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FIRST STEPS WITH BOTT FORMULA

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ABSTRACT. We use Bott's residue formula to infer that a general cubic surface in the projective space $\mathbb{P}^3_{\mathbb{C}}$ contains exactly twenty seven lines and that a general complete intersection of two quadrics in $\mathbb{P}^4_{\mathbb{C}}$ contains exactly sixteen lines.

1. Introduction

Bott formula allows one to compute the degree of certain zero cycles on a smooth complete variety X with an action of an algebraic torus in terms of local contributions supported on the components of the fixpoint set. Such a cycle should be a polynomial of Chern classes of linearized vector bundles with respect to the torus action. One advantage, if one can apply the formula, is that one may not need to know the intersetion theory of the variety X. The disadvantage could be that calculations may require the help of a computer. This note contains two explicit calculations showing how to use Bott formula (see Sections **4** and **6**). The original formula appeared first in the paper authored by Bott (1967). Atiyah and Bott (1984) generalised the formula expressing it in the language of equivariant cohomology. The formula used here (Theorem **3.1**) is the version published by Ellingsrud and Strømme (1996) which is an interpretation due to Carrell and Lieberman (1973, 1977) of that given by Atiyah and Bott (1984) and relies on the fact that, given an action of $\mathbb{C}^*$ on a variety X, one obtains a vector field s whose zero set is the fixed point set of the action. There are several works in the literature that use Bott formula and the reader interested in more elaborated situations should start having a look at Ellingsrud and Strømme (1996), where more situations with finite fixed point sets are considered, and also to the paper by Kontsevich (1995) where another version of the formula is used in a situation where the fixed point set has components of positive dimension.

2. The scheme of zeros of a global section

Let $\mathscr{E}$ be a rank r vector bundle over a smooth complete variety X over $\mathbb{C}$ of dimension n with Chern classes $c_1(\mathscr{E}),\ldots,c_r(\mathscr{E})$. Given a global section $s \in \Gamma(X,\mathscr{E})$ we have a homomorphism $s : \mathscr{O}_X \to \mathscr{E}$ sending 1 to s. The zeros scheme of s is defined to be the closed subscheme Y of X corresponding to the exact sequence

$$\mathscr{E}^\vee \xrightarrow{s^\vee} \mathscr{O}_X \longrightarrow \mathscr{O}_Y \to 0.$$

By the way we recall that when $\mathscr{E} = T_X = \Omega_X^\vee$ is the tangent sheaf of X one calls s a vector field on X. In terms of a local trivialization of $\mathscr{E}$ one can represent the section s as a vector of regular functions, say $s = (s_1, \dots, s_r)$, which locally generate the ideal sheaf of Y. This representation determines a Koszul complex $K_\bullet(\mathscr{E}, s)$ of sheaves on X:

$$0 \to \wedge^r \mathscr{E}^\vee \to \wedge^{r-1}\mathscr{E}^\vee \to \cdots \to \wedge^2 \mathscr{E}^\vee \to \mathscr{E}^\vee \xrightarrow{s^\vee} \mathscr{O}_X \longrightarrow \mathscr{O}_Y \to 0. \tag{1}$$

We also denote by Y the cycle associated to Y. If Y has codimension r then we have $c_r(\mathscr{E}) = Y$ in the Chow group $A^r(X)$ and the Koszul complex (1) is globally exact.

Theorem 2.1. *Let X be a complex smooth proyective variety that admits a vector field s whose zero scheme Y is nonempty and of codimension $r = \dim X$. Then X is rational and if $\mid p - q \mid > 0$ we have*

$$\dim_{\mathbb{C}} H^q(X, \Omega_X^p) = 0.$$

Proof. (see Carrell and Lieberman 1973, Theorem 1). ☐

As we have already mentioned at the introduction, given an action of $\mathbb{C}^*$ on X one obtains a vector field s whose zero set Y is the fixed point set of the action. If $\dim Y = 0$ the corresponding Koszul complex $K_\bullet(T_X, s)$ is then a locally free resolution of $\mathscr{O}_Y$.
For $i \geq 0$ let B_i be the cockernel of the Koszul map

$$\Omega_X^{i+1} \to \Omega_X^i.$$

In view of Theorem **2.1** one has that there are exact sequences for all i:

$$0 \to H^i(X, \Omega_X^i) \xrightarrow{p_i} H^i(X, B_i) \xrightarrow{r_i} H^{i+1}(X, B_{i+1}) \to 0. \tag{2}$$

