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DERIVATIONS OF THE STRESS-STRAIN RELATIONS FOR ANISOTROPIC VISCOANELASTIC MEDIA IN IRREVERSIBLE THERMODYNAMICS WITH INTERNAL VARIABLES

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ABSTRACT. In this paper the rheological and heat equations are derived in an anisotropic viscoanelastic media using the method of internal variables in the absence of external forces. Compared with an isotropic viscoanelastic media, where both the deformation and heat propagation are equal in all directions, this is only possible along certain privileged directions in the anisotropic viscoanelastic case. We have derived the tensor equations describing real behaviour according to privileged directions or main axes which depend on phenomenological and state coefficients. The results are of potential interest in material sciences, in material engineering, and in the analysis of biological tissues.

1. Introduction

In a recent paper V. Ciancio (2024) addressed the study of the heat transport equation in the case of ultra-fast non-linear irreversible processes. The heat transport equation was obtained by using internal variables under the assumption of isotropic viscoanelastic medium. This study has shown the importance of the relaxation phenomenon which allows a microscopic description of thermal conduction in solids. In this paper the study is extended to the case of an anisotropic viscoanelastic medium by the theory of thermodynamics of non-equilibrium (Meixner and Reik 1959; Prigogine 1961; de Groot and Mazur 1962; Kluitenberg 1962, 1967; Kluitenberg and V. Ciancio 1978; V. Ciancio 1979; V. Ciancio and Kluitenberg 1979, 1981a,b; V. Ciancio and Verhás 1990, 1991), using internal variables (Coleman and Gurtin 1991; Maugin and Muschik 1994a,b; A. Ciancio, V. Ciancio, and Farsaci 2007; V. Ciancio, A. Ciancio, and Farsaci 2008; A. Ciancio 2011; V. Ciancio 2022). For study of the behavior of rheological media, in the presence of viscous and anelastic phenomena, in addition to determining the stress-strain relationships it is important to obtain the heat propagation equation. For this purpose V. Ciancio (2024) introduced the tensor $\varepsilon^{(1)}_{\alpha\beta}$ (inelastic strain) and the vector ξ as internal variables which characterize both stress-strain relationships and which generalize the Fourier heat equation (parabolic type equation) and Maxwell-Cattaneo-Vernotte (hyperbolic type equation).

In particular, in Sects. 2, 3 and 4, are defined the affinity relations, derived the equation of Gibbs, the expression of the production of entropy, reported the relations of the phenomenological coefficients as well as the expression of the free energy. In Sect. 5, the equation stress, strain and temperature is derived for the general case. With reference to the physical phenomenon associated with heat transport, it is shown that the equation does not present tensors of odd order, unlike what happens with piezo-electric media. In Sect. 6 the physical significance of the internal variable as heat flux due to interactions between electrons and phonons within the crystal lattice of the metal is deduced. The result is that the total heat flow is related to both diffusion processes and microscopic phenomena. In Sect. 7, the heat equation is derived for an anisotropic homogeneous viscoanelastic medium. In Sect. 8, the heat transport equation obtained in Sect. 7 is applied to a gold plate of infinitesimal thickness which is heated by a very short-lived pulse emitted by a laser source. From the results obtained by applying the finite element method (FEM) using the COMSOL software (ver. 6.1). Finally, in Sect. 9 it is evident that the anisotropic nature of the metal is mainly correlated with relaxation times related to the internal variable state, in general tensor, which in the case of a homogeneous medium is a scalar. The importance of the results achieved and their possible applications are highlighted. This implies a potential interest in the diagnosis of biological tissues, which most of them are materially and functionally anisotropic (Ramírez-Torres *et al.* 2017). Other methodologies such as rational or extended thermodynamics (Müller and Ruggeri 1998; Amestoy *et al.* 2001; Jou, Casas-Vázquez, and Lebon 2010; Amestoy *et al.* 2019; Ruggeri and Sugiyama 2021; MUMPS 2025) are based on assumptions by defining a priori the nature of state and internal variables. This is a strong limitation since the results obtained are of purely mathematical value and do not find their way into applications. Another approach is based on a non conventional thermodynamical model describing heat transport in nanomaterials in the presence of superlattice irregularities caused by defects (Jou and Restuccia 2018, 2019). Finally another field of application is the use of fractional calculation which allows, as demonstrated in the case of isotropic viscoanelastic media (A. Ciancio, V. Ciancio, and Flora 2023), to determine rheological tensors on the basis of experimental data.

2. The balance equation of entropy

In the contest of irreversible processes an important role is played by the flow of heat which, classicaly, is not considered to be a state variable. Therefore we will suppose that the specific entropy s, depends not only on the specific internal energy u and the total strain $\varepsilon_{\alpha\beta}$, the tensor $\varepsilon^{(1)}_{\alpha\beta}$ describing the inelastic strain and also on a vectorial dynamic variable, ξ, that is odd function of microscopic particles velocities that have influence on the propagation phenomena which occur in the medium. Unless otherwise indicated, the representation of tensors with index follows the Einstein notation:

$$s = s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha) \tag{1}$$

where ξ_α ($\alpha = 1,2,3$) is the α-component of the vector ξ.

Definition 1. *Absolute temperature*

$$T^{-1} \overset{\text{def}}{=} \frac{\partial}{\partial u} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha) \tag{2}$$

Definition 2. *Equilibrium-stress tensor*

$$\tau^{(eq)}_{\alpha\beta} \stackrel{\text{def}}{=} -\rho T \frac{\partial}{\partial \varepsilon_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha) \tag{3}$$

Definition 3. *Affinity-stress* conjugate to $\varepsilon^{(1)}_{\alpha\beta}$

$$\tau^{(1)}_{\alpha\beta} \stackrel{\text{def}}{=} \rho T \frac{\partial}{\partial \varepsilon^{(1)}_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha) \tag{4}$$

Definition 4. *Vector j conjugate to the internal vector variable ξ conjugate to $\varepsilon^{(1)}_{\alpha\beta}$*

$$j_\alpha \stackrel{\text{def}}{=} \rho T \frac{\partial}{\partial \xi_\alpha} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha) \tag{5}$$

where $\rho = v^{-1}$ is the mass density. The tensor of total strain $\varepsilon_{\alpha\beta}$ is the symmetric part of the gradient of the displacement field. Hence this tensor is symmetric. We shall also assume that the elastic strain tensor and partial inelastic strain tensor are symmetric. Hence

$$\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha} \tag{6}$$

$$\varepsilon^{(1)}_{\alpha\beta} = \varepsilon^{(1)}_{\beta\alpha} \tag{7}$$

Moreover, we shall assume that

$$\tau^{(eq)}_{\alpha\beta} = \tau^{(eq)}_{\beta\alpha} \tag{8}$$

$$\tau^{(1)}_{\alpha\beta} = \tau^{(1)}_{\beta\alpha} \tag{9}$$

Theorem 1 (Gibbs relation).

By using (1) we can obtain the following Gibbs relation (see V. Ciancio 2024)

$$T ds = du - v\, \tau^{(eq)}_{\alpha\beta} d\varepsilon_{\alpha\beta} + v\, \tau^{(1)}_{\alpha\beta} d\varepsilon^{(1)}_{\alpha\beta} + v\, j_\alpha d\xi_\alpha \tag{10}$$

where v is the specific volume, $\tau^{(eq)}_{\alpha\beta}$ is the equilibrium stress tensor, $\tau^{(1)}_{\alpha\beta}$ is the affinity-stress conjugate to $\varepsilon^{(1)}_{\alpha\beta}$, T is the absolute temperature and j_α is the α-component of the vector j coniugate of $d\xi/dt$. In relation (10) and the following the usual summation convetion for dummy is used. From (10) we have:

$$\rho T \frac{ds}{dt} = \rho \frac{du}{dt} - \tau^{(eq)}_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} + \tau^{(1)}_{\alpha\beta} \frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt} \tag{11}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla \tag{12}$$

is the substantial derivative respect to time and v is the velocity field and ∇ is the gradient. To analyze phenomena due to viscous flows (analogus to those which occurs during flows in ordinary viscous liquids and gases) we introduce the following viscous stress tensor.

Definition 5.

$$\tau^{(vi)}_{\alpha\beta} \stackrel{\text{def}}{=} \tau_{\alpha\beta} - \tau^{(eq)}_{\alpha\beta} \tag{13}$$

where $\tau_{\alpha\beta}$ is the mechanical stress tensor which occurs in the equation of motion

$$\rho \frac{dv_\alpha}{dt} = \rho F_\alpha + \frac{\partial \tau_{\alpha\beta}}{\partial x^\beta} \tag{14}$$

and in the first law of thermodynamics

$$\rho \frac{du}{dt} = -\nabla \cdot J^{(q)} + \tau_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} \tag{15}$$

In (14) the force F_α is the volume force per unit of mass and in (15) the vector $J^{(q)}$ is the heat flux.

Theorem 2 (Entropy production).

By using the first law of thermodynamics (15) can be obtained the balance equation of the entropy

$$\rho \frac{ds}{dt} = -\nabla \cdot \left(\frac{J^{(q)}}{T} \right) + \sigma^{(s)} \tag{16}$$

and the entropy production (see V. Ciancio 2024)

$$\sigma^{(s)} = T^{-1} \left[J^{(q)} \cdot \left(-T^{-1} \nabla T \right) + \tau^{(vi)}_{\alpha\beta} \frac{d\,\varepsilon_{\alpha\beta}}{dt} + \tau^{(1)}_{\alpha\beta} \frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt} \right] \geq 0 \tag{17}$$

It is seen that the entropy production is due to three types of phenomena: the first term on the right-hand of (17) gives the contribution of the heat conduction phenomena, the second sum is the contribution of viscous phenomena, the last sum is the contribution of the variation of the dynamical variable. Each term is a production of flux $J^{(q)}$, $\tau^{(vi)}_{\alpha\beta}$, $d\varepsilon^{(1)}_{\alpha\beta}/dt$, j_α of the process and affinities conjugated to them: $T^{-1}\nabla T$, $d\,\varepsilon_{\alpha\beta}/dt$, $\tau^{(1)}_{\alpha\beta}$, $d\xi_\alpha/dt$, respectively.

