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G-GRAPH AND $\bar{G}$-GYRO-GRAPH

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ABSTRACT. In this short note, we survey several results concerning the structure of certain graphs, introduced by the fourth-named author, that can be associated to groups and gyro-groups, which are algebraic structures satisfying a twisted form of associativity. Furthermore, we include an original result, showing that the so-called $\bar{G}$-gyro-graph associated to a gyro-group $\bar{G}$ endowed with a subset $\bar{S} \subset \bar{G}$ is connected if and only if $\bar{S}$ is a left generating set of $\bar{G}$. Moreover some properties and examples of both G-graphs and $\bar{G}$-gyro-graphs are presented.

1. Introduction

Various characteristics of a group can be studied on the corresponding G-graph as shown by Ellison, Marinescu–Ghemeci, and Tanasescu (2014) and Badaoui, Bretto, and Mourad (2019) because, unlike Cayley graphs, G-graphs can be semi-regular and edge-transitive. Thus G-graphs are very useful in constructing symmetric and semi-symmetric networks (Bretto, Faisant, and Gillibert 2007). The structure of this new representation was first introduced by Bretto and Faisant (2005). Then, Bretto and Gillibert (2008) could obtain that many well-known graphs have structure of G-graphs associated with a finite group G and a generating set S. Moreover there is an algorithm to construct them and also for finding which graphs are G-graphs and which graphs are not. Ashrafi, Bretto, and Gholaminezhad (2019) constructed the G-graphs of an infinite class of special groups, and the automorphism group of the involution G-graph and Cayley graph. In this paper we also study the involution G-graph, its automorphism group and their hamiltonian property.

In the present paper, we will also consider certain graphs associated to gyro-groups. The structure of gyro-groups, first introduced by Ungar (1988), in order to deal with the lack of associativity of Einstein's addition law for velocities appearing in general relativity in the continuation of the study of Lorentz groups. Cayley graph of gyro-groups were defined by Bussaban, Kaewkhao, and Suantai (2019). But the interesting idea of studying the G-graph of gyro-groups was first expressed by Ashrafi and Bretto. Furthermore the new structures of gyro-groups were presented as: Gyro-cammutative Gyro-groups, Dihedralized Gyro-groups and 2-Gyro-groups (for further reading, see Suksumran and Wiboontion 2015; Mahdavi *et al.* 2021; Maungchang and Suksumran 2022).

In this article, we discuss a new structure of gyro-groups introduced by Mahdavi *et al.* (2021). This structure is called $G(n)$, $n \geq 3$ that is of order 2^n. All proper subgyro-groups of $G(n)$ are either cyclic or dihedral groups. In the following, after presenting the G-graph of finite groups, we define the structure of G-graph corresponding to a gyro-groups. We try to investigate the important properties of such structure in the form of lemmas and different theorems. Then some properties which are satisfied for the G-graph of group are not true for the G-graph of gyro-group. It is well-known that the Cayley graph of a group is connected if and only if the generating set spans a group. However, this fact need not be satisfied for the Cayley graph of a gyro-group. And by giving an example we show that the spanning condition does not guarantee connectedness of Cayley graphs of gyro-groups. In the following, we will see that the behavior of Cayley graph and G-graph of gyro-groups is similar to each other in this case. This leads us to provide a necessary and sufficient condition for connectivity of Cayley graph and G-graph of gyro-groups.

This short survey contains two main sections: G-graph of a finite group and G-graph of a gyro-group. Each section contains some preliminaries and definitions followed by the corresponding results. In the whole of this paper G is a finite group of size n with an identity element 1 and a non-empty subset S of size $k \geq 1$. $\bar{G}$ is a finite gyro-group with the operation $\oplus$ and the identity element 0. Moreover the set of all elements of order two in a group G, called involution set and is denoted by $Inv(G)$. The notation $L(\Gamma)$ will stand for the line graph of a given graph Γ. By definition, the vertices of $L(\Gamma)$ are the edges of Γ, and two vertices of $L(\Gamma)$ are joined by an edge of $L(\Gamma)$ if and only if the corresponding edges of Γ are adjacent. Also $Cay(G,S)$ is a Cayley graph corresponding to a group G with a non-empty subset S, in which G is vertex set and two vertices g and h are adjacent when there is at least one $s \in S$, such that $g = sh$. A graph is said to be symmetric, when it is both vertex-transitive and edge-transitive.

2. G-graph of groups

Definition 1. *(Bretto and Faisant 2005) For a given group G of order n with a non-empty subset S of size $k \geq 1$, consider the cycles*

$$(s)x = (x, sx, s^2x, s^3x, \ldots, s^{o(s)-1}x)$$

of the permutation $g_s : x \longmapsto sx$, $s \in S$ of the symmetric group S_G. Then $\Phi(G,S) = (V;E)$ is the G-graph of G, whose set of vertices V and set of edges E are defined as follows:

- $V = \bigsqcup_{s\in S} V_s = \bigsqcup_{s\in S}\{(s)x, x \in G\}$.
- *Between two vertices $(s)x, (t)y \in V$, $i \neq j$, there is a multi-edge of multiplicity $d \geq 1$, when $|\langle s\rangle x \cap \langle t\rangle y| = d \geq 1$.*

In the whole of this paper we prefer to consider the simple form of G-graph denoted by $\widehat{\Phi}(G,S)$. In the case of having only loops as multi edges we use the notation of $\tilde{\Phi}(G,S)$. There is a survey on G-graph of groups by Gholaminezhad and Ashrafi (2025), which contains many useful results on the structure of G-graphs, In the following, there are some properties of the structure of the G-graph of an S-group (G,S), each of them is illustrated by construction of G-graph (Bretto, Faisant, and Gillibert 2007; Badaoui, Bretto, and Mourad 2019).