There are natural maps

$$q_i = r_{i-1} \circ \cdots \circ r_0 : H^0(Y, \mathscr{O}_Y) \to H^i(X, B_i).$$

3. Representation of cohomology classes by functions

Recall that given a subvariety $V \subset X$ of codimension i one associates a cohomology class $\eta(V) \in H^i(X, \Omega_X^i)$ (see Hartshorne 1977, Chapter III, Ex. 7.4). Given a function $f : Y \to \mathbb{C}$ and a non-zero cohomology class $c \in H^i(X, \Omega_X^i)$ it is said that *f represents c* if

$$q_i(f) = p_i(c).$$

There is a filtration of the ring of complex-valued functions on Y

$$0 = A_{-1} \subseteq \mathbb{C} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r = H^0(Y, \mathscr{O}_Y),$$

where $A_i = \ker q_{i+1}$. One has that

$$A_i A_j \subseteq A_{i+j}$$

and the associated graded ring

$$\bigoplus_i A_i / A_{i-1}$$

is naturally isomorphic to the cohomology ring (see Carrell and Lieberman 1977, Theorem 2.6)

$$H^*(X,\mathbb{C}) \cong \bigoplus_i H^i(X,\Omega^i_X). \tag{3}$$

This gives a representation of the cohomology clases by functions on the fixed point set. The representation is unique upto addition of functions coming from cohomology classes of lower degree. This representation provides a way to evaluate zero-cycles if we know how to describe a function representing a given class and if we have an explicit formula to for the composition

$$\varepsilon_X : H^0(Y,\mathscr{O}_Y) \xrightarrow{q_n} H^n(X,\Omega^n_X) \underset{\simeq}{\xrightarrow{res_X}} \mathbb{C},$$

where $n = \dim(X)$, res_X corresponds to the trace map in the paper by Hartshorne (1966) and $res_X(\eta(V) = \deg V)$ for a 0-cycle $V \subset X$. Theorem **3.1** below deals with representation and evaluation issues. Let $\mathscr{E}$ be a $\mathbb{C}^*$-linearized vector bundle of rank r on X. For each fixed point $x \in Y$ the fibre of $\mathscr{E}$ splits as a direct sum of one-dimensional representations of $\mathbb{C}^*$; let $\tau_1(\mathscr{E},x),\ldots,\tau_r(\mathscr{E},x)$ denote the corresponding weights, and for all integers $k \geq 0$, let $\sigma_k(\mathscr{E},x) \in \mathbb{Z}$ be the k-th elementary symmetric function in the $\tau_i(\mathscr{E},x)$.

Theorem 3.1 (Bott Formula)**.** *Let $\mathscr{E}$ and the notation be as above. Then*

1. *The k-th Chern class $c_k(\mathscr{E}) \in H^k(X,\Omega^k_X)$ can be represented by the function $f : Y \to \mathbb{C}$ given by $f(x) = \sigma_k(\mathscr{E},x)$.*
2. *Given a function $f : Y \to \mathbb{C}$ one has*

$$\varepsilon_X(f) = \sum_{x\in Y} \frac{f(x)}{\sigma_n(T_X,x)}$$

4. Twenty seven lines through a cubic surface

Consider a cubic hypersurface $Q \subset \mathbb{P}^3$, say $Q = V(F)$ with F a cubic form in $\mathbb{C}[x_0,\ldots,x_3]$. Let $Q_{\mathbb{L}}$ be the set of lines in $\mathbb{P}^3$ contained in Q. The cardinality of $Q_{\mathbb{L}}$ is related to the degree of certain 0-cycle on the Grasmannian of lines in $\mathbb{P}^3$ as stated in Lemma **4.1**. We shall see how to use Theorem **3.1** to compute the degree of that cycle. Before that we need to introduce some background and the elements required to apply the theorem.

