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GRAD'S 14-MOMENTS MODEL FOR DENSE GASES OF SPHERES SUBJECTED TO INELASTIC COLLISIONS

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Abstract. A 14-moments model for a dense granular gas is developed in the context of Grad's theory. The gas consists of spheres subjected to inelastic collisions. This condition implies that the energy is not conserved, but a part of it is transformed into heat and this determines a dissipation of temperature. Furthermore, dense gases are characterized by two phenomena in inelastic collisions: the transfer and the transport of particle properties. The balance laws for mass, momentum, energy, stress tensor, heat flux and the fourth scalar moment are derived. The fluxes and the source terms are computed for each balance equation following the procedures elaborated by J. T. Jenkins and M. W. Richman [*Arch. Rational Mech. Anal.* **87**, 355 – 377 (1985)]. The results are compared with the dilute elastic and inelastic gases. This new set of field equations can be useful in different applications.

1. Introduction

The simplest case of gases studied in kinetic theory are dilute and elastic gases, that are gases whose particles have positions and velocities independent of each other and they are subject to elastic collisions. In dilute gases, when two particles collide, the contribution of the other particles is not taken into account because they are very spaced between them. Normally, two phenomena in binary collision can be identified: the transfer of particle properties, such as momentum and energy, and the transport of these properties. In the dilute case, the transport predominates and the transfer is neglected. In this work, we will focus on a more complex case, represented by dense and granular gases: the particles have positions and velocities that depend on each other and the collisions are inelastic. This last condition implies that the energy is not conserved, but a part of it is transformed into heat and this determines a temperature dissipation. In dense gases, the positions of the centers of two colliding particles are distinguished, and in collision the position of the two particles is affected by the presence of neighboring particles. Ultimately, the phenomenon of transfer of particle properties and transport are both considered. Mathematically, this means that there are terms in the balance equations, related to transport and transfer, that are considered.

The starting point for the present research is the paper of Jenkins and Richman (1985), in which a model for 13-moments dense granular gases is presented in the context of Grad's theory (Grad 1949, 1958). Using the techniques of kinetic theory fluxes associated with transport, fluxes associated with transfer and source terms are calculated for each balance equation. We amplify the model of Jenkins and Richman (1985) by adding a 14th variable. The 14th field is the non-equilibrium part of the 4th rank moment. Then we construct a model of differential equations for 14 moments and we determine all fluxes and source terms. Indeed it was shown (see, for example, Müller and Ruggeri 1998) that systems which contains additional moments as field variables can describe phenomena more rigorously than system with 13 moments.

In literature the 14-moments equations were already studied by Kremer and Marques Jr (2011) and Kremer (2019), who also amplified the 13-moments model by adding a 14th variable. In addition, they studied the decays of temperature, stress tensor, heat flux and this new 14th moment together with the stability of linear perturbation of a time-dependent solution. Kremer and Marques Jr (2011) and Kremer (2019) studied dilute or moderately dense gasses. In this paper also dense gases are considered. Gupta, Shukla, and Torrilhon (2018) studied the cases with more moments with the stability of the homogeneous time-dependent solution. On the basis of the solution obtained by Kremer and other authors (Brilliantov and Pöschel 2004; Kremer and Marques Jr 2011; Gupta, Shukla, and Torrilhon 2018; Kremer 2019), an analysis of the propagation of acceleration waves propagating into the spatially homogeneous case was conducted by Barbera and Pollino (2023).

In this paper, we follow the method of Jenkins and Richman (1985) without any truncation of the field equations and we determine a particular solution for the obtained set of field equations. More in detail the article is organized as follows. In Sect. 2 we define the first 14 moments of the kinetic theory, that are the unknown fields: the density, the velocity, the temperature, the stress tensor, the heat flux and the 14th scalar moment ρ_{llkk}. In Sect. 3 we derive the 14 balance laws from the Boltzmann equation. In Sections 4 and 5 we present the setting of the kinetic theory. In particular, in Sect. 4 we define the inelastic collisions analytically and in Sect. 5 we discuss the approximations of the distribution function necessary for calculating various production terms. Sect. 6 is devoted to the computations of all fluxes and source terms generalizing for the 14-moments model the formulas, obtained by Jenkins and Richman, which are linear approximations in the non-equilibrium fields. Finally, in Sect. 7 a particular solution is obtained and discussed.

2. Definition of moments

Kinetic theory (Chapman and Cowling 1991) deals with non-equilibrium problems concerning gases or gas mixtures. The starting point is the distribution function that is defined in the phase space consisting of macroscopic variables, such as time and space, and microscopic variables, such as velocity. Precisely, we say that the function $f(t,x,c)$ is the single particle distribution function, if

$$f(t,x,c)\,\mathrm{d}c \tag{1}$$

represents the number of particles that, at the time t, are in the position x with velocity in the range c and c+dc. In terms of this function, we define the so-called moments of the distribution function:

$$F_{i_1 i_2 \ldots i_N}(t,x) = m\int c_{i_1} c_{i_2} \ldots c_{i_N} f(t,x,c)\,\mathrm{d}c, \tag{2}$$

with m the molecular mass. These moments are related to macroscopic thermodynamic quantities (Chapman and Cowling 1991). In fact, the moment of zeroth and first order are related respectively to the density $\rho\,(t,x)$ and the macroscopic velocity $v\,(t,x)$ that are

$$\begin{aligned}\rho\,(t,x) &= m \textstyle\int f\,(t,x,c)\,\mathrm{d}c, \\ \rho\,(t,x)\,v_i\,(t,x) &= m \textstyle\int c_i\, f\,(t,x,c)\,\mathrm{d}c.\end{aligned} \tag{3}$$

Then, defining the relative velocity $C = c - v$, it is possible to introduce the internal moments as

$$\rho_{i_1 i_2 \ldots i_N}\,(t,x) = m \int C_{i_1} C_{i_2} \ldots C_{i_N} f\,(t,x,c)\, dC. \tag{4}$$

The first internal moments are related to well-known macroscopic quantities, that are the granular temperature, $\theta\,(t,x)$, the stress tensor ρ_{ij} and the heat flux q_i:

$$\begin{aligned}\theta\,(t,x) &= \tfrac{1}{3}\tfrac{m}{\rho} \textstyle\int C^2 f\,(t,x,c)\,\mathrm{d}c, \\ \rho_{ij}\,(t,x) &= m \textstyle\int C_i C_j f\,(t,x,c)\,\mathrm{d}c, \\ q_i\,(t,x) &= \tfrac{1}{2} m \textstyle\int C_i C^2 f\,(t,x,c)\,\mathrm{d}c.\end{aligned} \tag{5}$$

