frac{\partial p_{hd}}{\partial \eta} = -\rho_d g\frac{\partial z}{\partial \eta} = -\rho_d\frac{\partial \Phi}{\partial \eta}. \tag{101}$$

Taking into account that $\mu_d = p_{hd\eta} - p_{hdt}$ does not depend on η, we have

$$\frac{\partial p_{hd}}{\partial \eta} = \frac{\partial p_{hd}}{\partial(\mu_d\eta)}\frac{\partial(\mu_d\eta)}{\partial \eta} = \mu_d. \tag{102}$$

Substituting (102) in (101) we have

$$\frac{\partial \Phi}{\partial \eta} = -\alpha_d\mu_d, \tag{103}$$

i.e. equation (96) (see also equation (2.21) in in Skamarock *et al.* (2008) and (21) in Castorina *et al.* (2018)).

Equation (95), introducing μ_d, becomes:

$$\frac{\partial}{\partial t}\Big[(\mu_d u)\Big]_\eta + \nabla_\eta\cdot\Big(\mu_d u\mathbf{v}\Big) + \frac{\mu_d}{\rho}\Big(\frac{\partial p}{\partial x}\Big)_\eta + \frac{\rho_d}{\rho}\frac{\partial p}{\partial \eta}\Big(\frac{\partial \Phi}{\partial x}\Big)_\eta = F_x\frac{\mu_d}{\rho}, \tag{104}$$

or also:

$$\frac{\partial}{\partial t}\Big[(\mu_d u)\Big]_\eta + \nabla_\eta\cdot\Big(\mu_d u\mathbf{v}\Big) + \mu_d\alpha\Big(\frac{\partial p}{\partial x}\Big)_\eta + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\Big(\frac{\partial \Phi}{\partial x}\Big)_\eta = F_x\mu_d\alpha, \tag{105}$$

where $\alpha = \dfrac{1}{\rho}$, $\alpha_d = \dfrac{1}{\rho_d}$.

In the pressure coordinate system, *equation* (105) coincides with the equations (2.13) in Skamarock *et al.* (2008) and (14) in Castorina *et al.* (2018). In Ooyama 1990 (see also Ooyama 2001) it is seen also that if we assume that in the gas mixture the volume of its condensate components can be neglected compared to the volume of the dry air and water vapor, to which the ideal gas law can be applied, and assume that these two gases occupy the same volume without having physical interactions, the diagnostic (constitutive) equation for the total hydrodynamic pressure (vapor plus dry air) is given by

$$p = p_0\Big(\frac{R_d\theta_m}{p_0\alpha_d}\Big)^\gamma, \tag{106}$$

(106) where R_d is the gas constant for dry air, p_0 is a reference pressure, the potential temperature θ_m is given by

$$\theta_m = \theta[1 + (R_v/R_d)q_v] \approx \theta(1 + 1.61q_v). \tag{107}$$

The solution of (107) is obtained by particular numerical procedures (see (22) in Castorina *et al.* (2018) and (2.21) in Skamarock *et al.* (2008)).

5.2. Derivation of the second and third component of the momentum balance equation (85). In an anologous way as in Subsection 4.1 we can derive the second component of balance momentum of equations (85) in y-direction:

$$\frac{\partial}{\partial t}\Big[(\mu_d v)\Big]_\eta + \nabla_\eta \cdot \Big(\mu_d v \mathbf{v}\Big) = -\mu_d \alpha \Big(\frac{\partial p}{\partial y}\Big)_\eta - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\Big(\frac{\partial \Phi}{\partial y}\Big)_\eta + F_y \mu_d \alpha.$$

We derive now the evolution equation for the vertical component w of the velocity field in the η-system.

