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DIFFERENTIAL GEOMETRY AND DEMAND FUNCTION

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ABSTRACT. The presented work is devoted to the differential geometric analysis of the demand function. Analysis of the neoclassic approach leads to the introduction of the generalized demand function. The differential geometric structure is associated with this function and the correspondent interpretation is described.

The effective and multidirectional study of economic systems and corresponding processes is enough complicated because of the number of conditions having permanent but not stable impact on these systems and processes. In macroeconomics, they operate mainly with such objects as the volume of production, essential material funds, quantity and quality of employers etc. The creation of linear economy models comes with the discovery of some essential properties of economic systems. Nonlinear models are of a more high qualitative level of the mathematical study of economic processes but the introduction of these models is not so easy. Practically it is difficult to collect and classify all statistics necessary for the creation of such models. On the other hand, in many cases it is more easily to collect statistic data about behavior of the above-mentioned factors (price of production, material funds resources, quantity and qualification of employers etc). Studying relations between used funds and volume of production and taking into account labor resources they come to some interesting conclusions about production structure and its properties. Geometrically speaking macroeconomics studies the external part of the production system but it gives many facts that may be used for the study of some internal properties of this system (Haroutunian 2010).

Utility function depends on the same funds (capital and labor) but it is mainly oriented to the market (Haroutunian 2005). The demand function, which is the main object of the research in the present work, is connected with the market too but in contrast to the utility function, it is oriented mainly to itself.

The same idea is effective and realizable in modern differential geometry: analytical properties of objects give stimulation for the discovery of internal (geometric) properties of these objects. When we say differential geometric research of economic structures and processes, we have in mind the realization of the following program.

(A) Introduction of economic structure and its adequate differential geometric description.

Of course, economic structures have general mathematical description but in many cases, this description stays not completed up now. The differential geometric description presents all properties of the given structure.

(B) Identification of differential geometric structures which may be associated with the given economic structure by an invariant way and geometrical study of this structure.

This stage of the research is based on the idea that the modern differential geometry has an adequate totality of structures being able to describe essential economic structures. Economic structures are not very complicate and adequate differential geometric study of them belongs to the number of solvable problems.

(C) Economical interpretation of obtained geometrical results and its generalizations.

This stage is more interesting for economics and besides for the identification of further problems. We will illustrate the realization of this program for the demand functions considering these functions as economic structure.

The creation and development of the mathematical economy was connected with the Great Depression in USA in the first half of the second quarter in XX century. During this short but effective collaboration, many concepts of this economy were introduced and studied. For example, economist Paul Douglas and mathematician Charles Cobb (1928) introduced now well known Cobb-Douglas production function $Y = AK^aL^b$ where $a+b=1$. During this period some elements of the Probability Theory were introduced too, that is why this Theory is relatively more popular in Economy than for example in Geometry. Unfortunately, this collaboration was too short and experts had not time to analyze new concepts and check if they are correct and reasonable. The analysis of these concepts shows that sometimes they were not completed and did not reflected all possible situations in economy. For example, introduction of parameters in the production function (Haroutunian 2010) and in the utility functions (Haroutunian 2005) extracted the area of possible applications of these functions. Besides using these generalized versions, it is possible to discover relations between prices of welfares in the enterprise and their prices in market. The similar problem may be formulated for the demand function. The main goal of the present article is to analyze the concept of the demand function from the viewpoint of the Differential Geometry and present its some generalizations, following the main procedure of the Method of the modern differential geometric research, based on the Method of Exterior Forms. This Method was introduced by the French academician Elie Cartan (1936) in the beginning XX century and was well developed by Soviet Geometers in the second half of that century. We will consider this function as a differential geometric object on the manifold of all goods in market. Studying it from the viewpoint of the modern Differential Geometry, we will come to its geometric interpretations.