Another quantity that will be useful later is the non-equilibrium part of the so-called fourth moment, that is $\Delta\,(t,x) = \rho_{llss} - 15\rho\theta^2$ where, according to the definition (4), the following relation holds:

$$\rho_{llss} = m \int C^4 f\,(t,x,c)\,\mathrm{d}c. \tag{6}$$

Using the relation between the molecular and peculiar velocities, $c_i = C_i + v_i$, it is possible to derive the relation between the moments F's and the corresponding internal moments ρ's, so we split the moments in velocity-dependent parts and internal moments:

$$\begin{aligned}F_{ij} &= \rho_{ij} + \rho v_i v_j, \\ F_{ijk} &= \rho_{ijk} + 3\rho_{(ij} v_{k)} + \rho v_i v_j v_k, \\ F_{ikll} &= \rho_{ikll} + 4\rho_{(ikl} v_{l)} + 6\rho_{(ik} v_l v_{l)} + \rho v^2 v_i v_k, \\ F_{kllss} &= \rho_{kllss} + 5\rho_{(klls} v_{s)} + 10\rho_{(kll} v_s v_{s)} + 10\rho_{(kl} v_l v_s v_{s)} + \rho v^4 v_k.\end{aligned} \tag{7}$$

By this decomposition, it is equivalent to consider the F's as field variables or equivalently the corresponding internal moments ρ's. In this paper, we consider the moments ρ's, as it is usually done inside the Grad's theory (Grad 1949, 1958).

3. Balance equations

The distribution function $f\,(t,x,c)$ obeys the Boltzman equation

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} + f_i \frac{\partial f}{\partial c_i} = \mathbb{C}\,(f), \tag{8}$$

where f_i represents the specific body force acting on the particles. The term $\mathbb{C}\,(f)$ is the collisional operator, that is regarded as a measure of the change of the distribution function due to collisions between particles. The Boltzman equation governs the evolution in time and space of the distribution function and allows us to predict the behavior of the gas under investigation.

The Boltzman equation, that describes the gas at the microscopic level, can be also used to recover macroscopic equations. If we multiply it by the velocity c and integrate in the space of the velocities, it implies a set of balance equations for the moments. The purpose of this article is in fact to use the tools of kinetic theory to determine the fluxes and source terms of the balance laws, provided that the results here obtained are consistent with the continuum thermodynamic theories. So, multiplication of the Boltzman equation (8) by $mc_{i_1}c_{i_2}...c_{i_N}$ and integration over the whole range of $\mathbf{c}$ provides the balance equations for the macroscopic fields $F_{i_1 i_2 ... i_N}(t,x)$, which assume the compact form

$$\frac{\partial F_{i_1 i_2 ... i_N}}{\partial t} + \frac{\partial F_{k i_1 i_2 ... i_N}}{\partial x_k} - N F_{(i_1 i_2 ... i_{N-1} f_{i_N})} = P_{i_1 i_2 ... i_N}. \tag{9}$$

Here the round brackets indicate the symmetric part of a tensor. The terms in the right hand side represent the productions that will be discussed later.

We assume that the gas can be described by the first fourteen moments, that are density $\rho(t,x)$, velocity $v_i(t,x)$, temperature $\theta(t,x)$, stress tensor $\rho_{ij}(t,x)$, heat flux $q_i(t,x)$ and the non-equilibrium part of the double trace of the moment of rank four $\Delta(t,x)$. Then, the balance equations for these 14 macroscopic fields are

$$\begin{aligned}
&\tfrac{\partial \rho}{\partial t} + \tfrac{\partial \rho v_k}{\partial x_k} = 0,\\
&\tfrac{\partial \rho v_i}{\partial t} + \tfrac{\partial F_{ik}}{\partial x_k} = \rho f_i + P_i,\\
&\tfrac{\partial F_{ij}}{\partial t} + \tfrac{\partial F_{ijk}}{\partial x_k} = 2\rho v_{(i} f_{j)} + P_{ij},\\
&\tfrac{\partial F_{ill}}{\partial t} + \tfrac{\partial F_{ikll}}{\partial x_k} = 3F_{(il} f_{l)} + P_{ill},\\
&\tfrac{\partial F_{llss}}{\partial t} + \tfrac{\partial F_{kllss}}{\partial x_k} = 4F_{(lss} f_{l)} + P_{llss}.
\end{aligned} \tag{10}$$

Equations (10) represent the classical hierarchy of 14 equations for classical monatomic gases. This hierarchy together with the hierarchies corresponding to 13 and more moments for classical monoatomic gasses are described in detail by Müller and Ruggeri (1998).

4. Inelastic collisions

In order to evaluate the production terms, following the computations elaborated by Jenkins and Richman (1985), we assume that the microscopic particles are identical spheres of diameter σ, the collisions between them are binary but inelastic. In this way, indicating with c^1 and $c^{1\prime}$ the velocities of the particle "1" before and after the collisions and with c^2 and $c^{2\prime}$ those of the particle "2", one has

$$\begin{aligned}
mc_i^{1\prime} &= mc_i^1 - J_i,\\
mc_i^{2\prime} &= mc_i^2 + J_i.
\end{aligned} \tag{11}$$

If $g_i = c_i^1 - c_i^2$ and $g_i^\prime = c_i^{1\prime} - c_i^{2\prime}$ are the relative velocities of the centers of the spheres immediately before and after the collisions and k is the unit vector directed from the center of particle "1" to the center of "2" at contact, their components normal to the plane of contact are characterized by

$$\left(g^\prime \cdot k\right) = -e\left(g \cdot k\right), \tag{12}$$

where e is the so-called restitution coefficient. It is a parameter that verifies $0 \leq e \leq 1$. The particular case $e = 1$ corresponds to perfectly elastic collisions, while, when $e << 1$, a strong inelastic collision is considered. By combination of (11) and (12), it is possible to get the velocities of the particles "1" and "2" before the collision in terms of the velocity after this collision:

$$\begin{aligned} c_i^{1\prime} &= c_i^1 - \tfrac{1}{2}\,(1+e)\,(g\cdot k)\,k_i, \\ c_i^{2\prime} &= c_i^2 + \tfrac{1}{2}\,(1+e)\,(g\cdot k)\,k_i. \end{aligned} \tag{13}$$