In the z-system the component of (85) in z-direction is the following:

$$\Big[\frac{\partial(\rho_d w)}{\partial t}\Big]_z + \nabla_z \cdot (\rho_d \mathbf{v} w) = -\frac{\rho_d}{\rho}\Big(\frac{\partial P}{\partial z}\Big)_z + \frac{\rho_d}{\rho}F_z - \rho_d g. \tag{108}$$

By using (17), (20), (22) and (34) equation (108) can be written as follows:

$$\Big[\frac{\partial(\rho_d w)}{\partial t}\Big]_\eta - \frac{\partial(\rho_d w)}{\partial \eta}\frac{\partial \eta}{\partial z}\Big(\frac{\partial z}{\partial t}\Big)_\eta + \mathbf{v}_H \cdot \Big[\nabla^1_\eta(\rho_d w) - \frac{\partial(\rho_d w)}{\partial \eta}\frac{\partial \eta}{\partial z}\nabla^1_\eta z\Big] + \Big\{\dot{\eta} + \frac{\partial \eta}{\partial z}\Big[\Big(\frac{\partial z}{\partial t}\Big)_\eta + \mathbf{v}_H \cdot \nabla^1_\eta z\Big]\Big\}\frac{\partial(\rho_d w)}{\partial \eta} + (\rho_d w)\Big\{\nabla_\eta \cdot \mathbf{v} + \frac{\partial \eta}{\partial z}\Big[\frac{\partial}{\partial t}\Big(\frac{\partial z}{\partial \eta}\Big)_\eta + \mathbf{v}_H \cdot \nabla^1_\eta \frac{\partial z}{\partial \eta} + \dot{\eta}\frac{\partial}{\partial \eta}\Big(\frac{\partial z}{\partial \eta}\Big)\Big]\Big\} = -\frac{\rho_d}{\rho}\Big(\frac{\partial p}{\partial \eta}\Big)_\eta \frac{\partial \eta}{\partial z} + \frac{\rho_d}{\rho}F_z - \rho_d g, \tag{109}$$

or equivalently as:

$$\Big[\frac{\partial(\rho_d w)}{\partial t}\Big]_\eta + \mathbf{v} \cdot \nabla_\eta(\rho_d w) + (\rho_d w)\nabla_\eta \cdot \mathbf{v} + (\rho_d w)\frac{\partial \eta}{\partial z}\frac{d}{dt}\Big(\frac{\partial z}{\partial \eta}\Big) = -\frac{\rho_d}{\rho}\Big(\frac{\partial p}{\partial \eta}\Big)_\eta \frac{\partial \eta}{\partial z} + \frac{\rho_d}{\rho}F_z - \rho_d g. \tag{110}$$

Equation (63) multiplied by $\frac{\partial z}{\partial \eta}$ becomes:

$$\frac{\partial z}{\partial \eta}\Big[\Big(\frac{d(\rho_d w)}{dt}\Big)_\eta + (\rho_d w)\nabla_\eta \cdot \mathbf{v}\Big] + (\rho_d w)\frac{d}{dt}\Big(\frac{\partial z}{\partial \eta}\Big) = -\frac{\rho_d}{\rho}\Big(\frac{\partial p}{\partial \eta}\Big)_\eta + \frac{\rho_d}{\rho}\frac{\partial z}{\partial \eta}F_z - \frac{\partial z}{\partial \eta}\rho_d g, \tag{111}$$

or, using the definition of geopotential (94), we have:

$$\frac{d}{dt}\Big[(\rho_d w)\frac{\partial \Phi}{\partial \eta}\Big]_\eta + (\rho_d w)\frac{\partial \Phi}{\partial \eta}\nabla_\eta \cdot \mathbf{v} = -\frac{\rho_d}{\rho}g\Big(\frac{\partial p}{\partial \eta}\Big)_\eta + \frac{\rho_d}{\rho}\frac{\partial \Phi}{\partial \eta}F_z - \frac{\partial \Phi}{\partial \eta}\rho_d g. \tag{112}$$

Taking into account the hydrostatic pressure equation for the dry air (96), equation (112) can be written as follows:

$$\frac{\partial}{\partial t}\Big[(\mu_d w)\Big]_\eta + \nabla_\eta \cdot (\mu_d w \mathbf{v}) + g\Big[\mu_d - \frac{\partial p}{\partial \eta}\frac{\rho_d}{\rho}\Big] = -\frac{\rho_d}{\rho}\frac{\partial \Phi}{\partial \eta}F_z, \tag{113}$$

and introducing α_d and α we obtain:

$$\frac{\partial}{\partial t}\Big[(\mu_d w)\Big]_\eta + \nabla_\eta \cdot (\mu_d w \mathbf{v}) + g\Big[\mu_d - \frac{\partial p}{\partial \eta}\frac{\alpha}{\alpha_d}\Big] = \alpha \mu_d F_z. \tag{114}$$

The derived equation (114) coincides with the equations (12.15) in Skamarock *et al.* (2008) and (16) in Castorina *et al.* (2018), when we choose the hydrostatic pressure coordinate system.