3. Phenomenological equations

According to the usual procedure of non-equilibrium thermodynamics, by virtue of the form (17) for the entropy production, we have for anisotropic media the following phenomenological equations:

$$J^{(q)}_\alpha = L^{(q)(q)}_{\alpha\beta} \left(-T^{-1} \frac{\partial T}{\partial x^\beta} \right) + L^{(q)(0)}_{\alpha(\mu\nu)} \frac{d\varepsilon_{\mu\nu}}{dt} + L^{(q)(1)}_{\alpha(\mu\nu)} \tau^{(1)}_{\mu\nu} + L^{(q)(\xi)}_{\alpha\mu} \frac{d\xi_\mu}{dt} \tag{18}$$

$$\tau^{(vi)}_{\alpha\beta} = L^{(0)(q)}_{(\alpha\beta)\mu} \left(-T^{-1} \frac{\partial T}{\partial x^\mu} \right) + L^{(0)(0)}_{(\alpha\beta)(\mu\nu)} \frac{d\varepsilon_{\mu\nu}}{dt} + L^{(0)(1)}_{(\alpha\beta)(\mu\nu)} \tau^{(1)}_{\mu\nu} + L^{(0)(\xi)}_{(\alpha\beta)\mu} \frac{d\xi_\mu}{dt} \tag{19}$$

$$\frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} = L^{(1)(q)}_{(\alpha\beta)\mu} \left(-T^{-1} \frac{\partial T}{\partial x^\mu} \right) + L^{(1)(0)}_{(\alpha\beta)(\mu\nu)} \frac{d\varepsilon_{\mu\nu}}{dt} + L^{(1)(1)}_{(\alpha\beta)(\mu\nu)} \tau^{(1)}_{\mu\nu} + L^{(1)(\xi)}_{(\alpha\beta)\mu} \frac{d\xi_\mu}{dt} \tag{20}$$

$$j_\alpha = L^{(\xi)(q)}_{\alpha\beta} \left(-T^{-1} \frac{\partial T}{\partial x^\beta} \right) + L^{(\xi)(0)}_{\alpha(\mu\nu)} \frac{d\varepsilon_{\mu\nu}}{dt} + L^{(\xi)(1)}_{\alpha(\mu\nu)} \tau^{(1)}_{\mu\nu} + L^{(\xi)(\xi)}_{\alpha\beta} \frac{d\xi_\beta}{dt} \tag{21}$$

The tensors L are called phenomenological tensors and the indices of these tensors enclosed in round brackets mean that they are symmetrical because the tensors $\varepsilon_{\mu\nu}$, $\varepsilon^{(1)}_{\alpha\beta}$, $\tau^{(1)}_{\alpha\beta}$ and $\tau^{(vi)}_{\alpha\beta}$ are symmetric. The first of these equations may be regarded as a generalization of Fourier's law. Equation (19) describes the viscous flow phenomenon and it may be considered to be a generalization of Stockes-Navier's law.

3.1. Reciprocal Onsager-Casimir relations. Since the time derivatives $d\varepsilon_{\alpha\beta}/dt$, $d\varepsilon^{(1)}_{\alpha\beta}/dt$, the heat flow $J^{(q)}$ and j vector coniugate to the internal variabile ξ are odd functions of the microscopic particle velocities and the stress $\tau^{(vi)}_{\alpha\beta}$, $\tau^{(1)}_{\alpha\beta}$, $d\xi_\alpha/dt$ and the temperature gradient $\partial T/\partial t$ are even functions of these velocities. By virtue of (7) and (9), the phenomenological tensors may be chosen so that one obtain the Onsager's reciprocal relations:

$$L^{(q)(q)}_{\alpha\beta} = L^{(q)(q)}_{\beta\alpha}\,,\; L^{(\xi)(q)}_{\alpha\beta} = L^{(\xi)(q)}_{\beta\alpha}\,,\; L^{(q)(\xi)}_{\alpha\beta} = L^{(q)(\xi)}_{\beta\alpha}\,,\; L^{(\xi)(\xi)}_{\alpha\beta} = L^{(\xi)(\xi)}_{\beta\alpha} \tag{22}$$

$$L^{(q)(0)}_{(\alpha\beta)\mu} = L^{(q)(0)}_{\mu(\beta\alpha)}\,,\; L^{(0)(q)}_{(\alpha\beta)\mu} = L^{(0)(q)}_{\mu(\beta\alpha)}\,,\; L^{(q)(1)}_{(\alpha\beta)\mu} = L^{(q)(1)}_{\mu(\beta\alpha)}\,,\; L^{(1)(q)}_{(\alpha\beta)\mu} = L^{(1)(q)}_{\mu(\beta\alpha)} \tag{23}$$

$$L^{(0)(0)}_{(\alpha\beta)(\mu\nu)} = L^{(0)(0)}_{(\mu\nu)(\alpha\beta)}\,,\; L^{(1)(1)}_{(\alpha\beta)(\mu\nu)} = L^{(1)(1)}_{(\mu\nu)(\alpha\beta)} \tag{24}$$

$$L^{(1)(0)}_{(\alpha\beta)(\mu\nu)} = L^{(1)(0)}_{(\mu\nu)(\alpha\beta)}\,,\; L^{(0)(1)}_{(\alpha\beta)(\mu\nu)} = L^{(0)(1)}_{(\mu\nu)(\alpha\beta)} \tag{25}$$

The Casimir's reciprocal relations read (de Groot and Mazur 1962):

$$L^{(\xi)(q)}_{\alpha\beta} = -L^{(q)(\xi)}_{\beta\alpha} \tag{26}$$

$$L^{(0)(q)}_{(\alpha\beta)\mu} = -L^{(q)(0)}_{\mu(\beta\alpha)} \tag{27}$$

$$L^{(q)(1)}_{(\alpha\beta)\mu} = -L^{(1)(q)}_{\mu(\beta\alpha)} \tag{28}$$

$$L^{(1)(0)}_{(\alpha\beta)(\mu\nu)} = -L^{(0)(1)}_{(\mu\nu)(\alpha\beta)} \tag{29}$$

Equations (22)-(29) reduce the number of indipendent components of the phenomenological tensors.

3.2. Quadratic form of entropy production. Equations (18)-(21) are the phenomenological equations for the irreversible process of the dynamic degrees of freedom. Hence,substituting Eqs. (18)-(21) into (17), in the anistropic case one has:

$$\begin{aligned} T\sigma^{(s)} = {} & \left(L^{(q)(q)}_{\alpha\beta}\left(-T^{-1}\frac{\partial T}{\partial x^\beta}\right) + L^{(q)(0)}_{\alpha(\mu\nu)}\frac{d\varepsilon_{\mu\nu}}{dt} + L^{(q)(1)}_{\alpha(\mu\nu)}\tau^{(1)}_{\mu\nu} + L^{(q)(\xi)}_{\alpha\mu}\frac{d\xi_\mu}{dt}\right)\left(-T^{-1}\frac{\partial T}{\partial x^\alpha}\right) + \\ & + \left(L^{(0)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\nu)}\frac{d\varepsilon_{\mu\nu}}{dt} + L^{(0)(1)}_{(\alpha\beta)(\mu\nu)}\tau^{(1)}_{\mu\nu} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_\mu}{dt}\right)\left(\frac{d\varepsilon_{\alpha\beta}}{dt}\right) + \\ & + \left(L^{(1)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right) + L^{(1)(0)}_{(\alpha\beta)(\mu\nu)}\frac{d\varepsilon_{\mu\nu}}{dt} + L^{(1)(1)}_{(\alpha\beta)(\mu\nu)}\tau^{(1)}_{\mu\nu} + L^{(1)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_\mu}{dt}\right)\tau^{(1)}_{\alpha\beta} + \\ & + \left(L^{(\xi)(q)}_{\alpha\beta}\left(-T^{-1}\frac{\partial T}{\partial x^\beta}\right) + L^{(\xi)(0)}_{\alpha(\mu\nu)}\frac{d\varepsilon_{\mu\nu}}{dt} + L^{(\xi)(1)}_{\alpha(\mu\nu)}\tau^{(1)}_{\mu\nu} + L^{(\xi)(\xi)}_{\alpha\beta}\frac{d\xi_\beta}{dt}\right)\left(\frac{d\xi_\alpha}{dt}\right) \end{aligned} \tag{30}$$

By virtue of the Onsager-Casimir's relations, then production entropy is a quadratic form with positive definite character and (30) becomes :

$$\sigma^{(s)} = T^{-1}\left[L^{(q)(q)}_{\alpha\beta}\left(T^{-2}\frac{\partial^2 T}{\partial x^\alpha \partial x^\beta}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\nu)}\frac{d\varepsilon_{\alpha\beta}}{dt}\frac{d\varepsilon_{\mu\nu}}{dt} + \right. \\ \left. + L^{(1)(1)}_{(\alpha\beta)(\mu\nu)}\tau^{(1)}_{\alpha\beta}\tau^{(1)}_{\mu\nu} + L^{(\xi)(\xi)}_{\alpha\beta}\frac{d\xi_\alpha}{dt}\frac{d\xi_\beta}{dt}\right] \tag{31}$$

4. Linear equations of state

Let f be the specific free energy of the medium

$$f = u - T\,s \tag{32}$$

With the aid of Gibbs relation (10) we have:

$$df = v\,\tau^{(eq)}_{\alpha\beta}d\varepsilon_{\alpha\beta} - v\tau^{(1)}_{\alpha\beta}d\varepsilon^{(1)}_{\alpha\beta} - v\,j_\alpha\,d\xi_\alpha - sdT\,. \tag{33}$$

From this expression it can be said that free energy is a function of the total deformation, of the inelastic component of the deformation tensor, of the internal variable and temperature:

$$f = f(\varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha, T) \tag{34}$$

The total differential of free energy is given by:

$$df = \frac{\partial f}{\partial \varepsilon_{\alpha\beta}}d\varepsilon_{\alpha\beta} + \frac{\partial f}{\partial \varepsilon^{(1)}_{\alpha\beta}}d\varepsilon^{(1)}_{\alpha\beta} + \frac{\partial f}{\partial \xi_\alpha}d\xi_\alpha + \frac{\partial f}{\partial T}dT \tag{35}$$

From the comparison of (33) with (35) we obtain:

$$\frac{\partial f}{\partial \varepsilon_{\alpha\beta}} = v\,\tau^{(eq)}_{\alpha\beta} \tag{36}$$

$$-\frac{\partial f}{\partial \varepsilon^{(1)}_{\alpha\beta}} = v\,\tau^{(1)}_{\alpha\beta} \tag{37}$$

$$-\frac{\partial f}{\partial \xi_\alpha} = v\,j_\alpha \tag{38}$$

$$-\frac{\partial f}{\partial T} = s \tag{39}$$

Let us choose a temperature T_0. Furthermore, let us consider a state of the medium with the uniforme temperature T_0, in which the equilibrium-stress tensor, affinity-stress tensor and the flux j_α vanish. Hence:

$$\tau^{(eq)}_{\alpha\beta}(\varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha, T_0) = \rho\left(\frac{\partial f}{\partial \varepsilon_{\alpha\beta}}\right)_{\Sigma_0} = 0 \tag{40}$$

$$\tau^{(1)}_{\alpha\beta}(\varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha, T_0) = -\rho\left(\frac{\partial f}{\partial \varepsilon^{(1)}_{\alpha\beta}}\right)_{\Sigma_0} = 0 \tag{41}$$

$$j_\alpha(\varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha, T_0) = -\rho \left(\frac{\partial f}{\partial \xi_\alpha} \right)_{\Sigma_0} = 0 \tag{42}$$