The main arguments of the production function are the Capital (K) and the Labor (L). This function $X = F(K,L)$ describes the relation between volume of the production and K, L. In the case of classic utility function, they consider needs of one consumer in the market. Introduction of parameters for the production function and for the utility function significantly extracts the possible areas of applications and gives an opportunity to describe the relations between initial expenses to produce one unity of the given good in factory and variation of its prices in market (Haroutunian 2005, 2010). Taking into account the close relationship between production and corresponding utility function we can consider both depending on K and L.

The demand function reflects the dependence of the demand volume for some material welfare (goods) and services on the complex of different factors having impact on the demand. In more narrow sense the demand function shows the mutual dependence of the demand for the given welfare and the price of this welfare in market under condition that all other factors having impact on the demand are considering constant. The demand for a product (good) or service reflects the consumer's interests and means the willingness and ability of a consumer to buy some quantity of goods at a definitive price or at very limited variations of this price. One of characteristics of this function is the replacement of the given good by another but similar and equivalent. Besides, it is accepted that when the price of a welfare is going up the demand stays the same or goes down. Demand functions are homogeneous functions of the zero order by respect with prices and income of consumer. It follows from definition that the concept of the demand function can be determined for an individual consumer and if so, then it can be based on different variables as the total annual income of the consumer (or his (her) family), the rank of the material welfare and many different parameters. For example, in general the demand for the bookcase is higher for the consumers working in universities or schools than for people working in factories or similar enterprises and the demand for a cot is definitively higher in young families. Situation here is similar to corresponding situation with utility function: it was introduced for a consumer and was carrying all specialties of such approach. Moreover, in the neoclassic theory the demand function is connected with the utility function in a following way: the maximum of the utility function determines the demand function $D = D(D_1, \dots, D_P, R)$ where the components $D_h(p_1, \dots, p_N, R)$ are scalar functions depending on corresponding prices $p_1, \dots, p_N$ of welfares in market and income (R) of the consumer. If the totality X of goods is convex, closed and limited, contains a zero vector and $S = S(x, y)$ is a utility function then there exists a vector x^0, for which the utility function reaches maximum in X. If there exists only one such vector x^0, then the demand function $D = D(p, R)$ determining the vector x^0, is continuous for any vector $p = p(p_1, \dots, p_N)$ with only positive components and $R > 0$. In reality, the situation is a little more complicate. The demand function of a welfare is "running" in neighborhood of the vector x^0. It means that this function depends significantly on variables $p_1, \dots, p_N$, being functions on the utility function. It reflects the impact of the market on the demand of this welfare. As we can see this demand function (generalized demand function) and the generalized utility function (Haroutunian 2005) have some common properties. On the other hand, it is important to underline the differences of these concepts. In neoclassic theory, the utility function describes the interest of the individual consumer in market. In the case of the demand function, this approach is not acceptable. Here everything is concentrated on the welfare in the market. The demand for the given welfare itself depends on the prices, quality, situation in market, other factors and less depends on the income of consumers. For example, the demand for the winter jacket in July and November is not the same. It depends also on the presence and quantity of welfares in market able to replace the given welfare. Besides, in many cases different consumers can have the interest to buy the same welfare for different reasons. For example, one consumer is going to buy the car Mercedes Benz E320 for own travels and transportation of his (or her) children to (from) the school, another is going to use this car for transportation of some limited volume goods from one city to another. In contrast with the first half of the XX century, the modern production system is more and more oriented to the direction of multi-application welfares.

As indicated earlier the concept of the demand function was introduced and studied about one century ago. It is necessary to reconsider this concept and precise it. Besides, as indicated earlier the production and utility functions depend on capital K and labor L. Finally the demand function depends (through utility function) on these factors too. But here this relationship is not so visible.