5.3. Pseudodensity μ_d conservation equation in the η-system. In this Subsection we see that the conservation equation for the pseudodensity μ_d in the η-system, equivalent to the dry air conservation equation in the z-system, can be determined exactly as in the case where we have considered only a fluid, starting from $(84)_2$, so that we obtain:

$$\frac{\partial \mu_d}{\partial t} + \nabla_\eta \cdot (\mu_d \mathbf{v}) = 0. \tag{115}$$

Indeed, using (17), (22), (28), we transform $(84)_2$ in the following form:

$$\left(\frac{\partial \rho_d}{\partial t}\right)_\eta - \frac{\partial \rho_d}{\partial \eta}\frac{\partial \eta}{\partial z}\left(\frac{\partial z}{\partial t}\right)_\eta + \mathbf{v}_H \cdot \left[\nabla^1_\eta \rho_d - \frac{\partial \rho_d}{\partial \eta}\frac{\partial \eta}{\partial z}\nabla^1_\eta z\right] + \left\{\dot{\eta} + \frac{\partial \eta}{\partial z}\left[\left(\frac{\partial z}{\partial t}\right)_\eta + \mathbf{v}_H \cdot \right.\right.$$

$$\left.\left.\nabla^1_\eta z\right]\right\}\frac{\partial \rho_d}{\partial \eta} + \rho_d\left\{\nabla_\eta \cdot \mathbf{v} + \frac{\partial \eta}{\partial z}\left[\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \eta}\right)_\eta + \mathbf{v}_H \cdot \nabla^1_\eta \frac{\partial z}{\partial \eta} + \dot{\eta}\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \eta}\right)\right]\right\} = 0,$$

equivalent to the equation

$$\left(\frac{\partial \rho_d}{\partial t}\right)_\eta + \mathbf{v}\nabla_\eta \rho_d + \rho_d \nabla_\eta \cdot \mathbf{v} + \rho_d \frac{\partial \eta}{\partial z}\frac{d}{dt}\left(\frac{\partial z}{\partial \eta}\right) = 0. \tag{116}$$

Multiplying (116) by $\dfrac{\partial z}{\partial \eta}$ we have:

$$\frac{d}{dt}\left(\rho_d \frac{\partial z}{\partial \eta}\right)_\eta + \rho_d \frac{\partial z}{\partial \eta}\nabla_\eta \cdot \mathbf{v} = 0. \tag{117}$$

Using (96), (94), (101) and (103) equation (117) takes the form (115), conciding with the equations (2.17) in Skamarock *et al.* (2008) and (18) in Castorina *et al.* (2018).

5.4. Evolution equation for the geopotential Φ for a mixture. In this Subsection we see that the evolution equation for the geopotential Φ for a mixture is analogous as in the case when we consider only one fluid (see Subsection 4.8).

In fact, from the definition of geopotential (74), taking into account equations (75), (94) and (96) we derive in analogous way as in Subsection 4.8 the following expression:

$$\left(\frac{\partial \Phi}{\partial t}\right)_\eta + \mu_d^{-1}[\mathbf{V}_H \cdot \nabla_s \Phi - gW] = 0, \tag{118}$$

where we have defined

$$\mathbf{V}_H \equiv \mu_d \mathbf{v}_H \equiv \mu_d(u,v) = (U,V), \quad W \equiv \mu_d w. \tag{119}$$

In (118) and (119) we have continued to call $\mathbf{V}_H$ and W the defined quantities referred to the mixture.