For a given finite group G of size n, with a non-empty subset S of size $k \geq 1$,

1) $\Phi(G,S)$ is a k-partite and every vertex v has $o(s)$-loops if and only if $v \in V_s$.
2) $\Phi(G,S)$ has no multi edges if and only if for all $s,t \in S$, $\langle s\rangle \cap \langle t\rangle = \{1\}$.
3) (Bretto, Faisant, and Gillibert 2007, Proposition 4.3) $\Phi(G,S)$ is connected if and only if $G = \langle S\rangle$.
4) (Badaoui, Bretto, and Mourad 2019, Proposition 2.6) For every vertex $v = (s)x \in V(\Phi(G,S))$, $deg(v) = o(s)(k-1)$ and each level $V_s(\Phi(G,S))$ contains $no(s)$ vertices. Therefore

$$|V(\tilde{\Phi}(G,S))| = |G| \sum_{s\in S} \frac{1}{o(s)}, \quad |E(\tilde{\Phi}(G,S))| = \frac{1}{2}nk(k-1).$$

5) If for all $s \in S$, $o(s) = r$, then the G-graph $\widehat{\Phi}(G,S)$ is regular of degree $r(k-1)$.
6) Suppose that the elements of S are pairwise independent, then $\Phi(G,S)$ is a simple graph. By definition two elements s and t in G are independent if and only if for every $x,y \in G$ we have that $|\langle s\rangle x \cap \langle t\rangle y| \leq 1$.
7) (Bretto and Gillibert 2008, Proposition 3) Let $\tilde{\Phi}(G,S)$ be a connected bipartite and regular of degree p, for a prime number p, then either $\tilde{\Phi}$ is simple or it is of order 2, .
8) (Bretto, Faisant, and Gillibert 2007, Corollary 4.4) If k is odd and $G = \langle S\rangle$, then $\Phi(G,S)$ is eulerian, .
9) (Bretto, Faisant, and Gillibert 2007, Theorem 4.6) Consider an isomorphism f between two groups (G_1,S_1) and (G_2,S_2), such that $f(S_1) = S_2$, where S_1 and S_2 are the subsets of G_1 and G_2, respectively. Then $\Phi(G_1,S_1) \cong \Phi(G_2,S_2)$ Now let G_1 and G_2 be two abelian groups. These two groups are isomorphic if and only if $\Phi(G_1,G_1) \cong \Phi(G_2,G_2)$.
10) (Bretto and Gillibert 2008, Theorem 2) Let $\widehat{\Phi}(G,S = \{a,b\})$ be a simple connected G-graph and $S^* = (\langle a\rangle \cup \langle b\rangle) \setminus \{1\}$. Then $L(\widehat{\Phi}) \simeq Cay(G,S^*)$, where L denotes the line graph.

In the following there are some examples of the simple form of G-graphs of some groups.

Example 2. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the additive group of the ring $\mathbb{Z}/\mathbb{Z}_2$ and $S = \{a = (1,0), b = (0,1), c = (1,1)\}$ be it's generating set. $\widehat{\Phi}(G,S)$ is the octahedral graph which is the connected and $o(s) = 4$-regular, $s \in S$.*

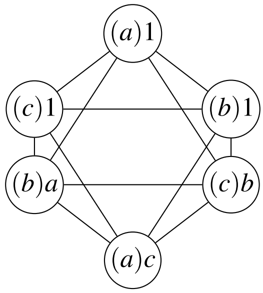

FIGURE 1. $\widehat{\Phi}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{a,b,c\})$.

Example 3. *Let $G = S_3$ be the symmetric group on a set of three elements and S be one of the following generating set.*

a) *If $S = Inv(G) = \{(1,2),(1,3),(2,3)\}$ is the set of all transpositions of S_3, the vertices of $\widehat{\Phi}(G,S)$ are as follows.*

$v_1 = \langle(1,2)\rangle 1 = \{1,(1,2)\}$, $v_2 = \langle(1,2)\rangle(1,3) = \{(1,3),(1,3,2)\}$, $v_3 = \langle(1,2)\rangle(2,3) = \{(2,3),(1,2,3)\}$.

$v_4 = \langle(1,3)\rangle 1 = \{1,(1,3)\}$, $v_5 = \langle(1,3)\rangle(1,2) = \{(1,2),(1,2,3)\}$, $v_6 = \langle(1,3)\rangle(2,3) = \{(2,3),(1,3,2)\}$.

$v_7 = \langle(2,3)\rangle 1 = \{1,(2,3)\}$, $v_8 = \langle(2,3)\rangle(1,2) = \{(1,2),(1,3,2)\}$, $v_9 = \langle(2,3)\rangle(1,3) = \{(1,3),(1,2,3)\}$.

So $\widehat{\Phi}(S_3,S)$ is isomorphic to a 3-partite, 4-regular graph with 9 vertices and 18 edges, as shown in Figure 2.

