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MODELLING RIVER FLOWS AS JUMP-DIFFUSION PROCESSES

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ABSTRACT. The streamflow of a river is modelled as a jump-diffusion process. The jump size is distributed as an exponential random variable. The various parameters of the model are estimated by using the method of moments. An application to the Mississippi River is presented.

1. Introduction

The problem of modelling the flow of rivers is an important one in hydrology. Various stochastic processes have been used to achieve this goal. In particular, diffusion processes, which are continuous-time and continuous-state processes, have been proposed as models. For instance, Lefebvre (2002) used geometric Brownian motion to forecast river flows.

Although river flows do evolve continuously over time and their values are continuous random variables, when we look at hydrographs, we often observe rapid increases in flow values, which are difficult to model as diffusion processes. Figures 1 and 2 present respectively the flow of the Mississippi River at St. Louis, MO, in 2020 and 2021. The flow (in cubic feet per second) was recorded every 30 minutes.

To account for these almost sudden increases, one can try to make use of jump-diffusion processes instead. Actually, jump-diffusion processes are widely used in fields like economics and mathematical finance. However, there are very few hydrological papers in which they appear. In Fig. 3, we show the closing values of the NASDAQ composite index during the year 2018. It can be seen that the variations in the index resemble those of the flow of the Mississippi River.

Konecny and Nachtnebel (1995) appear to be the first authors who tried to model daily streamflows by a jump-diffusion process. Their paper had already been published in conference proceedings in 1991. More recently, Tsai, Lin, and Hung (2016) made use of jump-diffusion processes in their study of the movement of sediment particles in response to extreme flow events. Similarly, Temgoua *et al.* (2017) used these processes to detect fluctuations in hydrologic processes in relationship with climate change. Finally, Song,

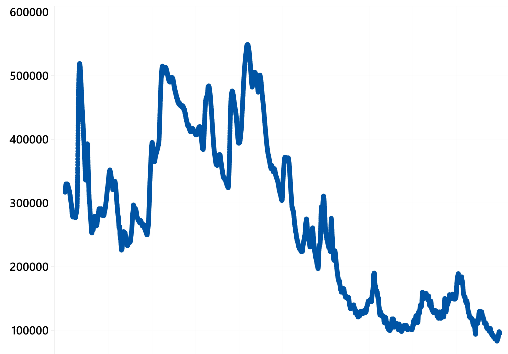

FIGURE 1. Flow of the Mississippi River at St. Louis, MO, in 2020.

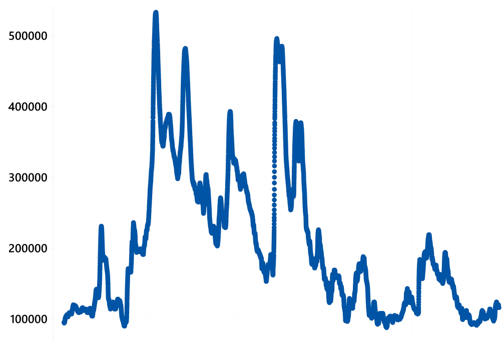

FIGURE 2. Flow of the Mississippi River at St. Louis, MO, in 2021.

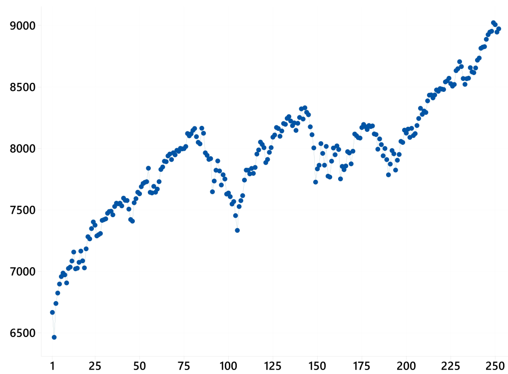

FIGURE 3. Closing values of the NASDAQ composite index during the year 2018.

Zhong, and Wang (2020) applied jump-diffusion processes to investigate the dynamic evolution of bankfull channel geometries.