The obtained equation (118) coincides with the equations (2.18) in Skamarock *et al.* (2008) and (19) in Castorina *et al.* (2018), when we choose the η-system (the pressure coordinate system).

5.5. Balance equation for the temperature θ for a mixture. In this Subsection we transform equation $(84)_4$ in the η-system.

Starting from $(84)_2$ and $(84)_4$ we have in z-system,

$$\frac{\partial}{\partial t}(\rho_d\theta)+\nabla_z\cdot(\rho_d\mathbf{v}\theta)=\rho_d F_\theta. \tag{120}$$

In η-system equation (120) takes the form:

$$\Big(\frac{\partial(\rho_d\theta)}{\partial t}\Big)_\eta-\frac{\partial\rho_d\theta}{\partial\eta}\frac{\partial\eta}{\partial z}\Big(\frac{\partial z}{\partial t}\Big)_\eta+\mathbf{v}_H\cdot\Big[\nabla^1_\eta(\rho_d\theta)-\frac{\partial(\rho_d\theta)}{\partial\eta}\frac{\partial\eta}{\partial z}\nabla^1_\eta z\Big]+\Big\{\dot\eta+\frac{\partial\eta}{\partial z}\Big[\Big(\frac{\partial z}{\partial t}\Big)_\eta \tag{121}$$

$$+\mathbf{v}_0\cdot\nabla^1_\eta z\Big]\Big\}\frac{\partial(\rho_d\theta)}{\partial\eta}+\rho_d\theta\Big\{\nabla_\eta\cdot\mathbf{v}+\frac{\partial\eta}{\partial z}\Big[\frac{\partial}{\partial t}\Big(\frac{\partial z}{\partial\eta}\Big)_\eta+\mathbf{v}_H\cdot\nabla^1_\eta\frac{\partial z}{\partial\eta}+\dot\eta\frac{\partial}{\partial\eta}\Big(\frac{\partial z}{\partial\eta}\Big)\Big]\Big\}=\rho_d F_\theta.$$

The previous equation is equivalent to the following one:

$$\Big[\frac{\partial(\rho_d\theta)}{\partial t}\Big]_\eta+\mathbf{v}\cdot\nabla_\eta(\rho_d\theta)+\rho_d\theta\nabla_\eta\cdot\mathbf{v}+\rho_d\theta\frac{\partial\eta}{\partial z}\frac{d}{dt}\frac{\partial z}{\partial\eta}=\rho_d F_\theta. \tag{122}$$

Multiplying (122) by $\dfrac{\partial z}{\partial\eta}$, we have:

$$\frac{d}{dt}\Big[(\rho_d\theta)\frac{\partial z}{\partial\eta}\Big]_\eta+(\rho_d\theta)\frac{\partial z}{\partial\eta}\nabla_\eta\cdot\mathbf{v}=\rho_d\frac{\partial z}{\partial\eta}F_\theta. \tag{123}$$

Finally, taking into account (96), (94), (101) and (83) we have:

$$\frac{\partial}{\partial t}(\mu_d\theta)+\nabla_\eta\cdot(\mu_d\mathbf{v}\theta)=\mu_d F_\theta, \tag{124}$$

being $\frac{\partial z}{\partial\eta}=\frac{1}{g}\frac{\partial\Psi}{\partial\eta}$ and $\frac{\partial\Psi}{\partial\eta}=-\frac{\mu_d}{\rho_d}$.

The worked out equation (124) concides with the equations (2.16) in Skamarock *et al.* (2008) and (17) in Castorina *et al.* (2018).

5.6. Balance equations for the densities of the mixture components. In this Subsection we derive in z-system the balance equations for the densities of the mixture components, starting from $(84)_3$ and taking into account of (80). Thus, we have

$$\frac{\partial}{\partial t}(\rho_d q_m)+\nabla_z\cdot(\rho_d\mathbf{v}q_m)=F_{\rho_m}. \tag{125}$$