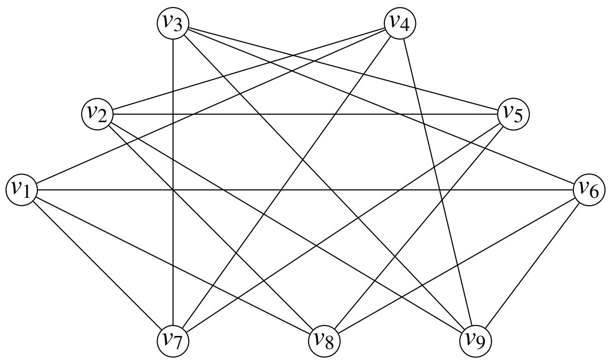

FIGURE 2. $\widehat{\Phi}(S_3,S)$.

b) *If $S = \{(1,2),(2,3)\}$, then $\widehat{\Phi}(S_3,S) \cong C_6$, that denotes the cycle graph on 6 vertices.*

c) *If $S = \{(1,2),(1,2,3)\}$, then $\widehat{\Phi}(S_3,S) \cong K_{2,3}$ which is a complete bipartite graph of 5 vertices.*

Example 4. *Consider the alternating group*

$A_4 = \{(1),(1,2)(3,4),(1,4)(2,3),(1,2,3),(1,3,2),(1,2,4),(1,4,2),(1,3,4),(1,4,3),(2,3,4),(2,4,3)\}$,

with the subset $S = \{s = (1,2,3), t = (1,2,4)\}$. *The vertices of* $\widehat{\Phi}(A_4,S)$ *are as follows.*

$$\begin{aligned}
v_1 &= \langle(1,2,3)\rangle 1 = \{1,(1,2,3),(1,3,2)\}\\
v_2 &= \langle(1,2,3)\rangle(1,2,4) = \{(1,2,4),(1,3)(2,4),(2,4,3)\}\\
v_3 &= \langle(1,2,3)\rangle(1,4,2) = \{(1,4,2),(1,4,3),(1,4)(2,3)\}\\
v_4 &= \langle(1,2,3)\rangle(1,3,4) = \{(1,3,4),(2,3,4),(1,2)(3,4)\}\\
v_5 &= \langle(1,2,4)\rangle 1 = \{1,(1,2,4),(1,4,2)\}\\
v_6 &= \langle(1,2,4)\rangle(1,2,3) = \{(1,2,3),(1,4)(2,3),(2,3,4)\}\\
v_7 &= \langle(1,2,4)\rangle(1,3,2) = \{(1,3,2),(1,3,4),(1,3)(2,4)\}\\
v_8 &= \langle(1,2,4)\rangle(2,4,3) = \{(2,4,3),(1,2)(3,4),(1,4,3)\}
\end{aligned}$$

then $\widehat{\Phi}(A_4,S)$ *is a bipartite, 3-regular, connected graph with 8 vertices and 12 edges.*

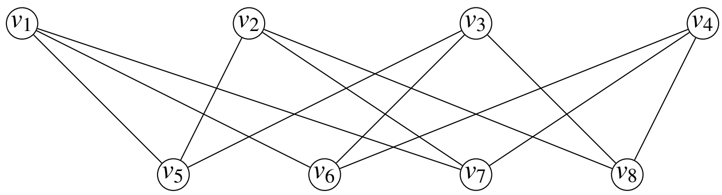

FIGURE 3. $\widehat{\Phi}(A_4,\{(1,2,3),(1,2,4)\})$

Example 5. *Table 1 contains the automorphism group of the G-graphs and the Cayley graphs of alternating groups* A_n, $3 \le n \le 8$. *For even number n, let* $S = \{(2,...,n),(1,2,3)\}$, *then* $\widehat{\Phi}(A_n,S)$ *is a bipartite graph of degree* $n-1$ *and* 3. *For an odd number n, let* $S = \{(1,...,n),(1,2,3)\}$, *then* $\widehat{\Phi}(A_n,S)$ *is a bipartite graph of degree n and* 3.

TABLE 1. $Aut(A_n)$, $Aut(\widehat{\Phi}(A_n,S))$ and $Aut(Cay(A_n,S))$, $n = 4,5,6,7,8,9$.

G	A_4	A_5	A_6	A_7	A_8	A_9
$Aut(G)$	S_4	S_5	$(A_6 : \mathbb{Z}_2) : \mathbb{Z}_2$	S_7	S_8	S_9
$Aut(\widehat{\Phi}(G,S))$	$\mathbb{Z}_2 \times S_4$	$\mathbb{Z}_2 \times A_5$	$\mathbb{Z}_2 \times A_6$	S_7	S_8	S_9
$Aut(Cay(G,S))$	$\mathbb{Z}_2 \times S_4$	$\mathbb{Z}_2 \times A_5$	$\mathbb{Z}_2 \times A_6$	S_7	S_8	S_9

According to this table, there is a conjecture by Ashrafi, Bretto, and Gholaminezhad (2019) which says $Aut(Cay(A_n,S)) \cong Aut(\widehat{\Phi}(A_n,S))$ *and for* $n \ge 7$ *the automorphism group is isomorphic to* S_n.

Bretto and Gillibert (2008) provided the tables of the structure of some small groups with the cubic symmetric G-graphs. A graph is symmetric when it is both vertex-transitive and edge-transitive. Moreover, Bretto, Faisant, and Gillibert (2007) presented some well-known graphs that have structure of G-graphs, like Heawood graph, Pappus graph, Ljubljana graph, Mobius–Kantor, Octahedral, and Gray graph.