From (13) it is easy to obtain some relations that will be useful later in the determination of the productions:

$$\begin{aligned} c_i^{1\prime}c_j^{1\prime} - c_i^1 c_j^1 \qquad &= \tfrac{1}{2}\,(1+e)\,(g\cdot k)\left[\tfrac{1}{2}\,(1+e)\,(g\cdot k)\,k_i k_j - \left(k_i c_j^1 + k_j c_i^1\right)\right], \\ c_i^{1\prime}c_j^{1\prime}c_p^{1\prime} - c_i^1 c_j^1 c_p^1 &= -\tfrac{1}{2}\,(1+e)\,(g\cdot k)\left[3k_{(i}c_j^1 c_{p)}^1 - \tfrac{3}{2}\,(1+e)\,(g\cdot k)\,k_{(i}k_j c_{p)}^1 + \right. \\ &\qquad\qquad \left. + \tfrac{1}{4}\,(1+e)^2\,(g\cdot k)^2\,k_i k_j k_p\right], \\ c_l^{1\prime}c_l^{1\prime}c_s^{1\prime}c_s^{1\prime} - c_l^1 c_l^1 c_s^1 c_s^1 &= -2\,(1+e)\,(g\cdot k)\left[k_l c_l^1 c_s^1 c_s^1 - \tfrac{1}{2}\,(1+e)\,(g\cdot k)\,\left(k_l c_l^1\right)^2 + \right. \\ &\left. - \tfrac{1}{4}\,(1+e)\,(g\cdot k)\,c_s^1 c_s^1 + \tfrac{1}{4}\,(1+e)^2\,(g\cdot k)^2\,k_l c_l^1 - \tfrac{1}{32}\,(1+e)^3\,(g\cdot k)^3\right]. \end{aligned} \tag{14}$$

The first two relations were obtained by Jenkins and Richman (1985), the third one is instead appropriate to the 14th moment. Similarly, it is possible to obtain the total change of a property φ due to the collision of the particles "1" and "2":

$$\Pi\,(\varphi) = \varphi^{1\prime} + \varphi^{2\prime} - \varphi^1 - \varphi^2, \tag{15}$$

that is

$$\begin{aligned} &\Pi(c_i c_j) = \tfrac{1}{2}\,(1+e)\,(g\cdot k)\,[(1+e)\,(g\cdot k)\,k_i k_j - (k_i g_j + k_j g_i)]\,, \\ &\Pi(c_i c_j c_k) = 3Q_{(i}\Delta(c_j c_{k)}), \\ &\Pi(c^4) = \tfrac{1}{8}\,(1+e)^4\,(g\cdot k)^4 - \tfrac{1}{2}\,(1+e)^3\,(g\cdot k)^4 + \\ &\qquad + (1+e)^2\,(g\cdot k)^2\left[Q^2 + \tfrac{1}{4}g^2 + 2\,(Q\cdot k)^2 + \tfrac{1}{2}\,(g\cdot k)^2\right] + \\ &\qquad - (1+e)\,(g\cdot k)\left[\tfrac{1}{2}\,(g\cdot k)\,g^2 + 2\,(g\cdot k)\,Q^2 + 4\,(g\cdot Q)\,(Q\cdot k)\right] \end{aligned} \tag{16}$$

with $Q_i = \frac{1}{2}\left(c_i^1 + c_i^2\right)$ the velocity of the center of mass of the two particles. Also here, the first two conditions were evaluated by Jenkins and Richman (1985), while the third is appropriate to the balance equation for the 14th moment. It can be easily seen that, by contracting i and j in $(16)_1$, we get

$$\Pi\,(E) = -\frac{1}{4}m(1-e^2)(g\cdot k)^2. \tag{17}$$

This term represents the energy production that is due to the collisions between the molecules. For inelastic collision this quantity is different from zero, while it vanishes in the elastic case, that is $e = 1$.

5. Determination of the production terms

The statistics of binary collisions is characterized by the complete pair distribution function $f^{(2)}\left(t,x^1,c^1,x^2,c^2\right)$, so

$$f^{(2)}\left(t,x^1,c^1,x^2,c^2\right)\mathrm{d}x^1\ \mathrm{d}c^1\mathrm{d}x^2\mathrm{d}c^2 \tag{18}$$

is the probable number of pairs of particles that at the time t are located between x^1 and $x^1+\mathrm{d}x^1$ and x^2 and $x^2+\mathrm{d}x^2$ with the velocities between c^1 and $c^1+\mathrm{d}c^1$ and c^2 and $c^2+\mathrm{d}c^2$, respectively. In the case of dilute gases, the positions and the velocities of the particles are assumed to be independent, therefore for rarefied gases the pair distribution function is assumed as the product $f^{(2)}\left(t,x^1,c^1,x^2,c^2\right)=f\left(t,x^1,c^1\right)f\left(t,x^2,c^2\right)$. For such gases the transfer of particles properties, like momentum and energy in collisions is neglected in comparison to the transport of particles properties.

Instead, for dense gases the transfer of particles properties in collisions is so important as the transport of particles properties. The probable position of two colliding particles is influenced by the presence of neighboord particles. To account for the interactions of neighboring particles when two particles collide, the equilibrium value of a radial distribution function g_0 is introduced into the relation that binds $f^{(2)}$ and the velocity distribution function of each particle f:

$$f^{(2)}\left(t,x^1,c^1,x^2,c^2\right)=g_0(x)\,f\left(t,x^1,c^1\right)f\left(t,x^2,c^2\right). \tag{19}$$

There are different evaluations of the radial distribution function g_0. For example, Carnahan and Starling (1969) evaluated the probability of collision in terms of the solid volume fraction $\nu=n\pi\sigma^3/6$ as

$$g_0(\nu)=\frac{1}{1-\nu}+\frac{3\nu}{2(1-\nu)^2}+\frac{\nu^2}{2(1-\nu)^3}. \tag{20}$$

In presence of dense gases Jenkins and Richman (1985) showed that that the production terms can be expressed as

$$P_{i_1i_2\ldots i_N}=\Psi_{i_1i_2\ldots i_N}-\frac{\partial\Theta_{ki_1i_2\ldots i_N}}{\partial x_k}. \tag{21}$$

The quantities $\Psi_{i_1i_2\ldots i_N}=\Psi\left(mc_{i_1}c_{i_2}\ldots c_{i_N}\right)$ are defined as

$$\begin{gathered}\Psi(\varphi)=\tfrac{1}{2}\int\int\int\left(\varphi^{1\prime}+\varphi^{2\prime}-\varphi^1-\varphi^2\right)f^{(2)}\left(x-\sigma k,c^1,x,c^2\right)\\ \sigma^2(g\cdot k)\,\mathrm{d}k\mathrm{d}c^1\mathrm{d}c^2.\end{gathered} \tag{22}$$