The state (with temperature T_0) characterized by (40)-(42), will be called the *reference state* Σ_0 . All strain will be measured with respect to this state, i.e., we choose the tensors $\varepsilon_{\alpha\beta}$, $\varepsilon^{(1)}_{\alpha\beta}$ and ξ_α so that they vanish in the reference state. Hence:

$$\tau^{(eq)}_{\alpha\beta} = 0 \quad , \quad \varepsilon_{\alpha\beta} = 0 \quad for \quad T = T_0 \tag{43}$$

$$\tau^{(1)}_{\alpha\beta} = 0 \quad , \quad \varepsilon^{(1)}_{\alpha\beta} = 0 \quad for \quad T = T_0 \tag{44}$$

$$j_\alpha = 0 \quad , \quad \xi_\alpha = 0 \quad for \quad T = T_0 \tag{45}$$

Considering the procedure used by V. Ciancio (2024), we obtain:

$$\begin{aligned}
\rho f = \frac{1}{2} & \left(a^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} \varepsilon_{\alpha\beta} \varepsilon_{\mu\gamma} + a^{(1)(1)}_{(\alpha\beta)(\mu\gamma)} \varepsilon^{(1)}_{\alpha\beta} \varepsilon^{(1)}_{\mu\gamma} + a^{(\xi)(\xi)}_{(\alpha\beta)} \xi_\alpha \xi_\beta \right) + \\
& + a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} \varepsilon_{\alpha\beta} \varepsilon^{(1)}_{\mu\gamma} + a^{(0)(\xi)}_{(\alpha\beta)\mu} \varepsilon_{\alpha\beta} \xi_\mu + \\
& + a^{(1)(\xi)}_{(\alpha\beta)\mu} \varepsilon^{(1)}_{\alpha\beta} \xi_\mu + a^{(T)(\xi)}_{\mu} (T - T_0) \xi_\mu + \\
& + a^{(0)(T)}_{(\alpha\beta)} (T - T_0) \varepsilon_{\alpha\beta} + a^{(1)(T)}_{(\alpha\beta)} (T - T_0) \varepsilon^{(1)}_{\alpha\beta} + \Psi(T)
\end{aligned} \tag{46}$$

$$\tau^{(eq)}_{\alpha\beta} = a^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} \varepsilon_{\mu\gamma} + a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} \varepsilon^{(1)}_{\mu\gamma} + a^{(0)(\xi)}_{(\alpha\beta)\mu} \xi_\mu + a^{(0)(T)}_{(\alpha\beta)} (T - T_0) \tag{47}$$

$$-\tau^{(1)}_{\alpha\beta} = a^{(1)(1)}_{(\alpha\beta)(\mu\gamma)} \varepsilon^{(1)}_{\mu\gamma} + a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} \varepsilon_{\mu\gamma} + a^{(1)(\xi)}_{(\alpha\beta)\mu} \xi_\mu + a^{(1)(T)}_{(\alpha\beta)} (T - T_0) \tag{48}$$

$$-j_\alpha = a^{(\xi)(\xi)}_{(\alpha\beta)} \xi_\beta + a^{(0)(\xi)}_{\alpha(\mu\gamma)} \varepsilon_{\mu\gamma} + a^{(1)(\xi)}_{\alpha(\mu\gamma)} \varepsilon^{(1)}_{\mu\gamma} + a^{(T)(\xi)}_{\alpha} (T - T_0) \tag{49}$$

Based on Eqs. (32) and (39), we have:

$$u = f - T \frac{\partial f}{\partial T} \tag{50}$$

Then

$$\begin{aligned}
u = & \nu \frac{1}{2} \left(a^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} \varepsilon_{\alpha\beta} \varepsilon_{\mu\gamma} + a^{(1)(1)}_{(\alpha\beta)(\mu\gamma)} \varepsilon^{(1)}_{\alpha\beta} \varepsilon^{(1)}_{\mu\gamma} + a^{(\xi)(\xi)}_{(\alpha\beta)} \xi_\alpha \xi_\beta \right) + \\
& + \nu \left(a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} \varepsilon_{\alpha\beta} \varepsilon^{(1)}_{\mu\gamma} + a^{(0)(\xi)}_{(\alpha\beta)\mu} \varepsilon_{\alpha\beta} \xi_\mu + a^{(1)(\xi)}_{(\alpha\beta)\mu} \varepsilon^{(1)}_{\alpha\beta} \xi_\mu \right) + \\
& - \nu T_0 \left(a^{(T)(\xi)}_{\mu} \xi_\mu + a^{(0)(T)}_{(\alpha\beta)} \varepsilon_{\alpha\beta} + a^{(1)(T)}_{(\alpha\beta)} \varepsilon^{(1)}_{\alpha\beta} \right) + \Psi(T) - T \frac{d\Psi}{dT}
\end{aligned} \tag{51}$$

The specific heat at constant deformation, c_ε, may be defined by

$$c_\varepsilon = \frac{\partial u \left(T, \varepsilon_{\alpha\beta} \right)}{\partial T} \tag{52}$$

and hence

$$\Psi(T) = c_\varepsilon T \, log \left(\frac{T}{T_0} \right) + s_0 T - c_\varepsilon (T - T_0) - u_0 \tag{53}$$

5. The stress-strain-temperature relations

Let us place:

$$\vartheta = T - T_0 \tag{54}$$

and hence

$$\frac{dT}{dt} = \frac{d\vartheta}{dt} \tag{55}$$

Then Eqs. (47) and (48) are written in the form

$$\tau^{(eq)}_{\alpha\beta} = a^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma} + a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\varepsilon^{(1)}_{\mu\gamma} + a^{(0)(\xi)}_{(\alpha\beta)\mu}\xi_\mu + a^{(0)(T)}_{(\alpha\beta)}\vartheta \tag{56}$$

$$-\tau^{(1)}_{\alpha\beta} = a^{(1)(1)}_{(\alpha\beta)(\mu\gamma)}\varepsilon^{(1)}_{\mu\gamma} + a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma} + a^{(1)(\xi)}_{(\alpha\beta)\mu}\xi_\mu + a^{(1)(T)}_{(\alpha\beta)}\vartheta \tag{57}$$

Theorem 3 (**Local termomechanical equation in anisotropic-viscoanelastic media).**

In anisotropic-viscoanelastic media the local termomechanics equation is:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt} + R^{(\tau)(0)}_{(\alpha\beta)(\mu\gamma)}\tau_{\mu\gamma} = R^{(\varepsilon)(2)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2} + R^{(\varepsilon)(1)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + R^{(\varepsilon)(0)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma} +\\ + R^{(\vartheta)(1)}_{(\alpha\beta)}\frac{d\vartheta}{dt} + R^{(\vartheta)(0)}_{(\alpha\beta)}\vartheta + \mathscr{F}_{\alpha\beta}(\xi, T)\end{aligned} \tag{58}$$

where the second order tensor $\mathscr{F}_{\alpha\beta}(\xi, T)$ is:

$$\begin{aligned}\mathscr{F}_{\alpha\beta}(\xi, T) = R^{(T)(1)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right) + R^{(T)(0)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right) + R^{(\xi)(2)}_{(\alpha\beta)\mu}\frac{d^2\xi_\mu}{dt^2} +\\ + R^{(\xi)(1)}_{(\alpha\beta)\mu}\frac{d\xi_\mu}{dt} + R^{(\xi)(0)}_{(\alpha\beta)\mu}\xi_\mu\end{aligned} \tag{59}$$

Proof. By virtue (13) and using Eq. (19) in Eq. (56) we obtain the expression for the total stress:

$$\begin{aligned}\tau_{\alpha\beta} = L^{(0)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + L^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\tau^{(1)}_{\mu\gamma} +\\ + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_\mu}{dt} + a^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma} + a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\varepsilon^{(1)}_{\mu\gamma} + a^{(0)(\xi)}_{(\alpha\beta)\mu}\xi_\mu + a^{(0)(T)}_{(\alpha\beta)}\vartheta\end{aligned} \tag{60}$$

Substituting the expression (57) of $\tau^{(1)}_{\mu\gamma}$ changed the sign in the equation in (60) we obtain:

$$\begin{aligned}\tau_{\alpha\beta} = L^{(0)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_\mu}{dt} + a^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma} +\\ + a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\varepsilon^{(1)}_{\mu\gamma} + a^{(0)(\xi)}_{(\alpha\beta)\mu}\xi_\mu + a^{(0)(T)}_{(\alpha\beta)}\vartheta + L^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\left(-a^{(1)(1)}_{(\mu\gamma)(\phi\omega)}\varepsilon^{(1)}_{\mu\gamma} -\right.\\ \left. - a^{(0)(1)}_{(\mu\gamma)(\phi\omega)}\varepsilon_{\mu\gamma} - a^{(1)(\xi)}_{(\phi\omega)\mu}\xi_\mu - a^{(1)(T)}_{(\phi\omega)}\vartheta\right)\end{aligned} \tag{61}$$

Let us place:

$$p^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} = -L^{(0)(1)}_{(\alpha\beta)(\phi\omega)}a^{(0)(1)}_{(\mu\gamma)(\phi\omega)} \tag{62}$$

$$p^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} = -L^{(0)(1)}_{(\alpha\beta)(\phi\omega)} a^{(1)(1)}_{(\mu\gamma)(\phi\omega)} \tag{63}$$

$$b^{(0)(\xi)}_{(\alpha\beta)\mu} = -L^{(0)(1)}_{(\alpha\beta)(\phi\omega)} a^{(1)(\xi)}_{(\phi\omega)\mu} \tag{64}$$

$$b^{(0)(T)}_{(\alpha\beta)} = -L^{(0)(1)}_{(\alpha\beta)(\phi\omega)} a^{(1)(T)}_{(\phi\omega)} \tag{65}$$

By virtue of the relations (62)-(65), we have:

$$\begin{aligned}\tau_{\alpha\beta} = {} & L^{(0)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_{\mu}}{dt} + \\ & + \left(a^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} + p^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\right)\varepsilon_{\mu\gamma} + \left(a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} + p^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\right)\varepsilon^{(1)}_{\mu\gamma} + \\ & + \left(a^{(0)(\xi)}_{(\alpha\beta)\mu} + b^{(0)(\xi)}_{(\alpha\beta)\mu}\right)\xi_{\mu} + \left(a^{(0)(T)}_{(\alpha\beta)} + b^{(0)(T)}_{(\alpha\beta)}\right)\vartheta\end{aligned} \tag{66}$$