2.1. Involution G-graph. All the elements of order two makes the set of involutions in a group, which has an important role in group theory, especially in constructing the simple groups. For instance, Bates *et al.* (2007) and Gholaminezhad (2014) give the structures of the commuting involution graphs. This motivates us to study the structure of Involution G-graph of some finite groups and comparing them with their corresponding Cayley graph. Moreover we study about the automorphism group and the hamiltonicity of involution G-graph and involution Cayley graph.

According to Ashrafi, Bretto, and Gholaminezhad (2019, Theorem 2.9), when G is a finite simple group such that $G \cong Aut\ (\Phi(G, Inv(G)))$, then the outer automorphism $Out(G)$ is an identity map. In the following example it is showed that the sporadic groups of M_{11} and M_{23} could be the examples for the converse of this fact.

Example 6. *For the Mathieu groups M_{11} and M_{23} (Conway* et al. *1985), from the sporadic group, $Out(G)$ is identity and since M_{11} and M_{23} are both simple groups generated by their involutions, then their G-graphs are connected. And we have that,*

$$Aut(M_{11}) \cong Aut(Cay(M_{11}, Inv(M_{11}))) \cong Aut(\widehat{\Phi}(M_{11}, Inv(M_{11}))) \cong M_{11},$$

$$Aut(M_{23}) \cong Aut(Cay(M_{23}, Inv(M_{23}))) \cong Aut(\widehat{\Phi}(M_{23}, Inv(M_{23}))) \cong M_{23}.$$

In the following we try to give some results and examples related to the structure of G-graphs of some finite groups, which are hamiltonian and isomorphic to the Cayley graphs of some other groups. For a given finite group G generated by its involutions, there is a relation which says that the line graph of involution Cayley graph is isomorphic to involution G-graph:

Lemma 7. $L(Cay(G, Inv(G))) \cong \widehat{\Phi}(G, Inv(G))$.

Theorem 8. *Let G be a finite group with a non-empty subset $Inv(G)$.*

i) *If $|Inv(G)| = 1$ then $Aut(Cay(G, Inv(G))) \cong \mathbb{Z}_2 \wr S_{\frac{|G|}{2}}$ and $Aut(\widehat{\Phi}(G, Inv(G))) \cong Aut(\overline{K_{|G|}}) \cong S_{|G|}$, where $\overline{K_{|G|}}$ is an edgeless graph of size $|G|$.*

ii) *If $|Inv(G)| \geq 2$, then $Aut(Cay(G, Inv(G))) \cong Aut(\widehat{\Phi}(G, Inv(G)))$.*

Ashrafi, Bretto, and Gholaminezhad (2019) conjectured that the automorphism group of involution Cayley graph and involution G-graph of the special linear group $G = L_2(p)$ for a prime number $p \geq 7$ is isomorphic to each other.

Theorem 9. *(Bretto and Faisant 2011, Theorem 12) Given a finite group G such that $|Inv(G)| \neq 3$, the graph $\widehat{\Phi}(G, Inv(G))$ is Hamiltonian.*

The next two examples, represent the structure of the group, that has the structure of G-graph, which is hamiltonian and also isomorphic to the Cayley graph of another group obtained by some functions.

For any $g \in G$, we associate the map $\delta : G \longrightarrow Aut(\Phi(G,S))$ to G that $\delta(g) = \delta_{g^{-1}}$, where $\delta_{g^{-1}} \in Aut(\Phi(G,S))$, such that $\delta_{g^{-1}}((s)x) = (s)xg^{-1}$. Furthermore $\delta(G)$ acts transitively on every level of vertices V_s (Bretto and Faisant 2005, Theorem 3.4).

Example 10. *Let $G = \mathbb{Z}_2{}^k$ be the product of k-many copies of $\mathbb{Z}_2$, for some integer $k \geq 2$. Then G is an elementary abelian group with set of involutions $Inv(G) = \{s_i : 1 \leq i \leq k\}$, where $s_i = (0,\dots,1,0,1,\dots,0)$ with the only 1 in the i-th position. For $x = \Sigma_{1\leq i\leq k} a_i s_i \in G$ and an automorphism $\alpha \in Aut(G)$, that $\alpha(s_i) = s_{i+1}$, we can see that the group $\langle\alpha\rangle$ acts transitively on $Inv(G)$. By the orbit stabilizer theorem,*

$$|\langle\alpha\rangle x| = |Inv(G)| = \frac{|\langle\alpha\rangle|}{|\langle\alpha\rangle_x|} = k,$$

which says that $|\langle\alpha\rangle_x| = 1$. Thus $\langle\alpha\rangle$ acts regularly on $Inv(G)$. Consider the subset

$$H = \{x = \Sigma_{1\leq i\leq t}\lambda_i s_i \in G | \Sigma_{1\leq i\leq k}\lambda_i = 0\},$$

of G that is clearly the subgroup of G and $H \cap \langle s_i\rangle = \{(0,0,\cdots,0)\}$, because if $x = \sum_{1\leq i\leq k}\lambda_i s_i = s_j$, then $\sum_{1\leq i\leq k}\lambda_i = 1$ which is a contradiction. $\alpha(H) \subset H$, because for $x \in H$, $\alpha(x) = \sum_{1\leq i\leq k}\lambda_i\alpha(s_i) = \sum_{1\leq i\leq k}\lambda_i s_{i+1}$ and $\sum_{1\leq i\leq k}\lambda_i = 0$, that means $\alpha(x) \in H$.