They represent the typical source terms that are already present in monatomic gases. The other terms, that are not present in kinetic theory of dilute gases, depend on the dense nature of the material under consideration. One has $\Theta_{si_1i_2\ldots i_N}=\Theta_s\left(mc_{i_1}c_{i_2}\ldots c_{i_N}\right)$ with

$$\begin{gathered}\Theta_s(\varphi)=-\tfrac{\sigma}{2}\int\int\int\left(\varphi^{1\prime}-\varphi^1\right)k_s\left(1-\tfrac{\sigma}{2!}k_j\tfrac{\partial}{\partial x_j}+\tfrac{\sigma^2}{3!}k_jk_m\tfrac{\partial}{\partial x_j\partial x_m}+\ldots\right)\\ f^{(2)}\left(x,c^1,x+\sigma k,c^2\right)\sigma^2(g\cdot k)\,\mathrm{d}k\mathrm{d}c^1\mathrm{d}c^2.\end{gathered} \tag{23}$$

By substitution of (21-23) into the balance equations for the moments (10), we get

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} = 0, \\
&\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_k}\left[F_{ik} + \Theta_{ik}\right] = \rho f_i, \\
&\frac{\partial F_{ij}}{\partial t} + \frac{\partial}{\partial x_k}\left[F_{ijk} + \Theta_{ijk}\right] = 2\rho v_{(i} f_{j)} + \Psi_{ij}, \\
&\frac{\partial F_{ill}}{\partial t} + \frac{\partial}{\partial x_k}\left[F_{ikll} + \Theta_{ikll}\right] = 3F_{(il} f_{l)} + \Psi_{ill}, \\
&\frac{\partial F_{llss}}{\partial t} + \frac{\partial}{\partial x_k}\left[F_{kllss} + \Theta_{kllss}\right] = 4F_{(lss} f_{l)} + \Psi_{llss}.
\end{aligned}
\tag{24}
$$

In the particular case of dilute gases the Θs vanish.

We define the internal quantities θ and ψ exactly like in the definitions (22) and (23) for the quantities Ψ and Θ, but using the peculiar velocity C instead of c. So, taking again into account the relation $C_i = c_i - v_i$, the following decomposition for the production terms holds

$$
\begin{aligned}
&\Theta_{ij} = \theta_{ij}, \\
&\Theta_{ijk} = \theta_{ijk} + 2\theta_{k(i} v_{j)}, \\
&\Theta_{ikll} = \theta_{ikll} + 3\theta_{k(il} v_{l)} + 3\theta_{k(i} v_l v_{l)}, \\
&\Theta_{kllss} = \theta_{kllss} + 4\theta_{k(lls} v_{s)} + 6\theta_{k(ll} v_s v_{s)} + 4\theta_{kl} v_l v_s v_s, \\
&\Psi_{ij} = \psi_{ij}, \\
&\Psi_{ill} = \psi_{ill} + 3\psi_{(il} v_{l)} \\
&\Psi_{ikll} = \psi_{ikll} + 4\psi_{(ikl} v_{l)} + \psi_{(ik} v_l v_{l)}.
\end{aligned}
\tag{25}
$$

Then, taking into account these decompositions and (7), it is possible to obtain the 14 balance equations for the 14 internal moments:

$$
\begin{aligned}
&\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \frac{\partial v_k}{\partial x_k} = 0, \\
&\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} + \frac{\partial}{\partial x_k}\left[\rho_{ik} + \theta_{ik}\right] = \rho f_i, \\
&\frac{\mathrm{d}\rho_{ij}}{\mathrm{d}t} + \frac{\partial}{\partial x_k}\left[\rho_{ijk} + \theta_{ijk}\right] + \rho_{ij}\frac{\partial v_k}{\partial x_k} + 2\left[\rho_{k(i} + \theta_{k(i}\right]\frac{\partial v_{j)}}{\partial x_k} = \psi_{ij}, \\
&\frac{\mathrm{d}\rho_{ill}}{\mathrm{d}t} + \frac{\partial}{\partial x_k}\left[\rho_{ikll} + \theta_{ikll}\right] + \rho_{ill}\frac{\partial v_k}{\partial x_k} + 3\left[\rho_{k(il} + \theta_{k(il}\right]\frac{\partial v_{l)}}{\partial x_k} + \\
&\qquad\qquad -3\frac{\rho_{(il}}{\rho}\frac{\partial}{\partial x_k}\left[\rho_{l)k} + \theta_{l)k}\right] = \psi_{ill}, \\
&\frac{\mathrm{d}\rho_{llss}}{\mathrm{d}t} + \frac{\partial}{\partial x_k}\left[\rho_{kllss} + \theta_{kllss}\right] + \rho_{llss}\frac{\partial v_k}{\partial x_k} + 4\left[\rho_{ksll} + \theta_{ksll}\right]\frac{\partial v_s}{\partial x_k} + \\
&\qquad\qquad -8\frac{q_l}{\rho}\frac{\partial}{\partial x_k}\left[\rho_{lk} + \theta_{lk}\right] = \psi_{llss}.
\end{aligned}
\tag{26}
$$

The first equation represents the conservation law of mass, the second is the balance law of momentum. The quantity $\rho_{ik} + \theta_{ik}$ is the total pressure tensor: the sum of the quantity due to the transport of momentum between collisions and those related to the transfer of momentum. The trace of the third equation represents the balance law of energy

$$
\frac{3}{2}\rho\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{1}{2}\frac{\partial\left[\rho_{kll} + \theta_{kll}\right]}{\partial x_k} + \left[\rho_{kl} + \theta_{kl}\right]\frac{\partial v_l}{\partial x_k} = \frac{1}{2}\psi_{ll},
\tag{27}
$$

where θ is the so-called granular temperature, the sum $\frac{1}{2}\left[\rho_{kll}+\theta_{kll}\right]$ is the heat flux, with the transport and collisional parts, the last term ψ_{ll} represents the dissipation due to the inelastic nature of the collision. The traceless part of equation $(26)_3$ is instead the balance law for the stress tensor and it assumes the form

$$\frac{\mathrm{d}\rho_{<ij>}}{\mathrm{d}t}+\frac{\partial\left[\rho_{<ij>k}+\theta_{<ij>k}\right]}{\partial x_k}+\rho_{<ij>}\frac{\partial v_k}{\partial x_k}+2\left[\rho_{k<i}+\theta_{k<i]}\right.\frac{\partial v_{j>}}{\partial x_k}=\psi_{<ij>}. \tag{28}$$

Square brackets in the indexes indicate traceless part of a tensor. The remaining equations are instead the balance laws for the heat flux and the 14th moment, since $q_i=\frac{1}{2}\rho_{ill}$ and $\Delta=\rho_{ssll}-15\rho\theta^2$. In the next section we will close the set of balance equations (26) by the Grad's method.