Let us place:

$$c^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} = a^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} + p^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} \tag{67}$$

$$c^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} = a^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} + p^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} \tag{68}$$

$$c^{(0)(\xi)}_{(\alpha\beta)\mu} = a^{(0)(\xi)}_{(\alpha\beta)\mu} + b^{(0)(\xi)}_{(\alpha\beta)\mu} \tag{69}$$

$$c^{(0)(T)}_{(\alpha\beta)} = a^{(0)(T)}_{(\alpha\beta)} + b^{(0)(T)}_{(\alpha\beta)} \tag{70}$$

Then Eq. (66) takes the following expression:

$$\begin{aligned}\tau_{\alpha\beta} = {} & L^{(0)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_{\mu}}{dt} \\ & + c^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma} + c^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\varepsilon^{(1)}_{\mu\gamma} + c^{(0)(\xi)}_{(\alpha\beta)\mu}\xi_{\mu} + c^{(0)(T)}_{(\alpha\beta)}\vartheta\end{aligned} \tag{71}$$

Deriving with respect to time both members of the equation (71), we have:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt} = {} & L^{(0)(q)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d^2\xi_{\mu}}{dt^2} + \\ & + c^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + c^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon^{(1)}_{\mu\gamma}}{dt} + c^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_{\mu}}{dt} + c^{(0)(T)}_{(\alpha\beta)}\frac{d\vartheta}{dt}\end{aligned} \tag{72}$$

From Eq. (20) we obtain:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt} = {} & L^{(0)(q)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d^2\xi_{\mu}}{dt^2} + \\ & + c^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + c^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\left(L^{(1)(q)}_{(\phi\omega)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(1)(0)}_{(\mu\gamma)(\phi\omega)}\frac{d\varepsilon_{\mu\gamma}}{dt} + \right. \\ & \left. + L^{(1)(1)}_{(\mu\gamma)(\phi\omega)}\tau^{(1)}_{\mu\gamma} + L^{(1)(\xi)}_{(\phi\omega)\mu}\frac{d\xi_{\mu}}{dt}\right) + c^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_{\mu}}{dt} + c^{(0)(T)}_{(\alpha\beta)}\frac{d\vartheta}{dt}\end{aligned} \tag{73}$$

Let us place:

$$q^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} = c^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\, L^{(1)(0)}_{(\mu\gamma)(\phi\omega)} \tag{74}$$

$$q^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} = c^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\, L^{(1)(1)}_{(\mu\gamma)(\phi\omega)} \tag{75}$$

$$d^{(0)(q)}_{(\alpha\beta)\,\mu} = c^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\, L^{(1)(q)}_{(\phi\omega)\mu} \tag{76}$$

$$d^{(0)(\xi)}_{(\alpha\beta)\,\mu} = c^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\, L^{(1)(\xi)}_{(\phi\omega)\mu} \tag{77}$$

By virtue of the relations (74)-(77), we have:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt} =& L^{(0)(q)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d^2\xi_{\mu}}{dt^2} + \\ &+ d^{(0)(q)}_{(\alpha\beta)\,\mu}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + \left(c^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} + q^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\right)\frac{d\varepsilon_{\mu\gamma}}{dt} + q^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\tau^{(1)}_{\mu\gamma} + \\ &+ \left(d^{(0)(\xi)}_{(\alpha\beta)\,\mu} + c^{(0)(\xi)}_{(\alpha\beta)\mu}\right)\frac{d\xi_{\mu}}{dt} + c^{(0)(T)}_{(\alpha\beta)}\frac{d\vartheta}{dt}\end{aligned} \tag{78}$$

Let us place:

$$w^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} = c^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} + q^{(0)(0)}_{(\alpha\beta)(\mu\gamma)} \tag{79}$$

$$w^{(0)(\xi)}_{(\alpha\beta)\mu} = c^{(0)(\xi)}_{(\alpha\beta)\mu} + d^{(0)(\xi)}_{(\alpha\beta)\,\mu} \tag{80}$$

Then Eq. (78) takes the following expression:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt} =& L^{(0)(q)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d^2\xi_{\mu}}{dt^2} + \\ &+ d^{(0)(q)}_{(\alpha\beta)\,\mu}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + w^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + q^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\tau^{(1)}_{\mu\gamma} + \\ &+ w^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_{\mu}}{dt} + c^{(0)(T)}_{(\alpha\beta)}\frac{d\vartheta}{dt}\end{aligned} \tag{81}$$

Substituting the expression (57) of $\tau^{(1)}_{\mu\gamma}$ changed the sign in the equation in (81) we obtain:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt} =& L^{(0)(q)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d^2\xi_{\mu}}{dt^2} + \\ &+ d^{(0)(q)}_{(\alpha\beta)\,\mu}\left(-T^{-1}\frac{\partial T}{\partial x^{\phi}}\right) + w^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + q^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\left(-a^{(1)(1)}_{(\mu\gamma)(\phi\omega)}\varepsilon^{(1)}_{\mu\gamma} - \right. \\ &\left. - a^{(0)(1)}_{(\mu\gamma)(\phi\omega)}\varepsilon_{\mu\gamma} - a^{(1)(\xi)}_{(\phi\omega)\mu}\xi_{\mu} - a^{(1)(T)}_{(\phi\omega)}\vartheta\right) + w^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_{\mu}}{dt} + c^{(0)(T)}_{(\alpha\beta)}\frac{d\vartheta}{dt}\end{aligned} \tag{82}$$

Let us place:

$$z^{(0)(1)}_{(\alpha\beta)(\mu\gamma)} = -q^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\, a^{(1)(1)}_{(\mu\gamma)(\phi\omega)} \tag{83}$$

$$z^{(1)(1)}_{(\alpha\beta)(\mu\gamma)} = -q^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\, a^{(0)(1)}_{(\mu\gamma)(\phi\omega)} \tag{84}$$

$$z^{(1)(\xi)}_{(\alpha\beta)\mu} = -q^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\, a^{(1)(\xi)}_{(\phi\omega)\mu} \tag{85}$$

$$z^{(1)(T)}_{(\alpha\beta)} = -q^{(0)(1)}_{(\alpha\beta)(\phi\omega)}\, a^{(1)(T)}_{(\phi\omega)} \tag{86}$$

By virtue of the relations (83)-(86), we have:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt} =& L^{(0)(q)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^{\mu}}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d^2\xi_\mu}{dt^2} + \\ &+ d^{(0)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^{\phi}}\right) + w^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt} + z^{(1)(1)}_{(\alpha\beta)(\mu\gamma)}\,\varepsilon^{(1)}_{\mu\gamma} + z^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma} + \\ &+ z^{(1)(\xi)}_{(\alpha\beta)\mu}\xi_\mu + z^{(1)(T)}_{(\alpha\beta)}\,\vartheta + w^{(0)(\xi)}_{(\alpha\beta)\mu}\frac{d\xi_\mu}{dt} + c^{(0)(T)}_{(\alpha\beta)}\frac{d\vartheta}{dt}\end{aligned} \tag{87}$$

The expression of the inelastic component of the strain tensor is obtained by multiplying on the left by the inverse $\underset{\sim}{c}^{(0)(1)}$ tensor both members of Eq. (71). The tensor is invertible since it is symmetric as can be seen from Eq. (68) and let us place:

$$\underset{\sim}{c}^{(1)(0)} = \left\{c^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}\right\}^{-1} \tag{88}$$

the inverse tensor of $\underset{\sim}{c}^{(0)(1)}$, hence, from Eq. (71) we obtain:

$$\begin{aligned}\varepsilon^{(1)}_{\mu\gamma} =& c^{(1)(0)}_{(\alpha\beta)(\mu\gamma)}\,\tau_{\alpha\beta} + \chi^{(1)(q)}_{(\mu\gamma)\phi}\left(T^{-1}\frac{\partial T}{\partial x^{\phi}}\right) + d_{10}\frac{d\varepsilon_{\mu\gamma}}{dt} + \chi^{(1)(\xi)}_{(\mu\gamma)\phi}\frac{d\xi_\phi}{dt} + \\ &+ e_{10}\,\varepsilon_{\mu\gamma} + \overline{\chi}^{(1)(\xi)}_{(\mu\gamma)\phi}\xi_\phi + \chi^{(1)(T)}_{(\mu\gamma)}\,\vartheta\end{aligned} \tag{89}$$

where

$$\begin{aligned}\chi^{(1)(q)}_{(\mu\gamma)\phi} &= c^{(1)(0)}_{(\alpha\beta)(\mu\gamma)}L^{(0)(q)}_{(\alpha\beta)\phi}\\ d_{10} &= -c^{(1)(0)}_{(\alpha\beta)(\mu\gamma)}L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\\ \chi^{(1)(\xi)}_{(\mu\gamma)\phi} &= -c^{(1)(0)}_{(\alpha\beta)(\mu\gamma)}L^{(0)(\xi)}_{(\alpha\beta)\phi}\\ e_{10} &= -c^{(1)(0)}_{(\alpha\beta)(\mu\gamma)}c^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}\\ \overline{\chi}^{(1)(\xi)}_{(\mu\gamma)\phi} &= -c^{(1)(0)}_{(\alpha\beta)(\mu\gamma)}c^{(0)(\xi)}_{(\alpha\beta)\phi}\\ \chi^{(1)(T)}_{(\mu\gamma)} &= -c^{(1)(0)}_{(\alpha\beta)(\mu\gamma)}c^{(0)(T)}_{(\alpha\beta)}\end{aligned} \tag{90}$$