2. Description of the model

Let $X(t)$ denote the flow of a certain river at time t. We assume that $X(t)$ satisfies

$$X(t) = X(0) + \mu t + \sigma B(t) + \sum_{n=1}^{N(t)} Y_n, \tag{1}$$

in which $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, $\{B(t), t \geq 0\}$ is a standard Brownian motion, $\{N(t), t \geq 0\}$ is a Poisson process with rate λ, and $Y_1, Y_2, \ldots$ are exponentially distributed random variables with parameter α that are independent between themselves and also of the Poisson process. Moreover $\{B(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are independent. Thus, the continuous part of the $\{X(t), t \geq 0\}$ process is a Wiener process with drift μ and variance parameter σ^2, whereas the discrete part is a compound Poisson process. The parameter λ is the rate at which hydrological events occur and $1/\alpha$ is the average size of the flow increase caused by a given event.

Remark. The continuous-time stochastic process $\{X(t), t \geq 0\}$ defined by

$$X(t) = X(0) + \sum_{n=1}^{N(t)} w(t, \tau_n, Y_n), \tag{2}$$

where the τ_n's are the arrival times of the events of $\{N(t), t \geq 0\}$ and $w(t, \tau_n, Y_n)$ is a *response function*, is known as a *filtered Poisson process* (see Parzen 1962). In hydrology, the function w could be of the form

$$w(t, \tau_n, Y_n) = Y_n e^{-(t-\tau_n)/c}, \tag{3}$$

where c is a parameter that is related to the speed at which the effect of an event on the river flow decreases. In Fig. 4, a possible realization of a filtered Poisson process with w defined as above is shown. The author used filtered Poisson processes and their generalization

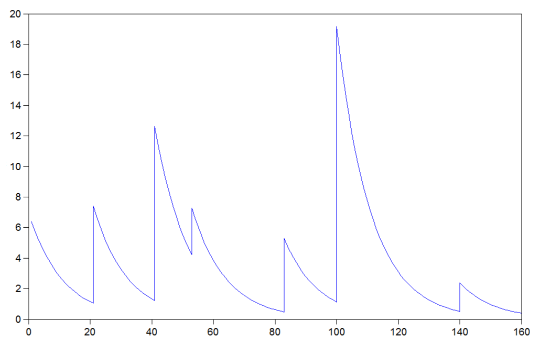

FIGURE 4. Possible realization of a filtered Poisson process.

to filtered renewal processes (when $\{N(t), t \geq 0\}$ is a general renewal process) in various hydrological applications (see Lefebvre 2005; Lefebvre and Guilbault 2008; Lefebvre and Bensalma 2014). Although these processes performed quite well, jump-diffusion processes are surely more realistic to model the variations of river flows over time. Indeed, as can be seen in Fig. 4, a filtered Poisson process behaves deterministically between the various jumps, which is not what is observed in practice.

In the next section, we will see how the four parameters in the model defined in Eq. (1) can be estimated. Then, in Section 4, we will present an application of the model to the Mississippi River, which is located in the United States. Finally, we will end with a few remarks in Section 5.

3. Parameters estimation

We will employ the method of moments to estimate the parameters μ, σ, λ and α that appear in our model. First, the moment-generating function of a Wiener process $\{W(t), t \geq 0\}$ with infinitesimal parameters μ and σ^2, and starting from zero, is given by

$$M_1(s) := E\left[e^{-sW(t)}\right] = \exp\left\{-\mu\, t\, s + \frac{1}{2}\sigma^2\, t\, s^2\right\}, \tag{4}$$

where $s > 0$. Next, the moment-generating function of a compound Poisson process $\{Z(t), t \geq 0\}$ defined by

$$Z(t) = \sum_{n=1}^{N(t)} Y_n \tag{5}$$

is (see Ross 2019)

$$M_2(s) := E\left[e^{-sZ(t)}\right] = \exp\left\{\lambda\, t\left[M_{Y_1}(s) - 1\right]\right\}, \tag{6}$$

where $M_{Y_1}(s)$ is the moment-generating function of the random variable Y_1. In our case, we have

$$M_{Y_1}(s) = \frac{\alpha}{\alpha + s}. \tag{7}$$

Let

$$X_0(t) := X(t) - X(0). \tag{8}$$

Making use of the above results, we will compute $E\left[X_0^k(t)\right]$ for $k = 1, 2, 3, 4$. We have

$$E\left[X_0(t)\right] = \mu\, t + \frac{\lambda}{\alpha}\, t, \tag{9}$$

$$E\left[X_0^2(t)\right] = \mu^2 t^2 + \sigma^2 t + \frac{\lambda\, t\,(\lambda\, t + 2)}{\alpha^2} + \frac{2\mu\,\lambda\, t^2}{\alpha}, \tag{10}$$

$$\begin{aligned} E\left[X_0^3(t)\right] &= \mu^3 t^3 + 3\mu\,\sigma^2 t^2 + \frac{\lambda\, t\,(\lambda^2 t^2 + 6\lambda\, t + 6)}{\alpha^3} \\ &\quad + \frac{3\mu\,\lambda\, t^2\,(\lambda\, t + 2)}{\alpha^2} + \frac{3\lambda\, t^2\,(\mu^2 t + \sigma^2)}{\alpha} \end{aligned} \tag{11}$$