6. Closure by Grad

Equations (26) represent a system of 14 partial differential equations for the 14 fields ρ, v_i, θ, $\rho_{<ij>}$, $q_k=1/2\rho_{kll}$ and Δ. Unfortunately, these equations are not closed for the occurrence of the quantities $\rho_{<ijk>}$, $\rho_{<ik>ll}$, ρ_{kllss}, θ_{ik}, θ_{ijk}, θ_{ikll}, θ_{kllss}, ψ_{ij}, ψ_{ill} and ψ_{llss}. In this section, we will obtain constitutive relations for these additional quantities using the methods of Grad (1949). First of all, the single distribution function is written as

$$f\left(t,x,c\right)=\left(1-a_i\frac{\partial}{\partial c_i}+a_{ij}\frac{\partial^2}{\partial c_i\partial c_j}-a_{ijk}\frac{\partial^3}{\partial c_i\partial c_j\partial c_k}+\ldots\right)f_0\left(t,x,c\right), \tag{29}$$

where $f_0\left(t,x,c\right)$ is the Maxwellian distribution function given by

$$f_0\left(t,x,c\right)=\frac{n}{\left(2\pi\theta\right)^{\frac{3}{2}}}e^{-\frac{c^2}{2\theta}}, \tag{30}$$

where n is the number density defined as $n=\rho/m$. Insertion of (30) into (29) yields, after simple calculations, the single distribution function in terms of the Hermite polynomials:

$$\begin{aligned}f\left(t,x,c\right)=f_0\Big\{1&+\tfrac{a_i}{\theta}C_i+\tfrac{a_{ij}}{\theta^2}\left[C_iC_j-\theta\delta_{ij}\right]+\tfrac{a_{ijk}}{\theta^3}\left[C_iC_jC_k-3\theta C_{(i}\delta_{jk)}\right]+\\&+\tfrac{a_{ijks}}{\theta^4}\left[C_iC_jC_kC_s-6\theta\delta_{(ij}C_kC_{s)}+3\theta^2\delta_{(ij}\delta_{ks)}\right]+\ldots\Big\}.\end{aligned} \tag{31}$$

The coefficients a can be evaluated by insertion of (31), truncated to the fourth order, into the definitions of the moments (3,5,6), so it follows

$$a_i=0,\quad a_{ij}=\tfrac{\rho_{<ij>}}{2\rho},\quad a_{ijk}=\tfrac{q_i}{5\rho}\delta_{jk},\quad a_{ijks}=\tfrac{\Delta}{120\rho}\delta_{ij}\delta_{ks}. \tag{32}$$

With these quantities, the single velocity distribution function, appropriate to 14 moments, takes the form

$$\begin{aligned}f\left(t,x,c\right)=f_0\Big\{1&+\tfrac{\rho_{<ij>}}{2\rho\theta^2}C_iC_j+\tfrac{q_iC_i}{5\rho\theta^3}\left[C^2-5\theta\right]+\\&\tfrac{\Delta}{120\rho\theta^4}\left[C^4-10\theta C^2+15\theta^2\right]\Big\}.\end{aligned} \tag{33}$$

Once this approximate distribution function is obtained, its insertion into the definition of the moments ρ_{ijk}, $\rho_{<ij>ll}$ and ρ_{illss} yields the determination of the first constitutive relations:

$$\begin{aligned}
&\rho_{ijk} = \tfrac{2}{5}\left(q_i\delta_{jk} + q_j\delta_{ik} + q_k\delta_{ij}\right), \\
&\rho_{<ij>ll} = 7\theta\rho_{<ij>}, \\
&\rho_{illss} = 28\theta q_i .
\end{aligned} \tag{34}$$

7. Collision integrals

Jenkins and Richman (1985) showed that the production terms can be evaluated from (22) and (23) obtaining

$$\begin{aligned}
\psi(\varphi) = \tfrac{g_0(x)}{2} \int\int\int \Pi(\varphi) f^{(1)}\left(x,c^1\right) f^{(1)}\left(x,c^2\right) \times \\
\times \left[1 + \tfrac{\sigma}{2} k_i \tfrac{\partial}{\partial x_i} \ln \tfrac{f^{(1)}\left(x,c^2\right)}{f^{(1)}\left(x,c^1\right)}\right] \sigma^2 (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2
\end{aligned} \tag{35}$$

for the source terms with Π defined in (15) and

$$\begin{aligned}
\theta_i(\varphi) = -\sigma \tfrac{g_0(x)}{2} \int\int\int \left(\varphi^{1\prime} - \varphi^1\right) k_i f^{(1)}\left(x,c^1\right) f^{(1)}\left(x,c^2\right) \times \\
\times \left[1 + \tfrac{\sigma}{2} k_j \tfrac{\partial}{\partial x_j} \left(\ln \tfrac{f^{(1)}\left(x,c^2\right)}{f^{(1)}\left(x,c^1\right)}\right)\right] \sigma^2 (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2 .
\end{aligned} \tag{36}$$

We provide approximate expression for $\theta_i(\varphi)$ and $\psi(\varphi)$ that are linear in the perturbations and the spatial gradients of velocity and temperature. So we obtain for the 14-moments case

$$\begin{aligned}
&\theta_i(\varphi) = A_i(\varphi) + B_i(\varphi) + \rho_{jk} B_{ijk}(\varphi) + q_j B_{ijll}(\varphi) + \Delta\hat{B}_i(\varphi), \\
&\psi(\varphi) = E(\varphi) + F(\varphi) + \rho_{jk} F_{jk}(\varphi) + q_j F_j(\varphi) + \Delta\hat{F}(\varphi),
\end{aligned} \tag{37}$$

where the integrals are given by

$$\begin{aligned}
&A_i = -\tfrac{\sigma^3}{2} g_0 \int\int\int \left(\varphi^{1\prime} - \varphi^1\right) k_i f_{01} f_{02} (g\cdot k) \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2, \\
&B_i = -\tfrac{\sigma^4}{4} g_0 \int\int\int \left(\varphi^{1\prime} - \varphi^1\right) k_i k_m f_{01} f_{02} \tfrac{\partial}{\partial x_m} \left(\ln \tfrac{f_{02}}{f_{01}}\right) (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2, \\
&B_{ijk} = -\tfrac{\sigma^3}{4\rho} g_0 \int\int\int \left(\varphi^{1\prime} - \varphi^1\right) k_i \left(f_{01} \frac{\partial^2 f_{02}}{\partial c_j^2 \partial c_k^2} + f_{02} \frac{\partial^2 f_{01}}{\partial c_j^1 \partial c_k^1}\right) (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2, \\
&B_{ijll} = \tfrac{\sigma^3 g_0}{10\rho} \int\int\int \left(\varphi^{1\prime} - \varphi^1\right) k_i \left(f_{01} \frac{\partial^3 f_{02}}{\partial c_j^2 \partial c_l^2 \partial c_l^2} + f_{02} \frac{\partial^3 f_{01}}{\partial c_j^1 \partial c_l^1 \partial c_l^1}\right) (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2, \\
&\hat{B}_i = -\tfrac{\sigma^3 g_0}{240\rho} \int\int\int \left(\varphi^{1\prime} - \varphi^1\right) k_i \left(f_{01} \frac{\partial^4 f_{02}}{\partial c_l^2 \partial c_l^2 \partial c_s^2 \partial c_s^2} + f_{02} \frac{\partial^4 f_{01}}{\partial c_l^1 \partial c_l^1 \partial c_s^1 \partial c_s^1}\right) (g\cdot k) \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2
\end{aligned} \tag{38}$$