Hence It is concluded that:

Proposition 11. *(Bretto and Faisant 2011) The involution G-graph $\widehat{\Phi}(G = \mathbb{Z}_2{}^k, Inv(G))$ is isomorphic to the hamiltonian Cayley graph of group $\delta(H) \rtimes \langle\alpha\rangle$.*

We can similarly see that some G-graphs associated to the symmetric groups $G = S_n$ can also be obtained as Cayley graphs of automorphism groups, as the following results show.

Proposition 12. *For $n \geq 2$, S_n is generated by the $(n-1)$ transpositions as follows:*

$$S = \{(1,2),(1,3),(1,4),\dots(1,n)\},$$

$$S' = \{(1,2),(2,3),(3,4),\dots,(n-1,n)\}.$$

Then $\widehat{\Phi}(S_n,S)$ is isomorphic to $(n-1)$- partite, $(2n-4)$-regular graph with $|V| = n(n-1)(n-1)!/2$ and $\widehat{\Phi}(S_n,S) \cong \widehat{\Phi}(S_n,S')$.

Example 13. *Consider the symmetric group $G = S_n$ with the generating set of transpositions $Inv(S_n) = \{(i,i+1), 1 \leq i \leq n\}$, $n \geq 4$. Define the map $\alpha = f_\rho$, $\rho = (1,2,\dots,n)$, such that*

$$\alpha((i,i+1)) = f_\rho(i,i+1) = \rho(i,i+1)\rho^{-1} = (i+1,i+2).$$

By this function, the group $\langle\alpha\rangle$ acts transitively on $Inv(G)$, because we can see that $\alpha^2((i,i+1)) = (i+2,i+3)$ and by induction $\alpha((i+t,i+t+1)) = (i+t+1,i+t+2)$. Thus for any two transpositions $(u,v),(w,z) \in Inv(G)$ there is m, such that $\alpha^m((u,v)) = (w,z)$. Also $\langle\alpha\rangle$ acts regularly on $Inv(G)$. Since $A_n \cap \langle\alpha\rangle = \{1\}$ and for every $x \in A_n$, $\alpha(x)$ has the same parity of x, then $\alpha(A_n) \subset A_n$.

Then, similarly to Example 10, we can see the the structure of $\widehat{\Phi}(G,Inv(G))$ is isomorphic to the hamiltonian Cayley graph of the group $\delta(A_n) \rtimes \langle\alpha\rangle$, where δ is the morphism mentioned before that for every $g \in G$, carries every vertex $(s)x \in V(\Phi(G,Inv(G)))$ to $(s)xg^{-1}$.

3. $\bar{G}$-gyro-graph

In this section we construct the structure of $\bar{G}$-gyro-graph in order to obtain the structure of G-graph associated to a given gyro-group $\bar{G}$.

Definition 14. *A magma $(\bar{G}, \oplus)$ is called a gyro-group if its binary operation satisfies the following axioms:*

a) *There is $0 \in \bar{G}$ such that for all $a \in \bar{G}$; $0 \oplus a = a$, which is the identity element.*
b) *For any $a \in \bar{G}$, there is $b \in \bar{G}$ such that $b \oplus a = 0$; the element b is indeed a unique two-sided inverse of a in G, denoted by $\ominus a$.*
c) *For any $a, b \in \bar{G}$, there is an automorphism $gyr[a,b] : \bar{G} \longmapsto \bar{G}$ such that for any $c \in \bar{G}$,*
$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a,b]c.$$
d) *For any $a, b \in \bar{G}$, $gyr[a,b] = gyr[a \oplus b, b]$.*

For further study, those interested in the concept of gyro-group can refer to Ungar (1988), Sabinin, Sabinina, and Sbitneva (1998), Suksumran and Wiboontion (2015), Suksumran (2016), and Maungchang *et al.* (2021), who examined in detail the features of gyro-groups, their basic theorems and propositions and also their corresponding Cayley graphs.

Let $\bar{G}$ be a gyro-group associated with a generating set $\bar{S}$ of size $k \geq 1$. A generating set means that $\bar{G}$ can be span by elements of $\bar{S}$. Consider the left action of the sub gyro-group $\langle s \rangle$ on $\bar{G}$, then there is a partition $\bar{G} = \bigsqcup_{s \in \bar{S}} \langle s \rangle \oplus x$, $x \in \bar{G}$. The cardinality of $\langle s \rangle$ is $o(s)$, the order of the element s. The structure of the $\bar{G}$-gyro-graph is defined similar to G-graph. For a given gyro-group $\bar{G}$ with a non-empty subset $\bar{S}$, consider the cycles

$$(s) \oplus x = (x, s \oplus x, (s \oplus s) \oplus x, \cdots (\underbrace{s \oplus \cdots \oplus s}_{o(s)-1}) \oplus x)$$

of the permutation $g_s : x \longmapsto s \oplus x$ of $\bar{G}$. A $\bar{G}$-gyro-graph denoted by $\Phi(\bar{G}, \bar{S})$ is defined in the following way:

I) The vertices of $\Phi(\bar{G}, \bar{S})$ are the cycles of g_s, $s \in S$, i.e., $V = \bigsqcup_{s \in \bar{S}} V_s$, with $V_s = \{(s) \oplus x, x \in \bar{G}\}$.
II) Two different vertices $(s) \oplus x$ and $(t) \oplus y$ are adjacent with $d \geq 1$ multi edges when
$$|(\langle s \rangle \oplus x) \cap (\langle t \rangle \oplus y)| = d.$$

Now there is a $\bar{G}$-gyro-graph of G_8 generated by $\bar{S} = \{1, 2\}$.