About notations. $p_1,\dots,p_N$ are prices for goods produced with volumes $x_1,\dots,x_N$ respectively. R is income of consumer (his financial opportunities in market), P, S, $X = (x_1,\dots,x_N,y_1,\dots,y_M)$ and R unifies hexogen values, x_h, y_k are endogen values, $D = D(D_1,\dots,D_A,R)$ is a general demand function in market with welfares $x_1,\dots,x_N$ and $D_h = D_h(p_1,\dots,p_N,R)$, $h = 1,2,\dots,A$ is a demand function of the consumer for the h^{th} welfare (or service). The beginning level of demand functions is known as a one-factor function when the demand depends only on income of consumer. Corresponding graphs are known as Engel's curves. Of course, it is difficult to study the real demand of consumer using such functions only. One can add some other factors and in this case, the study will be more adequate but also more complicate.

Let us go now to the geometric study of the demand function. Imagine that some consumer seeing that his computer does not satisfy new requirement is planning to buy new one in the fall of the year and pay about 500$ for it. In some months, his position was changed and his budget now is higher. If this budget is significantly higher, this consumer is able to buy a significantly more perfect computer and pay for example 1000$ for it. If the progress is not so serious then this consumer will be interested in buying computer by the price 500$ or near. We see that the acceptable price depends on the budget. Therefore the price of 500$ for the given computer belongs to some stripe of possible prices (for example from 480$ to 550$). On the other hand, if we fix the price of 500$ for the given computer we obtain a stripe for possible changes of the consumer's budget. We conclude from here that the following property holds

1) $\dfrac{\partial D}{\partial R} > 0.$

It means that the rising budget stimulates the rising demand. On the other hand, the supplementary analysis for each case is necessary. For example, the firm is preserving fruits and vegetables. During one year, the demand for this production is stable. Starting the second year, the demand for the same production, staying in the store, goes down very definitively in spite of the same level of quality. The price reaches the minimal level. The corresponding price-cutting does not stimulate the increasing demand for these products. The demand function $D = D(D_1,\dots,D_A,R)$ depends on components D_h, $h = 1,2,\dots,A$ which are functions on prices $p_1,\dots,p_N$.

$$D_h = D_h(p_1,\dots,p_N), \quad h = 1,2,\dots,A.$$

2) The rise in prices $p_1, \dots, p_N$ stimulates the reduction of the demand:

$$\frac{\partial D}{\partial p_m} < 0, \quad m = 1, 2, \dots, N.$$

In its turn, these prices depend on the capital and labor of production. Supplementary analysis for the case when some prices are rising and others are reducing is necessary. Suppose the prices $p_1, \dots, p_n$ are rising and prices $p_{n+1}, \dots, p_N$ are going down. Using the procedure applied by Haroutunian (2005) we can consider $N = 2n$. We obtain the presentation

$$D = D(p_1, \dots, p_n, p_{n+1}, \dots, p_{2n}, R). \tag{1}$$

Therefore, the generalized demand function depends finally on variables $(K)(x^1, x^2, \dots, x^n)$ and parameters of production $(L)(y_1, y_2, \dots, y_n)$. We have in the neighborhood of the "point" of maximum x^0

$$D = D(D_1, \dots, D_A, R) = D(D_1(p_1, \dots, p_N, R), \dots, D_A(p_1, \dots, p_N, R) = \\ = D(p_1, \dots, p_N, R) = D(x^1, x^2, \dots, x^n, y_1, y_2, \dots, y_n, R).$$

This function has the following properties.

3) $$\frac{\partial D}{\partial x^i} > 0, \quad \frac{\partial D}{\partial y_i} > 0, \quad \frac{\partial^2 D}{\partial x^i \partial x^j} < 0, \quad \frac{\partial^2 D}{\partial y_i \partial y_j} < 0, \quad \frac{\partial^2 D}{\partial x^i \partial y_j} = \frac{\partial^2 D}{\partial y_i \partial x^j} > 0.$$