TABLE 1. Point estimates $\hat{\mu}_k$, $k = 1,2,3,4$, for the Mississippi River in 2020

k	1	2	3	4
$\hat{\mu}_k$	−12.71	648,427	150,339,324	2.14688×10^{12}

and

$$E\left[X_0^4(t)\right] = \mu^4 t^4 + 6\mu^2\sigma^2 t^3 + 3\sigma^4 t^2 + \frac{4\mu\lambda t^2(\lambda^2 t^2 + 6\lambda t + 6)}{\alpha^3} + \frac{6\lambda t^2(\mu^2 t + \sigma^2)(\lambda t + 2)}{\alpha^2} + \frac{4\mu\lambda t^3(\mu^2 t + 3\sigma^2)}{\alpha} + \frac{\lambda t(\lambda^3 t^3 + 12\lambda^2 t^2 + 36\lambda t + 24)}{\alpha^4}. \tag{12}$$

If we have a set of data $x_1, \ldots, x_n$, we can define $y_i = x_{i+1} - x_i$ for $i = 1, \ldots, n-1$ and compute the point estimates

$$\hat{\mu}_k := \sum_{i=1}^{n-1} \frac{y_i^k}{n-1} \tag{13}$$

of the four moments $\mu_k := E[X_0^k(1)]$ for $k = 1,2,3,4$. Then, setting $\hat{\mu}_k = \mu_k$ for each value of k, we can in theory find point estimates of the parameters μ, σ, λ and α by solving (numerically) the system of four non-linear equations.

We can also proceed as follows: from Eq. (9) and the equation $\hat{\mu}_1 = \mu_1$, we deduce that

$$\mu = \hat{\mu}_1 - \frac{\lambda}{\alpha}. \tag{14}$$

Substituting this expression for μ into the equation $\hat{\mu}_2 = \mu_2$, we can solve for σ in terms of λ and α. Then, we can express λ in terms of α from the equation $\hat{\mu}_3 = \mu_3$. Finally, from the equation $\hat{\mu}_4 = \mu_4$, we obtain a polynomial equation of degree 9 in α. In the application to the Mississippi River presented in the next section, we will see that there is a single real root of the polynomial equation, and that this root is positive (as required). Hence, the other three parameters can be obtained by working backwards.

4. Application to the Mississippi River

On the website of the U.S. Geological Survey[1] it is possible to download various hydrological data for rivers located in the United States. We chose the discharge of the Mississippi River at St. Louis, MO (site number 07010000), for the years 2020 and 2021. The discharge was recorded (in cubic feet per second) every 30 minutes. There are respectively 17418 and 17181 observations in the data set (there are some missing data); see Figs. 1 and 2. First, from the data in 2020 we calculate the point estimates given in Table 1. Then, Eq. (14) implies that

$$\mu = -12.71 - \frac{\lambda}{\alpha}. \tag{15}$$

[1] https://waterdata.usgs.gov/nwis

TABLE 2. Point estimates $\hat{\mu}_k$, $k = 1,2,3,4$, for the Mississippi River in 2021

k	1	2	3	4
$\hat{\mu}_k$	1.19325	678,621	548,796,100	4.59356×10^{12}

Next, making use of the equation $\hat{\mu}_2 = \mu_2$, we find that

$$\sigma = \sqrt{648265.4559 - \frac{2\lambda}{\alpha^2}}, \tag{16}$$

which, when substituted into the equation $\hat{\mu}_3 = \mu_3$, enables us to write that

$$\lambda = 2.917662318 \times 10^7 \alpha^3. \tag{17}$$

Finally, replacing λ by the above expression in the equation $\hat{\mu}_4 = \mu_4$, we obtain that the parameter α is a root of the ninth-degree polynomial equation

$$\begin{aligned} &3 \times 10^{20}\,\alpha^9 + 1 \times 10^{14}\,\alpha^6 + 2 \times 10^{12}\,\alpha^5 + 3 \times 10^7\,\alpha^3 \\ &- 100000\,\alpha^2 + 8.944073660 \times 10^{11}\,\alpha - 7.002389563 \times 10^8 = 0. \end{aligned} \tag{18}$$