Substituting the expression (89) into (87), we have the stress-strain-temperature relations:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt}+R^{(\tau)(0)}_{(\alpha\beta)(\mu\gamma)}\tau_{\mu\gamma}=&R^{(\varepsilon)(2)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2}+R^{(\varepsilon)(1)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt}+R^{(\varepsilon)(0)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma}+\\&+R^{(T)(1)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right)+R^{(T)(0)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right)+\\&+R^{(\xi)(2)}_{(\alpha\beta)\mu}\frac{d^2\xi_\mu}{dt^2}+R^{(\xi)(1)}_{(\alpha\beta)\mu}\frac{d\xi_\mu}{dt}+R^{(\xi)(0)}_{(\alpha\beta)\mu}\xi_\mu+\\&+R^{(\vartheta)(1)}_{(\alpha\beta)}\frac{d\vartheta}{dt}+R^{(\vartheta)(0)}_{(\alpha\beta)}\vartheta\end{aligned}\tag{91}$$

where

$$\begin{aligned}&R^{(\tau)(0)}_{(\alpha\beta)(\mu\gamma)}=z^{(1)(1)}_{(\alpha\beta)(\mu\gamma)}c^{(1)(0)}_{(\alpha\beta)(\mu\gamma)};\,R^{(\varepsilon)(2)}_{(\alpha\beta)(\mu\gamma)}=L^{(0)(0)}_{(\alpha\beta)(\mu\gamma)};\,R^{(\varepsilon)(1)}_{(\alpha\beta)(\mu\gamma)}=w^{(0)(0)}_{(\alpha\beta)(\mu\gamma)}+d_{10}\,z^{(1)(1)}_{(\alpha\beta)(\mu\gamma)};\\&R^{(\varepsilon)(0)}_{(\alpha\beta)(\mu\gamma)}=\left(z^{(0)(1)}_{(\alpha\beta)(\mu\gamma)}+e_{10}\,z^{(1)(1)}_{(\alpha\beta)(\mu\gamma)}\right);\,R^{(T)(1)}_{(\alpha\beta)\mu}=L^{(0)(q)}_{(\alpha\beta)\mu};\,R^{(T)(0)}_{(\alpha\beta)\mu}=z^{(1)(1)}_{(\alpha\beta)(\phi\omega)}\chi^{(1)(q)}_{(\phi\omega)\mu};\\&R^{(\xi)(2)}_{(\alpha\beta)\mu}=L^{(0)(\xi)}_{(\alpha\beta)\mu};\,R^{(\xi)(1)}_{(\alpha\beta)\mu}=w^{(0)(\xi)}_{(\alpha\beta)\mu}+z^{(1)(1)}_{(\alpha\beta)(\phi\omega)}\chi^{(1)(\xi)}_{(\phi\omega)\mu};\,R^{(\xi)(0)}_{(\alpha\beta)\mu}=z^{(1)(\xi)}_{(\alpha\beta)\mu}+z^{(1)(1)}_{(\alpha\beta)(\phi\omega)}\overline{\chi}^{(1)(\xi)}_{(\phi\omega)\mu};\\&R^{(\vartheta)(1)}_{(\alpha\beta)}=c^{(0)(T)}_{(\alpha\beta)};\,R^{(\vartheta)(0)}_{(\alpha\beta)}=z^{(1)(T)}_{(\alpha\beta)}+z^{(1)(1)}_{(\alpha\beta)(\phi\omega)}\chi^{(1)(T)}_{(\phi\omega)}\end{aligned}\tag{92}$$

Let us place:

$$\begin{aligned}\mathscr{F}_{\alpha\beta}(\xi,T)=&R^{(T)(1)}_{(\alpha\beta)\mu}\frac{d}{dt}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right)+R^{(T)(0)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right)+\\&+R^{(\xi)(2)}_{(\alpha\beta)\mu}\frac{d^2\xi_\mu}{dt^2}+R^{(\xi)(1)}_{(\alpha\beta)\mu}\frac{d\xi_\mu}{dt}+R^{(\xi)(0)}_{(\alpha\beta)\mu}\xi_\mu\end{aligned}\tag{93}$$

Then the stress-strain-temperature (91) becomes:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt}+R^{(\tau)(0)}_{(\alpha\beta)(\mu\gamma)}\tau_{\mu\gamma}=&R^{(\varepsilon)(2)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2}+R^{(\varepsilon)(1)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt}+R^{(\varepsilon)(0)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma}+\\&+R^{(\vartheta)(1)}_{(\alpha\beta)}\frac{d\vartheta}{dt}+R^{(\vartheta)(0)}_{(\alpha\beta)}\vartheta+\mathscr{F}_{\alpha\beta}(\xi,T)\end{aligned}$$

that represents (58), i.e., the local thermomechanical equation in anisotropic-viscoanelastic media. □

By virtue of Eq. (93), we observe that the second order tensor $\mathscr{F}_{\alpha\beta}(\xi,T)$ is characterized by constant tensor coefficients of rank three. If the medium changes shape but not the volume, then these constant tensor coefficients of rank three are all null, i.e.:

$$\mathscr{F}_{\alpha\beta}(\xi,T)=0$$

and Eq. (58) becomes:

$$\begin{aligned}\frac{d\tau_{\alpha\beta}}{dt}+R^{(\tau)(0)}_{(\alpha\beta)(\mu\gamma)}\tau_{\mu\gamma}=&R^{(\varepsilon)(2)}_{(\alpha\beta)(\mu\gamma)}\frac{d^2\varepsilon_{\mu\gamma}}{dt^2}+R^{(\varepsilon)(1)}_{(\alpha\beta)(\mu\gamma)}\frac{d\varepsilon_{\mu\gamma}}{dt}+R^{(\varepsilon)(0)}_{(\alpha\beta)(\mu\gamma)}\varepsilon_{\mu\gamma}+\\&+R^{(\vartheta)(1)}_{(\alpha\beta)}\frac{d\vartheta}{dt}+R^{(\vartheta)(0)}_{(\alpha\beta)}\vartheta\end{aligned}\tag{94}$$

Hence, the second order tensor $\mathscr{F}_{\alpha\beta}(\xi,T)$ takes into account the change in volume due to both the internal variable and the instantaneous temperature.

6. Heat flux vector

We consider viscoanelastic anisotropic media with constant volume and strain on the time. If the shape changes but the volume remains invariate, then the odd order tensors cancel and those of even order are symmetrical. Therefore, Eqs. (18), (21) and (49) reduce, respectively, to:

$$J_\alpha^{(q)} = L_{\alpha\beta}^{(q)(q)}\left(-T^{-1}\frac{\partial T}{\partial x^\beta}\right) + L_{\alpha\beta}^{(q)(\xi)}\frac{d\xi_\beta}{dt} \tag{95}$$

$$j_\alpha = L_{\alpha\beta}^{(\xi)(q)}\left(-T^{-1}\frac{\partial T}{\partial x^\beta}\right) + L_{\alpha\beta}^{(\xi)(\xi)}\frac{d\xi_\beta}{dt} \tag{96}$$

$$-j_\alpha = a_{(\alpha\beta)}^{(\xi)(\xi)}\xi_\beta \tag{97}$$

Given the symmetry with respect to the main axes, without losing generality, one can choose the reference system coincident with them, then:

$$\left\{L_{\alpha\beta}\right\}^{(n)(m)} = \begin{bmatrix} \lambda_1^{(n)(m)} & 0 & 0 \\ 0 & \lambda_2^{(n)(m)} & 0 \\ 0 & 0 & \lambda_3^{(n)(m)} \end{bmatrix} \tag{98}$$

with $m,n \in \{q,\xi\}$ and

$$\left\{a_{\alpha\beta}\right\}^{(\xi)(\xi)} = \begin{bmatrix} A_1^{(\xi)(\xi)} & 0 & 0 \\ 0 & A_2^{(\xi)(\xi)} & 0 \\ 0 & 0 & A_3^{(\xi)(\xi)} \end{bmatrix} \tag{99}$$

Equations (95)-(97) can be written as:

$$J_\alpha^{(q)} = T\sum_{\beta=1}^{3}\lambda_\beta^{(q)(q)}\,\delta_{\alpha\beta}\left(-T^{-1}\frac{\partial T}{\partial x^\beta}\right) + \sum_{\beta=1}^{3}\lambda_\beta^{(q)(\xi)}\,\delta_{\alpha\beta}\,\frac{d\xi_\beta}{dt} \tag{100}$$

$$j_\alpha = T\sum_{\beta=1}^{3}\lambda_\beta^{(\xi)(q)}\,\delta_{\alpha\beta}\left(-T^{-1}\frac{\partial T}{\partial x^\beta}\right) + \sum_{\beta=1}^{3}\lambda_\beta^{(\xi)(\xi)}\,\delta_{\alpha\beta}\,\frac{d\xi_\beta}{dt} \tag{101}$$

$$-j_\alpha = \sum_{\beta=1}^{3}A_\beta^{(\xi)(\xi)}\,\delta_{\alpha\beta}\,\xi_\beta \tag{102}$$

where $\delta_{\alpha,\beta}$ is Kronecker's symbol and the sum of the β index from zero to three is considered. The symbols $\lambda_\beta^{(q)(q)}$; $\lambda_\beta^{(q)(\xi)}$; $\lambda_\beta^{(\xi)(q)}$; $\lambda_\beta^{(\xi)(\xi)}$; $A_\beta^{(\xi)(\xi)}$ are respectively the eigenvalues of the symmetric tensors $L_{\alpha\beta}^{(q)(q)}$; $L_{\alpha\beta}^{(q)(\xi)}$; $L_{\alpha\beta}^{(\xi)(q)}$; $L_{\alpha\beta}^{(\xi)(\xi)}$; $a_{(\alpha\beta)}^{(\xi)(\xi)}$. From Eqs. (100)-(102), we obtain:

$$J_\alpha^{(q)} = \lambda_\alpha^{(q)(q)}\left(-\frac{\partial T}{\partial x^\alpha}\right) + \lambda_\alpha^{(q)(\xi)}\,\frac{d\xi_\alpha}{dt} \tag{103}$$

$$j_\alpha = \lambda_\alpha^{(\xi)(q)}\left(-\frac{\partial T}{\partial x^\alpha}\right) + \lambda_\alpha^{(\xi)(\xi)}\,\frac{d\xi_\alpha}{dt} \tag{104}$$

$$-j_\alpha = A_\alpha^{(\xi)(\xi)}\ \xi_\alpha \tag{105}$$

with $\alpha = 1,2,3$ and in the following.

Theorem 4 (Heat flux vector decomposition).

The heat flux vector, $J_\alpha^{(q)}$, decomposes into two components, the first proportional to the temperature gradient (Fourier component) and the second to the internal variable (Cattaneo-Vernotte component):

$$J_\alpha^{(q)} = J_\alpha^{(0)} + J_\alpha^{(1)} \tag{106}$$

where

$$J_\alpha^{(0)} = -\left(\frac{\lambda_\alpha^{(\xi)(\xi)}\lambda_\alpha^{(q)(q)} - \lambda_\alpha^{(q)(\xi)}\lambda_\alpha^{(\xi)(q)}}{\lambda_\alpha^{(\xi)(\xi)}}\right)\left(\frac{\partial T}{\partial x^\alpha}\right) \tag{107}$$

and

$$J_\alpha^{(1)} = -\left(\frac{\lambda_\alpha^{(q)(\xi)}A_\alpha^{(\xi)(\xi)}}{\lambda_\alpha^{(\xi)(\xi)}}\right)\xi_\alpha \tag{108}$$

Proof. By replacing the flux associated to the internal variable, j_α, according to Eqs. (104) and (105), we obtain the expression of $\left(\frac{d\xi_\alpha}{dt}\right)$:

$$\frac{d\xi_\alpha}{dt} = -\left(\frac{\lambda_\alpha^{(\xi)(q)}}{\lambda_\alpha^{(\xi)(\xi)}}\right)\left(\frac{\partial T}{\partial x^\alpha}\right) - \left(\frac{A_\alpha^{(\xi)(\xi)}}{\lambda_\alpha^{(\xi)(\xi)}}\right)\xi_\alpha \tag{109}$$