and

$$
\begin{aligned}
E &= \tfrac{\sigma^2 g_0}{2} \int\int\int \Pi(\varphi)\, f_{01} f_{02}\, (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2,\\
F &= \tfrac{\sigma^3 g_0}{4} \int\int\int \Pi(\varphi)\, k_m f_{01} f_{02} \tfrac{\partial}{\partial x_m}\left(\ln \tfrac{f_{02}}{f_{01}}\right) \sigma^2\, (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2,\\
F_{ij} &= \tfrac{\sigma^2 g_0}{4\rho} \int\int\int \Pi(\varphi) \left(f_{01} \tfrac{\partial^2 f_{02}}{\partial c_i^2 \partial c_j^2} + f_{02} \tfrac{\partial^2 f_{01}}{\partial c_i^1 \partial c_j^1}\right) (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2,\\
F_{ijk} &= -\tfrac{\sigma^2 g_0}{10\rho} \int\int\int \Pi(\varphi) \left(f_{01} \tfrac{\partial^3 f_{02}}{\partial c_i^2 \partial c_j^2 \partial c_k^2} + f_{02} \tfrac{\partial^3 f_{01}}{\partial c_i^1 \partial c_j^1 \partial c_k^1}\right) (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2,\\
\hat{F} &= \tfrac{\sigma^2 g_0}{240\rho} \int\int\int \Pi(\varphi) \left(f_{01} \tfrac{\partial^4 f_{02}}{\partial c_l^2 \partial c_l^2 \partial c_s^2 \partial c_s^2} + f_{02} \tfrac{\partial^4 f_{01}}{\partial c_l^1 \partial c_l^1 \partial c_s^1 \partial c_s^1}\right) (g\cdot k)\, \mathrm{d}k \mathrm{d}c^1 \mathrm{d}c^2
\end{aligned}
\tag{39}
$$

with $f_{01} = f_0\left(x, c^1\right)$ and $f_{02} = f_0\left(x, c^2\right)$. The integrals $(38)_{1-4}$ and $(39)_{1-4}$ were already evaluated by Jenkins and Richman (1985). For completeness, we write here their values together with the new terms due to the presence of the 14th field Δ. By insertion of the Grad's distribution function (33) and by evaluation of well-known Gaussian integrals (Chapman and Cowling 1991), we get, after long computations, the following values for the flux and the source terms of each balance equation,

$$
\begin{aligned}
\theta_{ik} &= 2(1+e)\upsilon g_0 \rho \theta \delta_{ik} - \tfrac{2(1+e) g_0 \sqrt{\pi\theta} \rho^2 d_p^4}{15m} \left(2 \tfrac{\partial v_{(i}}{\partial x_{k)}} + \tfrac{\partial v_l}{\partial x_l} \delta_{ik}\right) + \tfrac{2(1+e) g_0 \pi \rho d_p^3}{15m} \rho_{<ik>},\\
\theta_{ijk} &= -\tfrac{2(1+e) g_0 \sqrt{\pi\theta} \rho^2 d_p^4}{5m} \tfrac{\partial \theta}{\partial x_{(i}} \delta_{jk)} + \tfrac{4(1+e) g_0 \pi \rho d_p^3}{75m} \left[q_i \delta_{jk} + \tfrac{9}{2} q_{(j} \delta_{k)i}\right],\\
\theta_{ikll} &= (1+e)\left(10 - 3e + 3e^2\right) \upsilon g_0 \rho \theta^2 \delta_{ik} +\\
&\quad - \tfrac{(1+e) g_0 \sqrt{\pi\theta}\, \theta \rho^2 d_p^4}{15m} \left[\left(23 - 3e + 8e^2\right) \tfrac{\partial v_{(i}}{\partial x_{k)}} + \left(19 - 4e + 4e^2\right) \tfrac{\partial v_l}{\partial x_l} \delta_{ik}\right]\\
&\quad + \tfrac{(1+e)\left(43 - 21e + 12e^2\right) g_0 \pi \theta \rho d_p^3}{30m} \rho_{<ik>} - \tfrac{(1+e)\left(16 - 3e + 3e^2\right) g_0 \pi \rho d_p^3}{180m} \Delta \delta_{ik},\\
\theta_{kllss} &= -\tfrac{2(1+e)\left(13 - 6e + 4e^2\right) g_0 \sqrt{\pi\theta}\, \theta \rho^2 d_p^4}{3m} \tfrac{\partial \theta}{\partial x_k} + \tfrac{4(1+e)\left(26 - 12e + 9e^2\right) g_0 \pi \theta \rho d_p^3}{15m} q_k,
\end{aligned}
\tag{40}
$$

and the final value of the source term,

$$
\begin{aligned}
\psi_i &= 0,\\
\psi_{ij} &= -\tfrac{4}{3} \tfrac{\left(1-e^2\right) g_0 \rho^2 d_p^2 \sqrt{\pi\theta}\, \theta}{m} \delta_{ij}\\
&\quad + \tfrac{2(1+e)(2-e) g_0 \rho^2 d_p^3 \pi \theta}{5m} \tfrac{\partial v_{(i}}{\partial x_{j)}} + \tfrac{(1+e)(1-3e) g_0 \rho^2 d_p^3 \pi \theta}{15m} \tfrac{\partial v_k}{\partial x_k} \delta_{ij}\\
&\quad - \tfrac{4(1+e)(3-e) g_0 \sqrt{\pi\theta} \rho d_p^2}{5m} \rho_{<ij>} - \tfrac{\left(1-e^2\right) g_0 \rho d_p^2 \sqrt{\pi\theta}}{60 m \theta} \Delta \delta_{ij},\\
\psi_{ill} &= \tfrac{(1+e)(13-9e) g_0 \rho^2 d_p^3 \pi \theta}{6m} \tfrac{\partial \theta}{\partial x_i} - \tfrac{2(1+e)(49-33e) g_0 \sqrt{\pi\theta} \rho d_p^2}{15m} q_i,\\
\psi_{llss} &= -4 \tfrac{\left(1-e^2\right)\left(9+2e^2\right) g_0 \rho^2 d_p^2 \sqrt{\pi\theta}\, \theta^2}{m} + \tfrac{\left(1-e^2\right)\left(19+5e^2\right) g_0 \rho^2 d_p^3 \pi \theta^2}{2m} \tfrac{\partial v_k}{\partial x_k}\\
&\quad - \left[\tfrac{\left(1-e^2\right) e^2}{2m} + \tfrac{(1+e)(271-207e)}{60m}\right] g_0 \rho d_p^2 \sqrt{\pi\theta} \Delta.
\end{aligned}
\tag{41}
$$