This equation has a single real root, namely

$$\alpha = 0.0007829083072. \tag{19}$$

It follows that

$$\lambda = 0.01400127966, \quad \sigma = 776.2604038 \quad \text{and} \quad \mu = -30.59367748. \tag{20}$$

Remark. The number of hydrological events in 2020 is given by $\lambda \times 17417 \simeq 243.86$. That is, according to the model, a non-negligible event was recorded on average every 1.5 days. Moreover, the average flow increase produced by an event is $1/\alpha \simeq 1277.29$ cubic feet per second ($\simeq 36.17$ m^3/s).

Notice that to estimate the parameters λ and α we did not have to define, rather subjectively, what is considered to be a *jump*. In comparison, Konecny and Nachtnebel (1995) applied their model to the Lafnitz River, in Austria, from 1961 to 1971. They defined a jump as "an increase in the discharge within a maximum of five days and at least 2.5 m^3/s increase per day and a peak exceeding 8 m^3/s. Minor fluctuations in the increasing limb were allowed. Finally 84 jumps were selected with a mean value of 14.2 m^3/s, a variance of 196.2 $(m^3/s)^2$ and a mean interarrival time of 43 days." Similarly, Lefebvre and Guilbault (2008) estimated the parameters λ and α by 0.1458 day^{-1} and 1.86×10^{-4} $(ft^3/s)^{-1}$ respectively in the case of the Delaware River in the time period from 1st October 2002 to 30th September 2003. Thus, for the Mississippi River, we retain approximately 4.6 times more jumps, whose size is on average about 4.2 times smaller.

Point estimates of the moments μ_k in the year 2021 are presented in Table 2. The polynomial equation satisfied by the parameter α is

$$\begin{aligned} &1.2 \times 10^{23}\,\alpha^9 - 1.3 \times 10^{16}\,\alpha^6 + 2 \times 10^8\,\alpha^4 + 300000\,\alpha^2 \\ &+ 3.209372806 \times 10^{12}\,\alpha - 2.185467240 \times 10^9 = 0. \end{aligned} \tag{21}$$

We find that $\alpha = 0.0006809639677$ is the only real root of the above equation. Then, we compute

$$\lambda = 0.02875447590, \quad \sigma = 744.7152963 \quad \text{and} \quad \mu = -41.03288422. \tag{22}$$

The point estimates of the parameters α, μ and σ are not too different than in 2020. However, $\hat{\lambda}$ is more than twice larger in 2021 than in 2020. This reflects the fact that the variations in the flow values were indeed more pronounced in 2021 than in 2020, as can be seen by comparing Fig. 1 and Fig. 2. The number of jumps during the year is equal to $17180\lambda \simeq 494.00$; that is, about 1.35 per day. Their average size is $1/\alpha \simeq 1468.51$ cubic feet per second.

Remark. From the above results, we can state that the proposed method for estimating the model parameters seems robust, in the statistical sense of the term. However, in order to make use of the model to forecast future flows, it is first necessary to estimate the parameters from data over a sufficiently large number of years, and then to use the estimated parameters to make the forecasts. The model could also be used to simulate flows over a long period, provided that the parameters are first estimated in this way.

5. Concluding remarks

In this note, we have seen how a jump-diffusion process could be used to model the fluctuations of river flows. This type of stochastic process is very popular in mathematical finance for modelling random phenomena that are quite similar to river flows. In our model, there are four parameters that must be estimated. By proceeding as explained in Section 3, we do not have to define rather subjectively what really constitutes a *jump* in the flow values. In addition, it eliminates the problem of deciding whether small upward variations constitute a new jump, or should be discarded.

As a follow-up to the work presented in this note, once the model parameters have been estimated, we could use the jump-diffusion process to attempt to forecast daily flow values, but also important hydrological quantities such as return periods. Moreover, we could measure the effects on the flow values of any changes in the model parameters. In particular, by increasing the value of $\hat{\lambda}$ and/or decreasing that of $\hat{\alpha}$, we can see what would happen if the rate at which hydrological events occur increases significantly, due to climate change, and/or the corresponding jumps in flow values also increase significantly.

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