Example 15. *(Gholaminezhad 2022) Let us compute explicitly the $\bar{G}$-gyro-graph associated to the gyro-group structure G_8 given on the set $\{0, 1, \cdots, 7\}$ with $A = (16)(25)$ and I which denotes the identity map, by the following tables.*

The previous example shows that even when $\bar{S}$ is the generating set for $\bar{G}$, the $\bar{G}$-gyro-graph of a gyro-group may not be connected. This shows that, the properties that are valid for G-graph of groups may not be valid for G-graph of gyro-groups.

The subset $\bar{S} = \{s_1, s_2, \cdots, s_k\}$, $k \geq 1$, of a gyro-group $\bar{G}$ is said to be symmetric if for each element $s_i \in \bar{S}$, $\ominus s_i \in \bar{S}$, $1 \leq i \leq k$. Moreover $\bar{G}$ is left-generated by $\bar{S}$, if $(\bar{S}\rangle = \bar{G}$, where

$$(\bar{S}\rangle = \{s_k \oplus (\cdots \oplus (\cdots s_3 \oplus (s_2 \oplus s_1)) \cdots) | s_1, s_2, \cdots s_k \in \bar{S}\}.$$

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	5	6	0	7	1	2	4
4	4	2	1	7	0	6	5	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

gyro	0	1	2	3	4	5	6	7
0	*I*	*I*	*I*	*I*	*I*	*I*	*I*	*I*
1	*I*	*I*	*A*	*A*	*A*	*A*	*I*	*I*
2	*I*	*A*	*I*	*A*	*A*	*I*	*A*	*I*
3	*I*	*A*	*A*	*I*	*I*	*A*	*A*	*I*
4	*I*	*A*	*A*	*I*	*I*	*A*	*A*	*I*
5	*I*	*A*	*I*	*A*	*A*	*I*	*A*	*I*
6	*I*	*I*	*A*	*A*	*A*	*A*	*I*	*I*
7	*I*	*I*	*I*	*I*	*I*	*I*	*I*	*I*

TABLE 2. The gyro-table of G_8

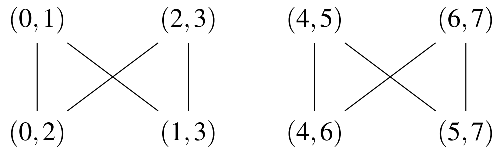

FIGURE 4. $\Phi(G_8, \bar{S} = \{1,2\})$

The right-generating set is defined in a similar fashion. More details about the left and right generator sets were given by Bussaban, Kaewkhao, and Suantai (2019).

A gyro-group $\bar{G}$ is called a gyro-commutative gyro-group, if for all $a,b \in \bar{G}$, $a \oplus b = gyr[a,b](b \oplus a)$. The gyro-commutative gyro-groups were introduced by Sabinin, Sabinina, and Sbitneva (1998) and Suksumran and Wiboontion (2015). Note that two isomorphic gyro-groups may not have isomorphic $\bar{G}$-gyro-graph. This even may happen for $\bar{G}$-gyro-graph of gyro-commutative gyro-groups.

Theorem 16. *(Gholaminezhad 2022; Moradi, Fath-Tabara, and Bretto 2024)* $\Phi(\bar{G}, \bar{S})$ *is connected if and only if* $\bar{G} = \langle \bar{S} \rangle$.

Proof. First assume that $\bar{S}$ is a left generating set. We show that there is a path between any two arbitrary vertices of $\bar{G}$- gyro-graph. Consider two vertices $(s) \oplus x \in V_s$ and $(s') \oplus y \in V_{s'}$, since $\bar{G} = \langle \bar{S} \rangle$, then

$$y = (s_n \oplus (\cdots \oplus (s_3 \oplus (s_2 \oplus s_1)) \cdots)) \oplus x.$$

Moreover we have:

$$\begin{aligned}
x &\in \langle s \rangle \oplus x \bigcap (s_1 \oplus x), \\
s_1 \oplus x &\in \langle s_1 \rangle \oplus x \bigcap \langle s_0 \rangle \oplus (s_1 \oplus x), \\
(s_2 \oplus s_1) \oplus x &\in \langle (s_2 \oplus s_1) \rangle \oplus x \bigcap \langle s_1 \rangle \oplus ((s_2 \oplus s_1) \oplus x)), \\
(s_{n-1} \oplus \cdots \oplus (s_3 \oplus (s_2 \oplus s_1))) \oplus x) &\in \langle (s_{n-1} \oplus \cdots \oplus (s_3 \oplus (s_2 \oplus s_1)) \oplus x) \rangle \\
&\bigcap \langle s_n \rangle \oplus (s_{n-1} \oplus \cdots \oplus (s_3 \oplus (s_2 \oplus s_1)) \oplus x), \\
y &\in \langle (s_{n-1} \oplus \cdots \oplus (s_3 \oplus (s_2 \oplus s_1)) \oplus x \rangle \bigcap \langle s' \rangle \oplus y.
\end{aligned}$$

Consequently, there is a path from $(s) \oplus x \in V_s$ to $(s') \oplus y \in V_t$. So $\Phi(\bar{G}, \bar{S})$ is a connected graph.