By replacing Eq. (109) into Eq. (103), we have Eq. (106). □

By placing:

$$\lambda_\alpha^{(0)} = \lambda_\alpha^{(q)(q)} - \left(\frac{\lambda_\alpha^{(q)(\xi)}\lambda_\alpha^{(\xi)(q)}}{\lambda_\alpha^{(\xi)(\xi)}}\right) \tag{110}$$

and

$$\lambda_\alpha^{(1)} = \frac{\lambda_\alpha^{(q)(\xi)}A_\alpha^{(\xi)(\xi)}}{\lambda_\alpha^{(\xi)(\xi)}} \tag{111}$$

Equation (106) is written in vector form as:

$$J^{(q)} = \underset{\sim}{\lambda}^{(0)}\,\nabla T + \underset{\sim}{\lambda}^{(1)}\,\xi \tag{112}$$

where

$$\underset{\sim}{\lambda}^{(i)} = \begin{bmatrix} \lambda_1^{(i)} & 0 & 0 \\ 0 & \lambda_2^{(i)} & 0 \\ 0 & 0 & \lambda_3^{(i)} \end{bmatrix} \tag{113}$$

with: $i = \{0,1\}$ and $\alpha = \{1,2,3\}$. As can be seen, the decomposition obtained allows to interpret the internal variable as a heat flux which depends on the nature of the medium in which the heat propagates and characterizes the behaviour at microscopic level. The remaining component obtained from Fourier characterizes the behaviour at macroscopic level and represents the heat flux which depends on the temperature gradient. The tensors $\underset{\sim}{\lambda}^{(0)}$ and $\underset{\sim}{\lambda}^{(1)}$, from a physical point of view, represent the coefficients of thermal conductivity of the medium along the principal axes. In particular, it is observed that $\underset{\sim}{\lambda}^{(0)}$ depends not only on the macroscopic thermal conductivity due to the tensor $\underset{\sim}{\lambda}^{(q)(q)}$, but also on the mutual thermal conductivity tensor $\underset{\sim}{\lambda}^{(q)(\xi)} \cdot \underset{\sim}{\lambda}^{(\xi)(q)} \cdot \left(\underset{\sim}{\lambda}^{(\xi)(\xi)}\right)^{-1}$ due to the internal variable. Conversely $\underset{\sim}{\lambda}^{(1)}$ depends on the state tensor $\underset{\sim}{A}^{(\xi)(\xi)}$ and mutual thermal conductivity $\underset{\sim}{\lambda}^{(q)(\xi)} \cdot \left(\underset{\sim}{\lambda}^{(\xi)(\xi)}\right)^{-1}$. It has also been shown that, in the anisotropic case, heat propagation is different along the main axes, unlike in an isotropic media where heat propagation is the same whatever the direction. By Eqs. (108) and (109) we get:

$$\underset{\sim}{t}_r \frac{dJ^{(1)}}{dt} + J^{(1)} = -\underset{\sim}{K}^{(1)} \nabla T \tag{114}$$

where

$$\underset{\sim}{t}_r = \begin{bmatrix} t_{1,r} & 0 & 0 \\ 0 & t_{2,r} & 0 \\ 0 & 0 & t_{3,r} \end{bmatrix} \tag{115}$$

and

$$\underset{\sim}{K}^{(1)} = \begin{bmatrix} \frac{\lambda_1^{(q,\xi)} \lambda_1^{(\xi,q)}}{\lambda_1^{(\xi,\xi)}} & 0 & 0 \\ 0 & \frac{\lambda_2^{(q,\xi)} \lambda_2^{(\xi,q)}}{\lambda_2^{(\xi,\xi)}} & 0 \\ 0 & 0 & \frac{\lambda_3^{(q,\xi)} \lambda_3^{(\xi,q)}}{\lambda_3^{(\xi,\xi)}} \end{bmatrix} \tag{116}$$

The $\underset{\sim}{t}_r$ matrix is made up of the thermal relaxation times due to the heat flux associated with the internal variable ξ. In the case of a homogeneous anisotropic viscoanelastic medium, the $\underset{\sim}{t}_r$ matrix is reduced to the scalar: $t_1 = \dfrac{\lambda^{(\xi,\xi)}}{A^{(\xi,\xi)}}$ and $\lambda_\alpha^{(1)} = t_1 \lambda_\alpha^{(q,\xi)}$.

7. Equation of heat transport for homogeneous anisotropic viscoanelastic media with constant volume and strain on the time.

In the case of strain and constant volume on the time and of homogeneous medium, we have:

- $\dfrac{d\varepsilon_{\alpha\beta}}{dt} = 0$, constant strain;
- $\dfrac{du}{dt} = c_v \left(\dfrac{dT}{dt}\right)$, constant volume;
- $t_1 = t_{1,r} = t_{2,r} = t_{3,r}$.

Therefore, Eq. (15) is written as:

$$\rho c_v \left(\frac{dT}{dt}\right) = -\nabla \cdot J^{(q)} \tag{117}$$

By replacing $J^{(q)}$ according to *Theorem* (4) and under the hypothesis that the reference system is fixed:

$$\frac{d}{dt} = \frac{\partial}{\partial t} \tag{118}$$

Eq. (117) becomes:

$$\rho c_v \left(\frac{\partial T}{\partial t}\right) = \nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right) + \nabla \cdot \left(\underset{\sim}{\lambda}^{(1)} \xi\right) \tag{119}$$

By deriving both members of Eq. (119) with respect to time and by multiplying to left for t_1 it follows:

$$\rho c_v \left(\frac{\partial^2 T}{\partial t^2}\right) t_1 = \frac{\partial}{\partial t}\left(\nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right)\right) t_1 + \frac{\partial}{\partial t}\left(\nabla \cdot \underset{\sim}{\lambda}^{(1)} \xi\right) t_1 \tag{120}$$

Upon adding member by member Eq. (119) and Eq. (120), we obtain:

$$\begin{aligned}\rho c_v \left(\frac{\partial^2 T}{\partial t^2}\right) t_1 + \rho c_v \left(\frac{\partial T}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right)\right) t_1 + \nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right) + \\ &+ \frac{\partial}{\partial t}\left(\nabla \cdot \underset{\sim}{\lambda}^{(1)} \xi\right) t_1 + \nabla \cdot \left(\underset{\sim}{\lambda}^{(1)} \xi\right)\end{aligned} \tag{121}$$

Therefore

$$\begin{aligned}\rho c_v \left(\frac{\partial^2 T}{\partial t^2}\right) t_1 + \rho c_v \left(\frac{\partial T}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right)\right) t_1 + \nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right) + \\ &+ \nabla \cdot \left(t_1 \frac{\partial}{\partial t}\left(\underset{\sim}{\lambda}^{(1)} \xi\right) + \underset{\sim}{\lambda}^{(1)} \xi\right)\end{aligned} \tag{122}$$

Being

$$\mathbf{J}^{(1)} = -\underset{\sim}{\lambda}^{(1)} \xi \tag{123}$$

Eq. (122) can then be written as:

$$\begin{aligned}\rho c_v \left(\frac{\partial^2 T}{\partial t^2}\right) t_1 + \rho c_v \left(\frac{\partial T}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right)\right) t_1 + \nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right) - \\ &- \nabla \cdot \left(t_1 \frac{\partial}{\partial t}\left(\mathbf{J}^{(1)}\right) + \mathbf{J}^{(1)}\right)\end{aligned} \tag{124}$$

Being

$$t_1 \frac{\partial}{\partial t}\left(\mathbf{J}^{(1)}\right) + \mathbf{J}^{(1)} = \left(\underset{\sim}{\lambda}^{(0)} - \underset{\sim}{\lambda}^{(q)(q)}\right) \nabla T \tag{125}$$

Eq. (124) is written as:

$$\rho c_v \left(\frac{\partial^2 T}{\partial t^2}\right) t_1 + \rho c_v \left(\frac{\partial T}{\partial t}\right) = \nabla \cdot \left(\underset{\sim}{\lambda}^{(q)(q)} \nabla T\right) + \frac{\partial}{\partial t}\left(\nabla \cdot \left(\underset{\sim}{\lambda}^{(0)} \nabla T\right)\right) t_1 \tag{126}$$

By dividing both members of Eq. (126) by ρc_v, we have:

$$\left(\frac{\partial^2 T}{\partial t^2}\right) t_1 + \left(\frac{\partial T}{\partial t}\right) = \nabla \cdot \left(\frac{\underset{\sim}{\lambda}^{(q)(q)}}{\rho c_v} \nabla T\right) + \frac{\partial}{\partial t}\left(\nabla \cdot \left(\frac{t_1 \underset{\sim}{\lambda}^{(0)}}{\rho c_v} \nabla T\right)\right) \tag{127}$$

Placing:

$$\underset{\sim}{\alpha}' = \frac{\underset{\sim}{\lambda}^{(q)(q)}}{\rho c_v} \tag{128}$$

we have

$$t_1\left(\frac{\partial^2 T}{\partial t^2}\right) + \left(\frac{\partial T}{\partial t}\right) = \nabla \cdot \left(\underset{\sim}{\alpha}'\nabla T\right) + \frac{\partial}{\partial t}\left(\nabla \cdot \left(t_1\,\underset{\sim}{\alpha}'\underset{\sim}{\eta}'\nabla T\right)\right) \tag{129}$$

where

$$\underset{\sim}{\eta}' = \underset{\sim}{\lambda}^{(0)}\left(\underset{\sim}{\lambda}^{(q)(q)}\right)^{-1} \tag{130}$$

Finally, by placing:

$$t_2 = t_1\eta_1' \quad ; \quad t_3 = t_1\eta_2' \quad ; \quad t_4 = t_1\eta_3' \tag{131}$$

the equation of heat transport, for viscoanelastic anisotropic media with constant volume and strain on the time, in scalar form is:

$$\begin{aligned} t_1\left(\frac{\partial^2 T}{\partial t^2}\right) + \left(\frac{\partial T}{\partial t}\right) = &\alpha_1'\left(\frac{\partial^2 T}{\partial x^2}\right) + \alpha_2'\left(\frac{\partial^2 T}{\partial y^2}\right) + \alpha_3'\left(\frac{\partial^2 T}{\partial z^2}\right) + \\ &+ \frac{\partial}{\partial t}\left(\alpha_1' t_2\left(\frac{\partial^2 T}{\partial x^2}\right) + \alpha_2' t_3\left(\frac{\partial^2 T}{\partial y^2}\right) + \alpha_3' t_4\left(\frac{\partial^2 T}{\partial z^2}\right)\right) \end{aligned} \tag{132}$$

In the case of isotropic media, the equation of heat transport is that obtained by V. Ciancio (2024). In fact, the relaxation times expressed by the relationships and the rates of change in thermal conductivity are equal under this hypothesis: $t_2 = t_3 = t_4$ being $\eta' = \eta_1' = \eta_2' = \eta_3'$ and $\alpha' = \alpha_1' = \alpha_2' = \alpha_3'$ Therefore, Eq. (132) is written as:

$$t_1\left(\frac{\partial^2 T}{\partial t^2}\right) + \left(\frac{\partial T}{\partial t}\right) = \alpha'\triangle T + \alpha'\,\eta'\,t_1\,\frac{\partial}{\partial t}\left(\triangle T\right) \tag{133}$$

In the next section we will show with an example the peculiar characteristics of the equation of heat transport compared to the models of Fourier.