Let W be a 4-dimensional complex vector space and consider the projective space $\mathbb{P}^3 = \mathbb{P}(W)$ and the Grassmanian $\mathbb{G}(2,W)$ of lines in $\mathbb{P}^3$. Let $S \longrightarrow \mathbb{G}(2,W)$ be the tautological bundle of $\mathbb{G}(2,W)$, then we have a universal line

$$\mathbb{L} = \mathbb{P}(S) \subset \mathbb{G}(2,W) \times \mathbb{P}^3.$$

This means that if $L \subset \mathbb{P}^3$ is a line and we represent by L the corresponding point in $\mathbb{G}(2,W)$ we have that

$$\mathbb{L} \cap \Big\{ \{L\} \times \mathbb{P}^3 \Big\} = \{L\} \times L \cong L.$$

Consider the sheaf

$$\mathscr{E} = p_{1*}(\mathscr{O}_{\mathbb{L}} \otimes p_2^*\mathscr{O}_{\mathbb{P}^3}(3)) \tag{4}$$

where p_1 and p_2 are the two projections of $\mathbb{G}(2,W) \times \mathbb{P}^3$. Write $\mathscr{F}$ to denote $\mathscr{O}_{\mathbb{L}} \otimes p_2^*\mathscr{O}_{\mathbb{P}^3}(3)$ (of course, there is some abuse of notation, specifically we mean the sheaf on $\mathbb{G}(2,W) \times \mathbb{P}^3$ given by $j_*(j^*(p_2^*\mathscr{O}_{\mathbb{P}^3}(3)))$ where j stands for the inclusion $\mathbb{L} \subset \mathbb{G}(2,W) \times \mathbb{P}^3$). Next we see that $\mathscr{E}$ is a locally free sheaf of rank 4. Given a point $L \in \mathbb{G}(2,W)$ consider the fibre $(\mathbb{G}(2,W)\times\mathbb{P}^3)_L$ under p_1 and the sheaf

$$\mathscr{F}_L := \mathscr{F}\mid_{(\mathbb{G}(2,W)\times\mathbb{P}^3)_L}.$$

Then $\mathscr{F}_L$ can be seen as a sheaf on $\{L\}\times L \cong L$ and since L is a line on $\mathbb{P}^3$ we can identify it with $\mathscr{O}_{\mathbb{P}^1}(3)$. Therefore

$$\dim H^0((\mathbb{G}(2,W)\times\mathbb{P}^3)_L,\mathscr{F}_L) = \dim H^0(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(3)) = 4 \tag{5}$$

for any $L \in \mathbb{G}(2,W)$. It follows from standard Cohomology and base change theory (Mumford 1970, Corollary 2, p. 50) that $\mathscr{E}$ is a locally free sheaf of rank 4.

Lemma 4.1. *If $Q_{\mathbb{L}}$ is finite then we have that $\#Q_{\mathbb{L}} \leq \deg c_4(\mathscr{E})$. If Q is a general cubic, then $\#Q_{\mathbb{L}} = \deg c_4(\mathscr{E})$.*

Proof. First one should notice that a cubic $Q \subset \mathbb{P}^3$ induces a non-zero section in $H^0(\mathbb{G}(2,W),\mathscr{E})$. From (4) one can see that

$$H^0(\mathbb{G}(2,W)\times\mathbb{P}^3,\mathscr{O}_{\mathbb{L}}\otimes p_2^*\mathscr{O}_{\mathbb{P}^3}(3)) = H^0(\mathbb{G}(2,W),\mathscr{E}).$$

Also, there is a natural homomorphism

$$H^0(\mathbb{G}(2,W)\times\mathbb{P}^3,p_2^*\mathscr{O}_{\mathbb{P}^3}(3)) \longrightarrow H^0(\mathbb{G}(2,W)\times\mathbb{P}^3,\mathscr{O}_{\mathbb{L}}\otimes p_2^*\mathscr{O}_{\mathbb{P}^3}(3))$$

given by restriction of sections, that is, $\sigma \longrightarrow \sigma\mid_{\mathbb{L}}$. Notice also that

$$H^0(\mathbb{G}(2,W)\times\mathbb{P}^3,p_2^*\mathscr{O}_{\mathbb{P}^3}(3)) = H^0(\mathbb{P}^3,p_{2*}p_2^*\mathscr{O}_{\mathbb{P}^3}(3)) = H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(3)).$$

Then, since a non-zero cubic σ cannot vanish on all lines in $\mathbb{P}^3$ we have in fact an injective map

$$H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(3)) \longrightarrow H^0(\mathbb{G}(2,W),\mathscr{E}).$$