Conversely, suppose that $\Phi(\bar{G}, \bar{S})$ is connected, using the fact that there is a path between any two arbitrary vertices, it can be seen that $\bar{S}$ is a left generating set. □

Example 17. *$M(1)$ and $G_{(8,5)}$ are two isomorphic gyro-groups on the set $\{0, \cdots, 7\}$; the authors refer interested readers to Ashrafi, Mavaddat Nezhaad, and Salahshour (2022) for more detailed explanations about isomorphism between them. Now consider two gyro-commutative gyro-groups $M(1)$ and $G_{(8,5)}$ with their following gyro-tables.*

$\oplus_{G_{(8,5)}}$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	5	7	6	2	4	3
2	2	5	0	6	7	1	3	4
3	3	6	7	5	0	4	2	1
4	4	7	6	0	5	3	1	2
5	5	2	1	4	3	0	7	6
6	6	4	3	2	1	7	0	5
7	7	3	4	1	2	6	5	0

TABLE 3. gyro table of $G_{(8,5)}$

$\oplus_{M(1)}$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	1	0	3	2
5	5	4	7	6	0	1	2	3
6	6	7	5	4	2	3	1	0
7	7	6	4	5	3	2	0	1

TABLE 4. gyro table of $M(1)$

In the following there are the figures of the structure of $\widehat{\Phi}(G_{(8,5)}, \bar{S} = \{7,8\})$ and $\widehat{\Phi}(M(1), \bar{S} = \{5,7\})$. This example shows that two isomorphic gyro-groups do not necessarily have isomorphic G-graphs.

3.1. $\bar{G}$-gyro-graphs of 2-gyro-groups. There is a class of gyro-groups denoted by $G(n)$, which is a class of 2-gyro-groups that was introduced by Ashrafi, Mavaddat Nezhaad, and Salahshour (2022). For $n \geq 3$, $G(n)$ is a 2-gyro-group of order 2^n, such that every proper subgyro-group of $G(n)$ is either a cyclic or a dihedral group. It has been proven that the subgyro-group lattice and normal subgyro-group lattice of $G(n)$ are isomorphic to the subgroup lattice and normal subgroup lattice of the dihedral group of order $2n$, which is the

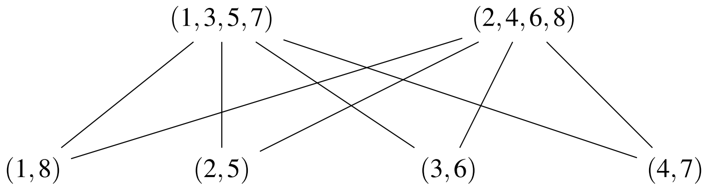

FIGURE 5. $\widehat{\Phi}(G_{(8,5)}, \bar{S} = \{7,8\})$.

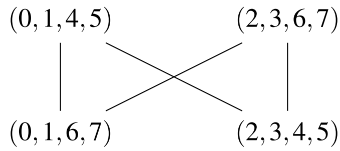

FIGURE 6. $\widehat{\Phi}(M(1), \bar{S} = \{5,7\})$.

reason for choosing the name dihedral gyro-group for this class of gyro-groups of order $2n$. Moreover, all proper subgyro-groups of $G(n)$ are subgroups.

For an integer $n \geq 3$, consider the following sets of order $m = 2^{n-1}$

$$P(n) = \left\{0, 1, 2, \cdots, 2^{n-1} - 1\right\},$$

$$H(n) = \left\{2^{n-1}, 2^{n-1} + 1, \cdots, 2^n - 1\right\},$$

then $G(n) = P(n) \bigcup H(n)$. In this paper, we tried to briefly examine the structure of $\bar{G}$-gyro-graph of such gyro-groups.

Theorem 18. *(Gholaminezhad 2022) The $\bar{G}$-gyro-graph $\Phi(G(n), H(n))$ is connected and hamiltonian.*

Proof. For every $j \in H(n) = \{m, m+1, \cdots, 2m-1\}$, where $m = 2^{n-1}$ and $n \geq 3$, $\langle j \rangle \cong \mathbb{Z}_2$ is an L-subgyro-group of $G(n)$, because for every $g \in G(n)$ and $s \in \langle j \rangle$, $gyr[g,s](\langle j \rangle) = \langle j \rangle$. Then the index of $\langle j \rangle$ in $G(n)$, which is denoted by $|G(n), \langle j \rangle|$ equals to $2^n/2 = m$ and $\Phi(G(n), H(n))$ is an m-partite $\bar{G}$-gyro-graph. Moreover each part has m vertices of the form $(j) \oplus x = (x, j \oplus x)$, $x \in G(n)$, which means that every vertex in one part has two elements in the intersections with the vertices in other parts. It implies that the graph is $2(m-1)$-regular. Also each V_j has a copy of $G(n)$ which induces that $\Phi(G(n), H(n))$ is connected. Furthermore we can find a closed path started from the first vertex in V_m, that passes all vertices in other levels, just once. Consequently we can find the hamiltonian cycle in this graph. □

In the following example, the structure of the $\bar{G}$-gyro-graph of $G(3)$ with the hamiltonian cycle is presented.