Equation (125) in η-system takes the form

$$\Big(\frac{\partial(\rho_d q_m)}{\partial t}\Big)_\eta-\frac{\partial\rho_d q_m}{\partial\eta}\frac{\partial\eta}{\partial z}\Big(\frac{\partial z}{\partial t}\Big)_\eta+\mathbf{v}_H\cdot\Big[\nabla^1_\eta(\rho_d q_m)-\frac{\partial(\rho_d q_m)}{\partial\eta}\frac{\partial\eta}{\partial z}\nabla^1_\eta z\Big]+$$

$$\Big\{\dot\eta+\frac{\partial\eta}{\partial z}\Big[\Big(\frac{\partial z}{\partial t}\Big)_\eta+\mathbf{v}_H\cdot\nabla^1_\eta z\Big]\Big\}\frac{\partial(\rho_d q_m)}{\partial\eta}+\rho_d q_m\Big\{\nabla_\eta\cdot\mathbf{v}+\frac{\partial\eta}{\partial z}\Big[\frac{\partial}{\partial t}\Big(\frac{\partial z}{\partial\eta}\Big)_\eta+$$

$$\mathbf{v}_H\cdot\nabla^1_\eta\frac{\partial z}{\partial\eta}+\dot\eta\frac{\partial}{\partial\eta}\Big(\frac{\partial z}{\partial\eta}\Big)\Big]\Big\}=F_{\rho_m}. \tag{126}$$

Equation (**5.6**) is equivalent to the following one:

$$\Big(\frac{\partial(\rho_d q_m)}{\partial t}\Big)_\eta+\mathbf{v}\cdot\nabla_\eta(\rho_d q_m)+\rho_d q_m\nabla_\eta\cdot\mathbf{v}+\rho_d q_m\frac{\partial\eta}{\partial z}\frac{d}{dt}\frac{\partial z}{\partial\eta}=F_{\rho_m}. \tag{127}$$

Multiplying (127) by $\dfrac{\partial z}{\partial \eta}$ we have:

$$\frac{d}{dt}\left[(\rho_d q_m)\frac{\partial z}{\partial \eta}\right]_\eta + (\rho_d q_m)\frac{\partial z}{\partial \eta}\nabla_\eta \cdot \mathbf{v} = \frac{\partial z}{\partial \eta} F_{\rho_m}. \tag{128}$$

Finally, taking into consideration equations (96), (94), (101) and (103) we obtain

$$\frac{\partial}{\partial t}(\mu_d q_m) + \nabla_\eta \cdot (\mu_d \mathbf{v} q_m) = \frac{\mu_d}{\rho_d} F_{q_m}. \tag{129}$$

The worked out equation (129) coincides with the equations (2.19) in Skamarock *et al.* (2008) and (20) in Castorina *et al.* (2018),

Conclusions

The objective of this paper was to formulate in detail way a mathematical-physical approach to derive governing equations of a meteorological model, in a non-inertial *local* rectangular Cartesian reference frame solidal to the rotating Earth, where the Coriolis force is neglected, called z-system, and also in a new η-system, different only for the vertical coordinate, introduced by means of the hydrostatic pressure, used in several applications. In the obtained model, called mesoscale limited area model WRF, detailed calculations, special assumptions and approximations and the meaning of some physical quantities (such as the geopotential, the pseudodensity and others) are given. The derived governing equations coincide with the equations presented in Skamarock *et al.* (2008) (see also Holton (2004) and Ferrari (2012)) and used to describe several meteorological situations as in Castorina *et al.* (2018) and Caccamo *et al.* (2017), where simulations of heavy snowfall and rainfall events in Sicily were done. These equations describe the motions and the behaviour of the atmosphere, at troposphere layer and at mesoscale range, by a model for a mixture of components (water vapor, cloud water, rainwater, ice, ...), defined by own mixing ratios, referred to the dry air, that satisfies a conservation law. The model equations are the momentum balance equation, the pseudodensity conservation equation, the evolution equation for the geopotential, the balance equation for the temperature and the balance equations for the densities of the mixture components. In the first part of the paper a derivation of the model equations was given assuming that the atmosphere is modeled as a fluid, in particular a perfect gas, the dry air. This model can also be applied to other meteorological scenarios, such as forecasting extreme weather events, studying atmospheric dynamics in specific geographical regions, or assessing climate variability over time, with minor adjustments.

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