Equations (26) together with the constitutive relations (34) and the relations (40) and (41) form a closed set of 14 field equations for the 14 fields. These equations are appropriate to a dense gas whose molecules are subjected to inelasic collisions. We compared the constitutive relations (40) and (41) with the 14-moments calculations of Kremer (2019). He calculated not all the terms in (40) and used a different Grad's approximation. Anyway, the common terms coincides. Also Chen *et al.* (2012) carried out similar calculations as a second-order moment method. These authors evaluated some terms of (40) and (41) and the corresponding terms coincide. Gupta, Shukla, and Torrilhon (2018) studied the dilute case but they considered more moments. Also in this case, the corresponding terms coincide.

8. A numerical application

In order to show a simple application of the previous calculations, we study system (26), together with the relations (40) and (41), in the one-dimensional stationary case, assuming that the fields depend only on x and the velocity vanishes. In this case, the field equations become

$$
\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}x}\left[(1+\chi)\,p+\left(1+\tfrac{2}{5}\chi\right)\rho_{<11>}\right]=0,\\
&\tfrac{\mathrm{d}}{\mathrm{d}x}\left[\left(1+\tfrac{3}{5}\chi\right)q_1-\chi\varphi\tfrac{\mathrm{d}\theta}{\mathrm{d}x}\right]=-6\tfrac{(1-e)\chi\varphi}{d_p^2}\theta-\tfrac{3}{40}\tfrac{(1-e)\chi\varphi}{d_p^2}\tfrac{\Delta}{\rho\theta},\\
&\tfrac{8}{15}\tfrac{\mathrm{d}}{\mathrm{d}x}\left[\left(1+\tfrac{9}{10}\chi\right)q_1-\chi\varphi\tfrac{\mathrm{d}\theta}{\mathrm{d}x}\right]=-\tfrac{12}{5}\tfrac{(3-e)\chi\varphi}{d_p^2}\tfrac{\rho_{<11>}}{\rho},\\
&\tfrac{\mathrm{d}}{\mathrm{d}x}\Big[\left(5+\tfrac{10-3e+3e^2}{2}\chi\right)p\theta+\left(7+\tfrac{43-21e+4e^2}{10}\chi\right)\theta\rho_{<11>}+\\
&\qquad+\tfrac{1}{3}\left(1+\tfrac{16-3e+3e^2}{20}\chi\right)\Delta\Big]=-\tfrac{2}{5}\tfrac{(49-33e)\chi\varphi}{d_p^2}\tfrac{q_1}{\rho}+\tfrac{(13-9e)\chi}{2}\rho\theta\tfrac{\mathrm{d}\theta}{\mathrm{d}x},\\
&\tfrac{\mathrm{d}}{\mathrm{d}x}\left[\left(28+4\tfrac{26-12e+9e^2}{5}\chi\right)\theta q_1-2\left(16-6e+4e^2\right)\chi\varphi\theta\tfrac{\mathrm{d}\theta}{\mathrm{d}x}\right]=\\
&\qquad=-12\tfrac{(1-e)\left(9+2e^2\right)\chi\varphi}{d_p^2}\theta^2-\left[\tfrac{3(1-e)e^2}{2}+\tfrac{271-207e}{20}\right]\tfrac{\chi\varphi\Delta}{d_p^2\rho}
\end{aligned}
\tag{42}
$$

with

$$
\begin{aligned}
&\chi=\tfrac{\pi}{3}(1+e)\tfrac{\rho}{m}d_p^3g_0,\\
&\varphi=\rho d_p\sqrt{\tfrac{\theta}{\pi}}.
\end{aligned}
\tag{43}
$$

These quantities depend on the gas under consideration. χ is an dimensionless parameter that will be discussed later. For simplicity, we introduce the following dimensionless quantities

$$
\begin{aligned}
&\hat{x}=\tfrac{x}{L},\qquad \hat{p}=\tfrac{p}{P},\qquad \hat{\theta}=\tfrac{\theta}{\theta_0},\quad \hat{\sigma}=\tfrac{\rho_{<11>}}{P},\\
&\hat{q}=\tfrac{q}{P\sqrt{\theta_0}},\quad \hat{\Delta}=\tfrac{\Delta}{P\theta_0\sqrt{\theta_0}},\quad \lambda=\tfrac{d_p}{\sqrt{\pi}L}
\end{aligned}
\tag{44}
$$

where L is the length of the domain, P the boundary pressure and θ_0 the boundary temperature.

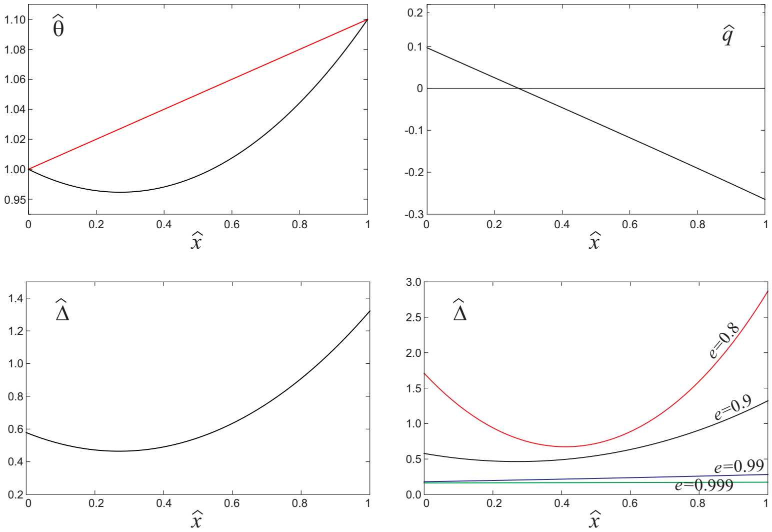

FIGURE 1. Solution for the temperature, heat flux and the variable $\hat{\Delta}$ obtained from the field equations (45) with the values $e = 0.9$, $\hat{\theta}_0 = 1$, $\hat{\theta}_1 = 1.1$, $\lambda = 0.05$ and $\chi = 0.1$. Fig. 1d: solution for $\hat{\Delta}$ obtained with different labeled values of the restitution coefficient e.