Example 19. *(Gholaminezhad 2022) The $\bar{G}$-gyro-graph of $\widehat{\Phi}(G(3), H(3))$ is a $|H(3)| = m = 2^{3-1} = 4$-partite connected graph with $m^2 = 16$ vertices as: $V = V_4 \cup V_5 \cup V_6 \cup V_7$, such* *that* $|V_4| = |V_5| = |V_6| = |V_7| = 4$ *and every vertex is of degree* $2(m-1) = 2(3-1) = 4$ *and*

$$V = \{(4)\oplus x\}\bigcup\{(5)\oplus y\}\bigcup\{(6)\oplus z\}\bigcup\{(7)\oplus t\}$$
$$= \{(0,4),(1,5),(2,6)(3,7)\}\bigcup\{(0,5),(1,6),(2,7),(4,3)\}$$
$$\bigcup\{(0,6),(1,7),(2,4),(3,5)\}\bigcup\{(0,7),(1,4),(2,5),(3,6)\}.$$

Now there is a cycle which passes every vertex just once:

$$(0,4)-(4,3)-(3,5)-(3,6)-(2,6)-(1,6)-(1,7)-(0,7)-$$
$$(2,7)-(2,4)-(1,4)-(1,5)-(0,5)-(0,6)-(0,7)-(0,4)$$

It shows that this graph is hamiltonian.

Example 20. *(Moradi, Fath-Tabara, and Bretto 2024) Consider the gyro-group* $\bar{G} = G(4) = P(4)\bigcup H(4)$. *We can see that* $\widehat{\Phi}(G(4),H(4))$ *has* 8 *parts of the vertex set:*

$$V = V_8\bigcup V_9\bigcup V_{10}\bigcup V_{11}\bigcup V_{12}\bigcup V_{13}\bigcup V_{14}\bigcup V_{15}.$$

Also clearly this graph is $2(m-1) = 2(8-1) = 14$*-regular.*

Another topic which has been discussed is that although every Cayley graph is vertex-transitive, this is not the case with gyro-groups.The interested reader can look at Example 3.7 of Bussaban, Kaewkhao, and Suantai (2019) for a prototypical instance of this phenomenon.

In this paper we also investigate the symmetry of the $\bar{G}$-gyro-graph and by providing a new definition called $\bar{L}$-$\bar{G}$-gyro-graph and $\bar{R}$-$\bar{G}$-gyro-graph, we express the necessary condition for transitivity of $\bar{G}$-gyro-graphs.

Definition 21. *Let* $\bar{G}$ *be a gyro-group with a non-empty subset* $\bar{S}$. *If* $\bar{S}$ *is a left generating set, i.e* $\langle\bar{S}\rangle = \bar{G}$, *we call its corresponding* $\bar{G}$*-gyro-graph* $\widehat{\Phi}(\bar{G},\bar{S})$ *as* $\bar{L}$-$\bar{G}$*-gyro-graph and if* $\bar{S}$ *is the right generating set, we say it* $\bar{R}$-$\bar{G}$*-gyro-graph.*

In the following, we state the necessary condition for $\bar{G}$-gyro-graphs to be symmetric. But before that, we will state the theorem and propositions that will be useful in the future. It is recommended to refer to Moradi, Fath-Tabara, and Bretto (2024) for more details.

Theorem 22. *Let* $\bar{G}$ *be a finite gyro-group and* $\bar{S}$ *be a symmetric subset of* $\bar{G}$. *If* $gyr[g,s]$ *is the identity map for all* $g\in\bar{G}$, $s\in\bar{S}$. *Then* $\bar{L}$-$\bar{G}$*-gyro-graph is vertex- transitive.*

Proposition 23. *Let* $\bar{G}$ *be a gyro-group and* $\bar{S} = \{s,t\}$ *with* $\langle\bar{S}\rangle = \bar{G}$ *and* $\langle s\rangle\cap\langle t\rangle = \{0\}$, *then* $\Lambda = \bar{L}$-$\bar{G}$*-gyro-graph is a simple graph and* $L(\Lambda)\cong Cay(\bar{G},A)$, *where* $L(\Lambda)$ *is the line graph of* Λ *and* $A = (\langle s\rangle\cup\langle t\rangle)\setminus\{0\}$.

Proposition 24. *(Newman* et al. *2019) A connected graph is edge-transitive if and only if its line graph is vertex-transitive.*

Theorem 25. *(Moradi, Fath-Tabara, and Bretto 2024) Let* $\bar{G}$ *be a gyro-group and* $\bar{S} = \{s,t\}$ *with* $\langle\bar{S}\rangle = \bar{G}$ *and* $\langle s\rangle\cap\langle t\rangle = \{0\}$ *and also* $gyr[s,g]$ *is identity map for* $g\in\bar{G}$ *and* $s\in\bar{S}$, *then* $\bar{L}$-$\bar{G}$*-gyro-graph is symmetric.*

Proof. According to theorem 22, we conclude that $\bar{G}$-gyro-graph is transitive. By using proposition 23, we have that $L(\Lambda)$ is transitive, and finally, according to proposition 24, it is a symmetric graph. □

We especially try to use the structure of G-graph and $\bar{G}$-gyro-graph to study the well-known classical problem that is referred to as the *Lovász conjecture* (Guy *et al.* 1970) on a hamiltonian path in graph, which says that every finite connected vertex-transitive graph contains a hamiltonian path. For instance, Pak and Radoičić (2009) gave some informations about the hamiltonian Cayley graph. To prove this result, the authors of this survey will try to work on hamiltonian G-graph and $\bar{G}$-gyro-graph, and also on the involution $\bar{G}$-gyro-graphs.

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