8. Numerical results

In this section, we apply in a practical case the heat transport equation obtained in the homogeneous anisotropic medium. In particular, consider a square Au metal plate on which a laser source is applied at the vertex $O(0,0)$ (see Fig. 1). In this case Eq. (132) is written

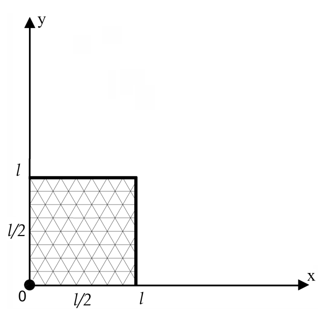

FIGURE 1. Geometry of the 2D-problem

as:

$$t_1\left(\frac{\partial^2 T}{\partial t^2}\right)+\left(\frac{\partial T}{\partial t}\right)=\alpha_1'\left(\frac{\partial^2 T}{\partial x^2}\right)+\alpha_2'\left(\frac{\partial^2 T}{\partial y^2}\right)+\frac{\partial}{\partial t}\left(\alpha_1' t_2\left(\frac{\partial^2 T}{\partial x^2}\right)+\right.$$
$$\left.+\alpha_2' t_3\left(\frac{\partial^2 T}{\partial y^2}\right)\right)+\frac{\alpha_m'}{K}\left(S(t,x,y)+t_1\frac{\partial}{\partial t}S(t,x,y)\right) \tag{134}$$

where K is the thermal conductivity of metal plate, $S(t,x,y)$ is the aproximate intensity function of the laser pulse and the time derivative of $S(t,x,y)$ in Eq. (134) is the apparent heat source resulting from the fast-transient effect of thermal inertial described by t_1. From an experimental and computational point of view the function $S(t,x,y)$ is analytically expressible by the following equation:

$$S(t,x,y)=S_0\,Exp\left(-\frac{x+y}{\phi}\right)I(t) \tag{135}$$

where $I(t)$ is the light intensity of the laser beam, ϕ denotes the optical depth of penetration, and S_0 is the intensity of the laser absorption rate, measured in $\frac{[\text{Watt}]}{[\text{meters}]^3}$, that depends on the laser fluence $J\left(\frac{[\text{Joule}]}{[\text{meters}]^2}\right)$, on the radiative reflectivity of the sample to the laser beam (R), and on the full width at half-maximum pulse duration (t_p):

$$S_0=0.94\,J\left(\frac{1-R}{t_p\,\phi}\right) \tag{136}$$

In this case we consider a femto-laser stimulator, i.e., $t_p = 100$ (femto-seconds). The analytical expression of $I(t)$ which best approximates the real trend on the basis of experimental data is:

$$I(t)=Exp\left(-a\,\frac{|t-2t_p|}{t_p}\right) \tag{137}$$

For a gold plate, it has been found experimentally that suitable values are $t_1 = 8.5$ ps and $t_2 = 90$ ps. The values of the dimensional parameters are given in Table 1. The simulations were obtained by applying the finite element method. We used the scientific software COMSOL version 6.1, and in particular its MUltifrontal Massively Parallel Sparse direct solver (MUMPS 2025) with a Newton automatic termination method. For more details, see Amestoy *et al.* (2001, 2019) and MUMPS (2025). The hardware used is a processor Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz, 1.99 GHz, RAM 16 GB, with paralell programming up to eight threads. Using the procedure to dimensionalize the physical model, we derive the following equation of heat without dimension, where β represents the dimensionless time and δ_1 and δ_2 the dimensionless spatial coordinates of x and y respectively and ϑ represents the dimensionaless temperature. Introducing the following dimensionless variables:

$$\vartheta=\frac{T-T_0}{T_w-T_0};\quad \delta_1=\frac{x}{l};\quad \delta_2=\frac{y}{l};\quad \beta=\frac{t}{l^2/\alpha_m'}$$

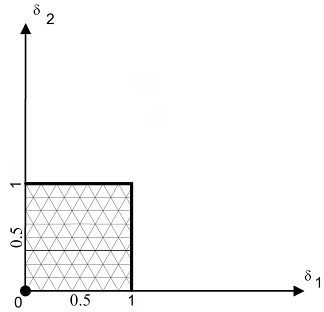

FIGURE 2. Geometry of the dimensionless 2D-problem

respectively, temperature, space and time (see Fig. 2), we obtain the dimensionless equation:

$$z_1\left(\frac{\partial^2\vartheta}{\partial\beta^2}\right)+\left(\frac{\partial\vartheta}{\partial\beta}\right)=m_1\left(\frac{\partial^2\vartheta}{\partial\delta_1^2}\right)+m_2\left(\frac{\partial^2\vartheta}{\partial\delta_2^2}\right)+\frac{\partial}{\partial t}\left(z_2\left(\frac{\partial^2\vartheta}{\partial\delta_1^2}\right)+\right.$$
$$\left.+z_3\left(\frac{\partial^2\vartheta}{\partial\delta_2^2}\right)\right)+Q(\beta,\delta_1,\delta_2)+z_1\frac{\partial Q(\beta,\delta_1,\delta_2)}{\partial\beta} \tag{138}$$

where

$$z_1=\frac{t_1}{(l^2/\alpha'_m)} \tag{139}$$

$$z_2=\frac{t_2}{(l^2/\alpha'_m)} \tag{140}$$

$$z_3=\frac{t_3}{(l^2/\alpha'_m)} \tag{141}$$

$$Q(\beta,\delta_1,\delta_2)=Q_0\,Exp\left(-\frac{\delta_1+\delta_2}{\phi/l}\right)I(\beta) \tag{142}$$

with

$$m_1=\frac{\alpha'_1}{\alpha'_m}\qquad;\qquad m_2=\frac{\alpha'_2}{\alpha'_m} \tag{143}$$

$$Q_0=\frac{l^2\,S_0}{K\,(T_w-T_0)} \tag{144}$$

and

$$I(\beta)=Exp\left(-a\,\frac{|\beta-2z_p|}{z_p}\right) \tag{145}$$

The values of the dimensionless parameters are given in Table 2. The simulations are reported in Figs. 3-8.

TABLE 1. Dimensional parameter values

Parameter	Unit of measure	Value
ϕ: optical penetration depth	meters	15.3×10^{-9}
R: radiative reflectivity of the sample to the laser beam	---	0.93
t_p: time duration of the laser pulse 1	seconds	10.0×10^{-12}
J: laser fluence	Joule/meters^2	13.4
T_0: equilibrium temperature	Celsius	23
T_w: initial temperature value at point O(0,0)	Celsius	600
l: length of the side of the square plate	meters	1.0×10^{-7}
K: mean value of the coefficient of thermal conductivity point O(0,0)	Watt$\times$ Kelvin$^{-1}\times$ meters^{-1}	315
α_m': mean value of the equivalent thermal diffusivity	meters$^2\times$ seconds^{-1}	1.24950×10^{-4}
α_1': value of the equivalent thermal diffusivity along the x-direction	meters$^2\times$ seconds^{-1}	4.99400×10^{-8}
α_2': value of the equivalent thermal diffusivity along the x-direction	meters$^2\times$ seconds^{-1}	6.24500×10^{-9}
t_1: phase lag of temperature gradient	seconds	8.5×10^{-12}
t_2: phase lag of the heat flux vector ξ	seconds	90.0×10^{-12}

TABLE 2. Dimensionless parameter values

Parameter	MCV	Fourier	V. Ciancio	V. Ciancio - defects Au metal
z_1	0.10621	0.00000	0.10621	0.10621
z_2	0.00000	0.00000	1.12460	0.00000
z_3	0.00000	0.00000	0.32455	1.12460
m_1	1.99920×10^{-4}	0.80000	0.10000	1.99920×10^{-4}
m_2	4.99800×10^{-5}	0.20000	0.90000	4.99800×10^{-5}
Q_0	31.70710	31.70710	31.70710	31.70710
a	1.99200	1.99200	1.99200	1.99200

9. Conclusions

In Fig. 3, the simulation result is reported considering the two relaxation times $z_2 = 0$ and $z_3 = 0$ null induced by heat flow due to the internal variable on the Maxwell-Cattaneo-Vernotte's model. It is observed that the distribution is not uniform and also how the temperature slowly returns to its initial state. Figure 4 shows the simulation result using the Fourier's model. Fourier's model is without relaxation times. As you can see a rapid

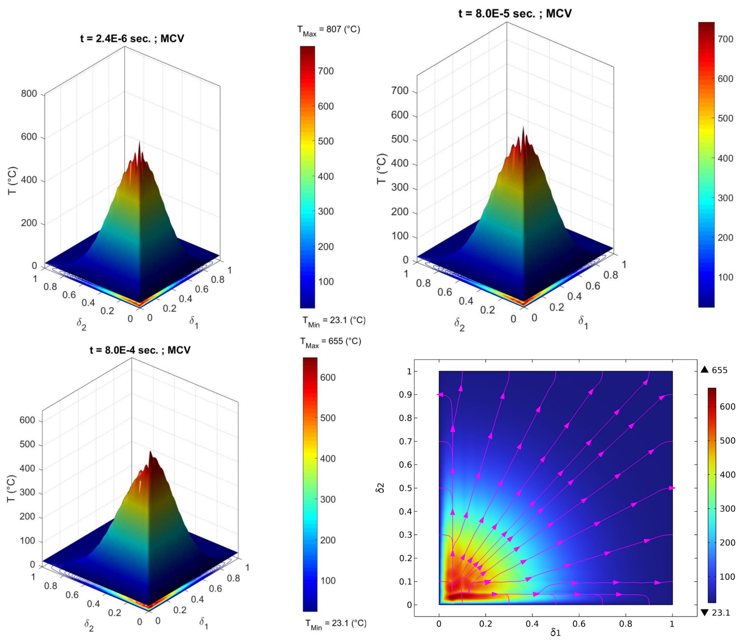

FIGURE 3. Maxwell-Cattaneo-Vernotte model. $z_1 = 0.10621$; $z_2 = 0$; $z_3 = 0$; $m_1 = 1.9992 \times 10^{-4}$; $m_2 = 4.998 \times 10^{-5}$; $Q_0 = 31.70710$; $a = 1.99200$

decrease in temperature occurs until the initial equilibrium value is recovered. In this phase, we observe in the right lower panel of the flux linees that the temperature tends to be distributed rapidly and evenly, unlike what is observed using the Ciancio's model (see Fig. 5). The simulation result is reported considering different relaxation times z_2 and z_3 induced by heat flow due to the internal variable on the V. Ciancio's model. It is observed that the distribution is not uniform and after a first temperature increase the system returns to its initial temperature with a strongly anisotropic distribution and irreversible process. We can therefore deduce that the dimensionaless relaxation times, z_2 and z_3, due to the internal variable, characterize the heat propagation directions.