Finally, given a cubic form $F \in H^0(\mathbb{P}^3,\mathscr{O}_{\mathbb{P}^3}(3))$ the set of zeros of the section $F_{|\mathbb{L}}$ of $\mathscr{E}$ induced by F corresponds to the set $Q_{\mathbb{L}}$ of lines in $\mathbb{P}^3$ on which F vanish. It follows from section **2** that if $Q_{\mathbb{L}}$ is finite then we have that as schemes $Q_{\mathbb{L}} = c_4(\mathscr{E})$ in the Chow ring of $\mathbb{G}(2,W)$ because $\dim(\mathbb{G}(2,W)) = 4$. Since a point in $Q_{\mathbb{L}}$ could have multiplicity ≥ 1 in $c_4(\mathscr{E})$ one has $\#Q_{\mathbb{L}} \leq \deg c_4(\mathscr{E})$. By the Kleiman's Bertini Theorem (Kleiman 1974) on has that for a general cubic Q the zero scheme $Q_{\mathbb{L}}$ is finite and smooth and then $\#Q_{\mathbb{L}} = \deg c_4(\mathscr{E})$. □

Let W have coordinates $(x_0,\ldots,x_3)$. Define an action of $\mathbb{C}^*$ on W given by $t\cdot(x_0,\ldots,x_3) = (t^{a_0}x_0,\ldots,t^{a_3}x_3)$. This induces an action on $\mathbb{G}(2,W)$ which have six fixed points(if $a_i \neq a_j$ for $i \neq j$), namely, the lines through every pair of canonical points $\bar{e}_i = (\underbrace{0:\cdots:1}_{i}:\cdots:0) \in \mathbb{P}^3$. Let $L_{i,j}$ the line through $\bar{e}_i$ and $\bar{e}_j$. Then tangent space to $\mathbb{G}(2,W)$ at $L_{i,j}$ is given by

$$T_{\mathbb{G}(2,W),L_{i,j}} = Hom(S_{i,j},Q_{i,j}) = S_{i,j}^{\vee}\otimes Q_{i,j} \tag{6}$$

where $S_{i,j} \subset W$ is subspace generated by $\{e_i,e_j\}$ and $Q_{i,j} = W/S_{i,j}$. Then we have splittings $S_{i,j} \cong V(a_i)\oplus V(a_j)$ and $Q_{i,j} \cong V(k)\oplus V(l)$, where the action of $\mathbb{C}^*$ on $V(i)\cong\mathbb{C}$ is multiplication by t^i. So

$$T_{\mathbb{G}(2,W),L_{i,j}} \cong V(k-a_i)\oplus V(k-a_j)\oplus V(l-a_i)\oplus V(l-a_j) \tag{7}$$

and we use the vector of weights

$$(\tau_1(T_{\mathbb{G}(2,W)}, L_{i,j}), \ldots, \tau_4(T_{\mathbb{G}(2,W)}, L_{i,j})) = (k - a_i, k - a_j, l - a_i, l - a_j)$$

to compute

$$\sigma_4(T_{\mathbb{G}(2,W)}, L_{i,j}) = (k - a_i)(k - a_j)(l - a_i)(l - a_j). \tag{8}$$

We want to represent $c_4(\mathscr{E})$ by a function so we have also to determine the weights of the action on the fibres of $\mathscr{E}$ at the fixed points. With the notation as in (5) we have, following Mumford (1970, Corollary 2, (ii), p. 50), that the fibre at $L_{i,j}$ is given by

$$\mathscr{E}_{L_{i,j}} = H^0((\mathbb{G}(2,W) \times \mathbb{P}^3)_{L_{i,j}}, \mathscr{F}_{L_{i,j}}) = H^0(L_{i,j}, \mathscr{O}_{L_{i,j}}(3)) = Sym^3(S_{i,j}^\vee). \tag{9}$$

Therefore we have

$$Sym^3(S_{i,j}^\vee) \cong \bigoplus_{\substack{p,q \geq 0, \\ p+q=3}} V(-pa_i - qa_j), \tag{10}$$

and by Theorem **3.1** we have that $c_4(\mathscr{E})$ is represented by the function f

$$L_{i,j} \longrightarrow \sigma_4(\mathscr{E}, L_{i,j}) = \prod_{\substack{p,q \geq 0, \\ p+q=3}} (-pa_i - qa_j). \tag{11}$$