Thus equations (42) become

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}\hat{x}}\left[(1+\chi)\,\hat{p}+\left(1+\tfrac{2}{5}\chi\right)\hat{\sigma}\right]=0,\\
&\tfrac{\mathrm{d}}{\mathrm{d}\hat{x}}\left[\left(1+\tfrac{3}{5}\chi\right)\hat{q}-\chi\lambda\hat{\rho}\sqrt{\hat{\theta}}\,\tfrac{\mathrm{d}\hat{\theta}}{\mathrm{d}\hat{x}}\right]=-\tfrac{3(1-e)\chi\hat{\rho}\sqrt{\hat{\theta}}}{\pi\lambda}\left[2\hat{\theta}+\tfrac{1}{40}\tfrac{\hat{\Delta}}{\hat{\rho}\hat{\theta}}\right],\\
&\tfrac{8}{15}\tfrac{\mathrm{d}}{\mathrm{d}\hat{x}}\left[\left(1+\tfrac{9}{10}\chi\right)\hat{q}-\chi\lambda\hat{\rho}\sqrt{\hat{\theta}}\,\tfrac{\mathrm{d}\hat{\theta}}{\mathrm{d}\hat{x}}\right]=-\tfrac{12}{5}\,\tfrac{(3-e)\chi\sqrt{\hat{\theta}}}{\pi\lambda\hat{\sigma}},\\
&\tfrac{\mathrm{d}}{\mathrm{d}\hat{x}}\left[\left(5+\tfrac{10-3e+3e^2}{2}\chi\right)\hat{p}\hat{\theta}+\left(7+\tfrac{43-21e+4e^2}{10}\chi\right)\hat{\theta}\hat{\sigma}+\right.\\
&\qquad\left.+\tfrac{1}{3}\left(1+\tfrac{16-3e+3e^2}{20}\chi\right)\hat{\Delta}\right]=-\tfrac{2}{5}\,\tfrac{(49-33e)\chi}{\pi\lambda}\sqrt{\hat{\theta}}\hat{q}+\tfrac{(13-9e)\chi}{2}\hat{\rho}\,\hat{\theta}\,\tfrac{\mathrm{d}\hat{\theta}}{\mathrm{d}\hat{x}},\\
&\tfrac{\mathrm{d}}{\mathrm{d}x}\left[\left(28+4\tfrac{26-12e+9e^2}{5}\chi\right)\hat{\theta}\hat{q}-2\left(16-6e+4e^2\right)\chi\lambda\hat{p}\sqrt{\hat{\theta}}\,\tfrac{\mathrm{d}\hat{\theta}}{\mathrm{d}\hat{x}}\right]=\\
&\qquad=-\tfrac{12(1-e)\left(9+2e^2\right)\chi\hat{p}\hat{\theta}\sqrt{\hat{\theta}}}{\pi\lambda}-\left[\tfrac{3(1-e)e^2}{2}+\tfrac{271-207e}{20}\right]\tfrac{\chi\sqrt{\hat{\theta}}\hat{\Delta}}{\pi\lambda}.
\end{aligned}\tag{45}$$

In Fig. 1 we show the solution of these field equations obtained with $e = 0.9$, $\hat{\theta}_0 = 1$, $\hat{\theta}_1 = 1.1$, $\lambda = 0.05$ and $\chi = 0.1$. Due to some numerical problems some non-linear effect in the equations are not taken into account. In Fig. 1a the solution is shown together with the line connecting the two boundary temperatures. The solution shows that the temperature greatly reduces in the center of the domain due to its dissipation in the collision and then it grows due to the boundary temperature. The decrease in temperature is stronger than that reported by Barbera and Pollino (2025a) since here a denser case is considered. The new variable Δ is proportional to the temperature and it depends on the restitution coefficient e. For values of $e \approx 1$ this field variable tends to vanish, as shown in Fig. 1d where the field Δ is depicted for different labeled values of e. In Fig. 1b the heat flux is shown and its behavior is approximately linear. The stress tensor σ is instead approximately constant and, for the chosen values of the constants $\hat{\sigma} \approx 0.0651$ holds. This is also an interesting result since in this case the equations herein obtained imply stress tensor components although the gas is at rest. This is a consequence of the inelastic collisions, as also shown in dilute granular gases, but here the effect is stronger since again the gas is more dense.

9. Concluding remarks

In this paper we started from the Grad's 13-moments model for dense and granular gases of Jenkins and Richman (1985) and we extended it to a model of 14 moments, adding the 14th fields Δ and the balance equation for this new variable. The θ_i fluxes, related to the transfer of the particle moments, and the ψ_i source terms for each balance equation have been determined analytically, generalizing those obtained by Jenkins and Richman (1985). A particular solution is determined in order to understand, in a particular case, the effect of this new variable Δ.

A future research perspective would be to treat the model on the basis of continuum thermodynamic theories, in particular the Rational Extended Thermodynamics (Müller and Ruggeri 1998; Ruggeri and Sugiyama 2015, 2021). It has been shown that Extended Thermodynamics is able to recover some results in agreement with the Grad's theory. Furthermore, there is a connection between Grad's theory, at least for the first 13 moments (Grad 1949, 1958), the Maximum Entropy Principle and the Entropy Principle used in Extended Thermodynamics (see Arima and Ruggeri 2024 for a historical summary). It would be interesting to see whether Rational Extended Thermodynamics is able to obtain the results of the present manuscript. Other authors have found a connection, also for the dense gases, between the method of moments and the Maximum Entropy Principle (see, for example, Romero-Salazar, Mayorga, and Velasco 1997) Some preliminary results in this direction are being presented by Barbera and Pollino (2025a) for dilute gases and by Barbera and Pollino (2025b) for slightly dense ones, but the study of the more general case requires a more complex tool.

It would also be remarkable to see the applicability of this model to more realistic and complex cases. Indeed, in the determination of the numerical solution some non-linear effects are not taken into account. Also the study of more realistic gases and conditions should be addressed. Furthermore, it has been shown that the solutions of Grad's theory for rarefied gasses become more interesting and different from the Classical Thermodynamic fields when the gas is in curved domains or in particular situations (see Barbera and Brini 2018 and the references therein).

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