Figure 6 shows the maximum temperature trend over time in relation to the models considered in the simulations. Figure 7 shows the temperature trend in the centre of the metal plate for the various models. The MCV model has a different behaviour than the others. In particular, in the anisotropic model of Ciancio it is observed that the temperature tends to reach the equilibrium temperature quickly at first and then stabilize after 0.8 milliseconds at a value of about 6 degrees Celsius higher. This higher value is due to the microscopic effects that help maintain a higher temperature. The Fourier model reaches equilibrium temperature in the same time interval at minus one tenth of a degree Celsius. In fact, precisely because of the absence of relaxation times in the Fourier model, the system response is memory-free. The MCV model is based on a single phase lag due to thermal inertia, does not take into account microscopic phenomena and therefore is not suitable for physically representing thermal conduction. This is why in the MCV model

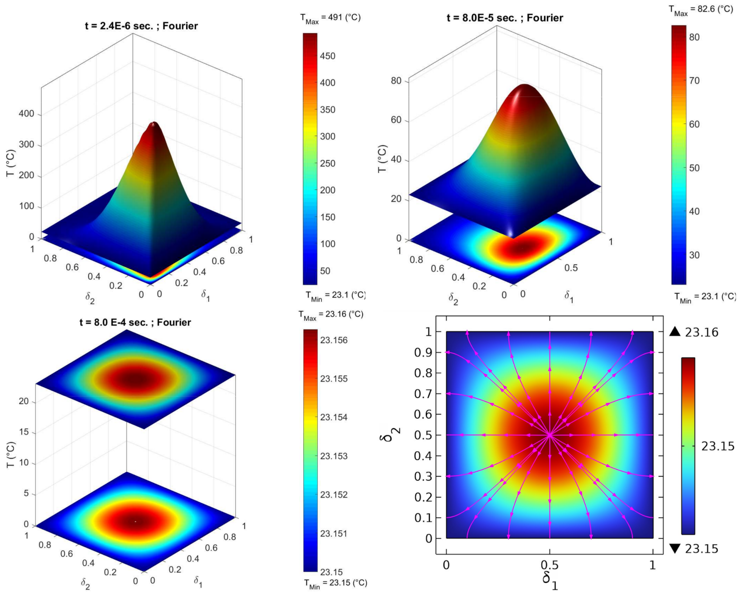

FIGURE 4. Fourier model. $z_1 = 0$; $z_2 = 0$; $z_3 = 0$; $m_1 = 0.8$; $m_2 = 0.2$; $Q_0 = 31.70710$; $a = 1.99200$

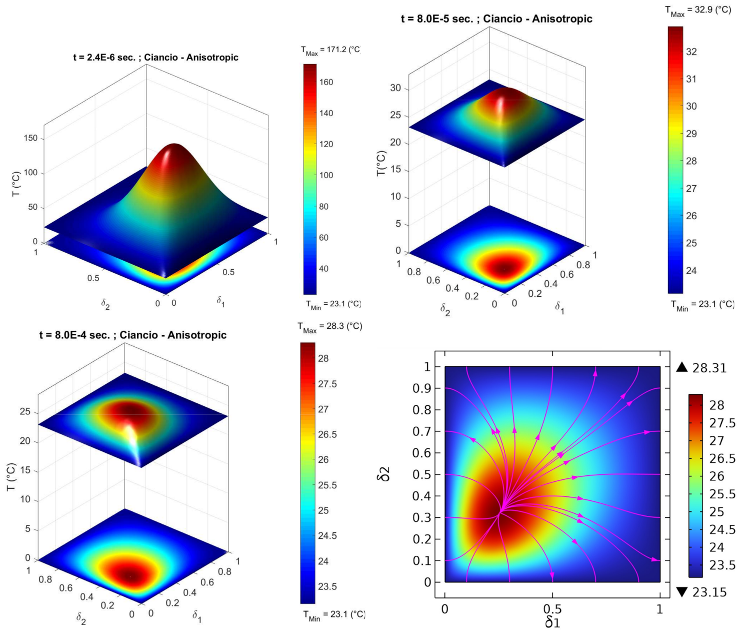

FIGURE 5. V. Ciancio model. $z_1 = 0.10621$; $z_2 = 1.12460$; $z_3 = 0.32455$; $m_1 = 0.1$; $m_2 = 0.9$; $Q_0 = 31.70710$; $a = 1.99200$

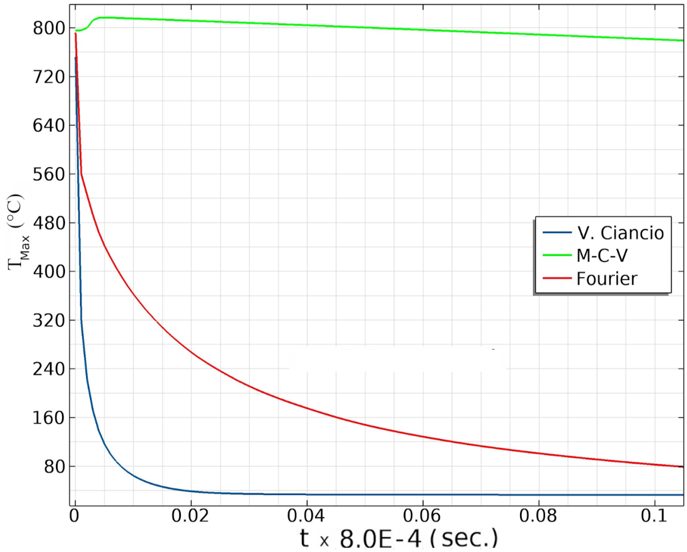

FIGURE 6. Maximum temperature trend comparison

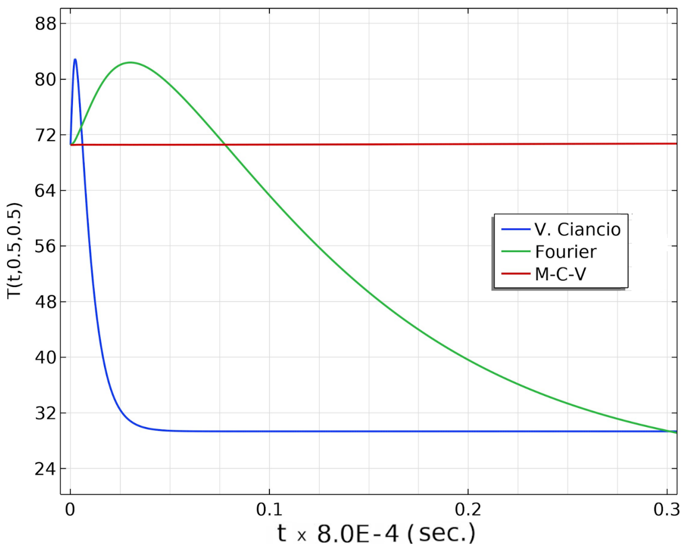

FIGURE 7. Trend of Temperature in the point (0.5,0.5)

the system reaches equilibrium temperature over very long periods of time which are not physically representative of the conduction phenomenon under consideration. In this sense, the MCV model is valid from a mathematical point of view but does not describe the thermal conduction phenomenon as the anisotropic Ciancio's model. The reason is that unlike amorphous or liquid substances such as glass or water, solids are not isotropic but also not totally anisotropic. It is shown unequivocally that the Fourier equation is valid from a physical point of view because, as demonstrated by V. Ciancio (2024), the solution corresponds exactly with that of the diffusion equation, in the case where the relaxation times due to thermal inertia and internal variable coincide. From the results obtained it can be stated that, unlike the Fourier equation closely related to the diffusion equation, the Ciancio's model not only describes the microscopic behaviour of a system subject to an irreversible process of a thermomechanical nature but also shows how relaxation times are correlated with the anisotropic nature of the medium. This implies a potential interest in the diagnosis of biological tissues (Ramírez-Torres *et al.* 2017).

Another field of application is the study of change in material properties. Irregularities due to defects or impurities or imperfections in the crystals and non-homogeneous superetiles characterising a change of material properties, for example from viscoelastic to viscoanelastic, isotropic to anisotropic that are most prominent in nanomaterials, particularly in metals such as gold and semiconductors. From the results obtained in Sect. 8, this results, from a physical point of view, in a variation of relaxation times more pronounced along an axial direction than across (see Fig. 8). Using the standard theory of thermodynamics of irreversible processes with internal variables, the simulations obtained confirm what was achieved by another approach (Jou and Restuccia 2018, 2019).

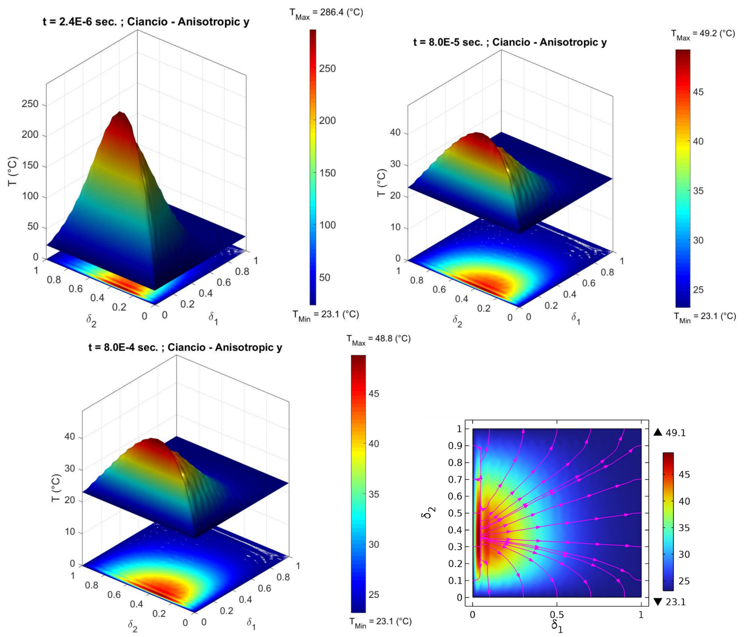

FIGURE 8. V. Ciancio model - Defects Au metal. $z_1 = 0.10621$; $z_2 = 0.0$; $z_3 = 1.2460$; $m_1 = 1.99920 \times 10^{-4}$; $m_2 = 4.99800 \times 10^{-5}$; $Q_0 = 31.70710$; $a = 1.99200$

Finally a field of application is the use of fractional calculation which allows, as demonstrated in the case of isotropic viscoanelastic media (A. Ciancio, V. Ciancio, and Flora 2023), to determine rheological tensors on the basis of experimental data.

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