The last property was justified by Haroutunian (2005). We suppose here that the market is able to satisfy any demand of the consumer (or consumers). Geometrically we can consider this function as a scalar field $D(p_1, \dots, p_n, p_{n+1}, \dots, p_{2n})$ on $2n$ dimensional manifold M of variables determined by the component K and parameters determined by the component L, parameterized by R. For any fixed value of R this scalar field under some natural limitations (Haroutunian 2010) induces the structure of the Rashevsky pseudoriemannian space (Rashevsky 1948) on M. The pseudoriemannian metrics on M is determined by nondegenerate matrix $\left(\frac{\partial^2 D}{\partial x^i \partial y_j} \right)$. The specialty of this metrics is visible from its canonical form. Here n squares have a positive and n squares negative sign. In order to obtain the structure equations of the Rashevsky pseudoriemannian space we associate with M a moving tangent frame $(m, e^1, e^2, \dots, e^n, e^{n+1}, e^{n+2}, \dots, e^{2n})$, $m \in M$. In cobasis of linear differential forms $\omega^1, \omega^2, \dots, \omega^n, \omega_1, \omega_2, \dots, \omega_n$ (the forms $\omega_1, \omega_2, \dots, \omega_n$ correspond the vector fields $e_{n+1}, e_{n+2}, \dots, e_{2n}$ respectively) the structure equations of the manifold M have the form (Haroutunian 2020)

$$\begin{gathered} d\omega^i = \omega^i_k \wedge \omega^k. \\ d\omega_i = -\omega^k_i \wedge \omega_k. \\ d\omega^i_k = \omega^i_p \wedge \omega^p_k + R^{it}_{kp} \omega^p \omega_p, \end{gathered} \tag{2}$$

where the secondary forms ω^i_k are defined on the manifold $M^{(2)}$ of the second order frames on M, R^{it}_{kp} are non-trivial components of the curvature tensor. Fixation of R separates an individual consumer or the totality of all consumers with the same budget. We come to the following statement.

Theorem 1. *The generalized demand function* (1) *under condition* $\det\left(\frac{\partial^2 D}{\partial x^i \partial y_j}\right) \neq 0$ $(i = 1,2,\dots,n)$ *and fixed value of R induces the structure of the pseudoriemannian Rashevsky space on* $2n$ *dimensional manifold of prices* $p_1,\dots,p_n,p_{n+1},\dots,p_{2n}$ *in an invariant way.*

It follows from here that if the budget R is a variable then we have a $2n+1$ dimensional foliation with one dimensional base (R). The construction of this foliation is enough complicate.

Theorem 2. *The sheets of the foliation D are Rashevsky pseudoriemannian spaces* (2) *of dimension* $2n$*. Different sheets of this foliation are not diffeomorphic: each value of the budget R affects parameters and they affect the curvature tensor* R^{it}_{kp}.

Suppose now that the budget R is not fixed. As mentioned above the structure of the manifold associated with demand function $D = D(D_1,\dots,D_{2n},R)$ is $2n+1$ dimensional foliation with one dimensional base R and $2n$ dimensional sheets. This structure has a specialty; the base is an ordered manifold. This algebraic property does supplementary information about demand function. The dependence of the demand function on budget is continuous. It means in particular that in the enough limited neighborhood $(r^0-\varepsilon, r^0+\varepsilon)$ of the point $r^0 \in R$ the demand is changing not significantly. It is more interesting to interpret the structure of sheets corresponding significantly different values of the variable R.

Theorem 3. *The more is the value of the variable R the more is the number of nonzero components of the curvature tensor of the corresponding Rashevsky pseudoriemannian space.*

It means geometrically that the increasing budget corresponds to the more reach structure of sheets in the total space of foliation D. Of course, here consumer has the better and vaster opportunities of choice. On the other hand, the procedure of orientation in market becomes more complicate. We arrive to the strange prima facie conclusion.

Corollary. The existence of reach consumers in the isolated, pure or under serious sanctions country is not justified.

In such country the market is limited and corresponds to consumers with the limited financial opportunities. Consumers with higher demand cannot satisfy their interests in market. Some politicians of the previous century were sure that in the new political society no people were rich. Such approach comes to the not correct interior politics and stimulates the economical destruction of this country.

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