Let $f_{i,j}$ be the contribution of $L_{i,j}$ to $\varepsilon_X(f)$, that is,

$$f_{i,j} = \frac{f(L_{i,j})}{\sigma_4(T_{\mathbb{G}(2,W)}, L_{i,j})}.$$

So from (8) and (11) $f_{i,j} = c(a_i, a_j, k, l)$ where k, l depend on i, j and

$$c(a_i, a_j, k, l) = \frac{9a_i a_j \, (a_j + 2a_i) \, (2a_j + a_i)}{(k - a_i) \, (k - a_j) \, (l - a_i) \, (l - a_j)}.$$

When $(i, j) = (0, 1)$ one can set $(k, l) = (a_2, a_3)$ so

$$f_{0,1} = c(a_0, a_1, a_2, a_3) = \frac{9a_0 a_1 \, (a_1 + 2a_0) \, (2a_1 + a_0)}{(a_2 - a_0) \, (a_2 - a_1) \, (a_3 - a_0) \, (a_3 - a_1)}.$$

Similarly, one has for the other fixed points

$$f_{0,2} = c(a_0, a_2, a_1, a_3) = \frac{9a_0 a_2 \, (a_2 + 2a_0) \, (2a_2 + a_0)}{(a_1 - a_0) \, (a_1 - a_2) \, (a_3 - a_0) \, (a_3 - a_2)}.$$

$$f_{0,3} = c(a_0, a_3, a_1, a_2) = \frac{9a_0 a_3 \, (a_3 + 2a_0) \, (2a_3 + a_0)}{(a_1 - a_0) \, (a_2 - a_0) \, (a_1 - a_3) \, (a_2 - a_3)}.$$

$$f_{1,2} = c(a_1, a_2, a_0, a_3) = \frac{9a_1 a_2 \, (a_2 + 2a_1) \, (2a_2 + a_1)}{(a_0 - a_1) \, (a_0 - a_2) \, (a_3 - a_1) \, (a_3 - a_2)}.$$

$$f_{1,3} = c(a_1, a_3, a_0, a_2) = \frac{9a_1 a_3 \, (a_3 + 2a_1) \, (2a_3 + a_1)}{(a_0 - a_1) \, (a_2 - a_1) \, (a_0 - a_3) \, (a_2 - a_3)}.$$

$$f_{2,3} = c(a_2, a_3, a_0, a_1) = \frac{9a_2 a_3 \, (a_3 + 2a_2) \, (2a_3 + a_2)}{(a_0 - a_2) \, (a_1 - a_2) \, (a_0 - a_3) \, (a_1 - a_3)}.$$

Now one can verify that

$$\varepsilon(f) = f_{0,1} + f_{0,2} + f_{0,3} + f_{1,2} + f_{1,3} + f_{2,3} = 3^3.$$

5. The case of k-planes in projective varieties

Consider now a subvariety $Y \subset \mathbb{P}^n$ where Y is a complete intersection of hypersurfaces $V(F_{d_1}),\dots,V(F_{d_s}) \subset \mathbb{P}^n$ of degrees $d_1,\dots,d_s$. Let $Y_{\mathbb{K}}$ be the set of k-planes in $\mathbb{P}^n$ that are contained in Y. Assume $\mathbb{P}^n = \mathbb{P}(W)$. We now take $S \longrightarrow \mathbb{G}(k+1,W)$ to be the tautological bundle of $\mathbb{G}(k+1,W)$ and consider the universal k-plane

$$\mathbb{K} = \mathbb{P}(S) \subset \mathbb{G}(k+1,W) \times \mathbb{P}^n.$$

Now, the vector bundle

$$\mathscr{E}_d = p_{1*}(\mathscr{O}_{\mathbb{K}} \otimes p_2^* \mathscr{O}_{\mathbb{P}^n}(d)) \tag{12}$$

has $rank(\mathscr{E}_d) = \binom{k+d}{d}$. Notice that $(k+1)(n-k) = \dim \mathbb{G}(k+1,W)$. As in the proof of Lemma **4.1** taking the section induced by $(F_{d_1},\dots,F_{d_s})$ one has the following

Proposition 5.1. *Let $Y \subset \mathbb{P}^n$ be the complete intersection of p general hypersurfaces in $\mathbb{P}^n$ of degrees $d_1,\dots,d_p$ respectively. Assume that $\sum_i \binom{k+d_i}{d_i} = (k+1)(n-k)$. Then the number of k-planes contained in Y is finite and equals*

$$\deg(c_N(\mathscr{E}))$$

where $N = (k+1)(n-k)$ and

$$\mathscr{E} = \bigoplus_i \mathscr{E}_{d_i}.$$

6. Sixteen lines through the intersection of 2 quadrics

Now we can consider the case of a surface that is the complete intersection of 2 general quadrics in $\mathbb{P}^4 = \mathbb{P}(W)$. Similarly to section **4** one takes an action of $\mathbb{C}^*$ on W given by $t \cdot (x_0,\dots,x_4) = (t^{a_0}x_0,\dots,t^{a_4}x_4)$ with $a_i \neq a_j$ for $i \neq j$. For a fixed point $L_{i,j} \in \mathbb{G}(2,W)$ one has $S_{i,j} \cong V(a_i) \oplus V(a_j)$ and $Q_{i,j} \cong V(k) \oplus V(l) \oplus V(m)$ so that

$$T_{\mathbb{G}(2,W),L_{i,j}} \cong V(k-a_i) \oplus V(k-a_j) \oplus V(l-a_i) \oplus V(l-a_j) \oplus V(m-a_i) \oplus V(m-a_j) \tag{13}$$

and then

$$\sigma_6(T_{\mathbb{G}(2,W)}, L_{i,j}) = (k-a_i)(k-a_j)(l-a_i)(l-a_j)(m-a_i)(m-a_j). \tag{14}$$

Here we take

$$\mathscr{E} = \mathscr{E}_2 \oplus \mathscr{E}_2.$$

The fibre of $\mathscr{E}_2$ at $L_{i,j}$ is given by

$$(\mathscr{E}_2)_{L_{i,j}} = H^0(L_{i,j}, \mathscr{O}_{L_{i,j}}(2)) = Sym^2(S_{i,j}^{\vee}) \cong \bigoplus_{\substack{p,q \geq 0,\\ p+q=2}} V(-pa_i - qa_j). \tag{15}$$

And then we have that $c_6(\mathscr{E})$ is represented by the function f

$$L_{i,j} \longrightarrow \sigma_6(\mathscr{E}, L_{i,j}) = \prod_{\substack{p,q \geq 0,\\ p+q=2}} (-pa_i - qa_j)^2. \tag{16}$$

From (14) and (16) the contribution $f_{i,j}$ of $L_{i,j}$ to $\varepsilon_X(f)$ is of the form

$$
\begin{aligned}
f_{i,j} &= \mathrm{c}\left(a_i, a_j, k, l, m\right) \\
&:= \frac{\left(2a_i\right)^2 \left(a_i + a_j\right)^2 \left(2a_j\right)^2}{\left(k - a_i\right)\left(k - a_j\right)\left(l - a_i\right)\left(l - a_j\right)\left(m - a_i\right)\left(m - a_j\right)}.
\end{aligned}
$$

For $f_{i,j}$, $0 \leq i < j \leq 4$ one has

$$
\begin{aligned}
f_{0,1} &= c(a_0, a_1, a_2, a_3, a_4) \\
&= \frac{16 a_0^2 a_1^2 \left(a_1 + a_0\right)^2}{\left(a_2 - a_0\right)\left(a_2 - a_1\right)\left(a_3 - a_0\right)\left(a_3 - a_1\right)\left(a_4 - a_0\right)\left(a_4 - a_1\right)} \\
&\vdots \\
f_{3,4} &= c(a_3, a_4, a_2, a_0, a_1) \\
&= \frac{16 a_3^2 a_4^2 \left(a_4 + a_3\right)^2}{\left(a_0 - a_3\right)\left(a_1 - a_3\right)\left(a_2 - a_3\right)\left(a_0 - a_4\right)\left(a_1 - a_4\right)\left(a_2 - a_4\right)}
\end{aligned}
$$

and one can verify that

$$
\varepsilon(f) = \sum_{0 \leq i < j \leq 4} f_{i,j} = 2^4.
$$

The same was obtained by Griffiths and Harris (1994, p. 199) using Schubert calculus (see also Griffiths and Harris 1994, Chap. 4, Sec. 4).

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