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OPTIMALITY CONDITIONS AND LAGRANGE MULTIPLIERS FOR SHAPE AND TOPOLOGY OPTIMIZATION PROBLEMS

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ABSTRACT. We discuss first order optimality conditions for geometric optimization problems with Neumann boundary conditions and boundary observation. The methods we develop here are applicable to large classes of state systems or cost functionals. Our approach is based on the implicit parametrization theorem and the use of Hamiltonian systems. It establishes equivalence with a constrained optimal control problem and uses Lagrange multipliers under a simple constraint qualification. In this setting, general functional variations are performed, that combine classical topological and boundary variations, in a natural way.

1. Introduction

Let $\mathscr{O}$ be a family of subdomains in $D \subset R^2$, a given bounded domain. We discuss the first order necessary optimality conditions for the following optimal design problem:

$$\underset{\Omega \in \mathscr{O}}{\mathrm{Min}} \int_{\partial\Omega} j(\sigma, y_\Omega(\sigma))d\sigma, \tag{1}$$

subject to

$$-\Delta y_\Omega + y_\Omega = f \quad \text{in } \Omega, \tag{2}$$

$$\frac{\partial y_\Omega}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{3}$$

Here, $f \in L^2(D)$ is given and $j(\cdot,\cdot)$ is a Carathéodory mapping . More assumptions will be imposed as needed. We work in dimension two since the Poincaré-Bendixson theorem (Pontriaguine 1968; Hirsch, Smale, and Devaney 2004) is employed in our periodicity argument for the associated Hamiltonian systems appearing in the description of the unknown geometry. Dimension two is an important case in optimal design. Except this (key) detail, the proofs are valid in arbitrary dimension. Moreover, they can be easily extended to other boundary conditions, elliptic operators, cost functionals (see Murea and Tiba (2022a,c), where computational approaches are discussed), and our choice for the case (1)-(3) is motivated by its intrinsic difficulty.

The literature on topology optimization is very rich and we quote just the monographs by Bendsøe and Sigmund (2004), Allaire (2007), and Novotny and Sokołowski (2013) and references therein, devoted to this subject. There are many approaches: the topological derivative based on the asymptotic analysis of shape functionals in domains with holes, the SIMP method which uses a relaxation procedure, the homogenization approach, level set techniques. The question of the necessary optimality conditions is a central point in optimization theory and they are essential for the indirect methods. In shape/topology optimization, comprehensive investigations are reported in this direction, for instance, by Sokołowski and Zolesio (1992) and Novotny and Sokołowski (2013), but our methodology is different and combines both aspects.

In this paper, we also use level set functions, introduced by Osher and Sethian (1988) in the setting of free boundary problems (in optimal design a first application was apparently given by Mäkinen, Neittaanmäki, and Tiba (1992)). However, no Hamilton-Jacobi equation is needed here and no "evolution" of the level sets is taken into account. Instead, we employ just simple ordinary Hamiltonian differential equations and the implicit parametrization theorem, (Tiba 2018b). As it is natural in optimization problems, we define a family of variations, called functional variations of the geometry. We refer to Neittaanmäki, Pennanen, and Tiba (2009), Neittaanmäki and Tiba (2012), and Tiba (2018b) where such ideas were introduced. The generality of this concept and its relationship with classical boundary or topological variations is briefly discussed in Example 3.1. Applications to numerical approaches in optimal design can be found in the papers by Tiba (2018a) and Murea and Tiba (2019, 2022a).

This paper uses as well recent developments achieved by Murea and Tiba (2022c), for instance the differentiability of the period for ordinary differential Hamiltonian systems, with respect to functional variations. The key point in this article is the application of a Lagrange multipliers rule, while in the paper by Murea and Tiba (2022c), the cost penalization plays an essential role. We also quote Amstutz and Ciligot-Travain (2010), where Lagrange multipliers are obtained under the presence of supplementary constraints, by using interiority conditions. Moreover, we are not employing topological asymptotic properties for elliptic equations as done by Hassine, Jan, and Masmoudi (2007) and Novotny and Sokołowski (2013), and we use another technique (Neittaanmäki and Tiba 2012; Murea and Tiba 2022c) that allows the application of gradient algorithms (Murea and Tiba 2019, 2022a). In this framework, the equivalence of shape/topology optimization problems with certain optimal control problems involving mixed constraints of a special form, is obtained. The relationship of optimal design problems with optimal control problems is a natural question and the pioneering mapping method of Murat and Simon (1976) offered a first equivalence result (without allowing topology changes). Our technique may be as well viewed as a fixed domain approach, but here we have even an equivalence property, not just approximation properties as usual in the literature (Neittaanmäki and Tiba 2012; Novotny and Sokołowski 2013).

Another important point is that our methodology, using functional variations, provides at the computational level simultaneous topological and boundary variations of the unknown geometry. They are not prescribed, but automatically chosen by the algorithm (Murea and Tiba 2022b,c). As a fixed domain approach it also has clear implementation advantages: the mesh, the mass matrix need not to be updated during the iterations. Such questions are discussed as well, for instance by Novotny and Sokołowski (2013), for approximation procedures via different techniques.

It is here that we investigate, for the first time, the impact of general functional variations (Neittaanmäki, Pennanen, and Tiba 2009; Neittaanmäki and Tiba 2012), in the derivation of the necessary optimality conditions in optimal design problems. These variations are richer with respect to classical boundary or topological variations, see Example 3.1. The obtained necessary conditions have a purely analytic character, that is the geometry is absent from their formulation. However, certain terms (but not all) admit a geometric interpretation, see (43), (44). The form of the optimality system is unexpected since both Lagrange multipliers and Fermat type conditions are involved, even in the case of unconstrained geometric optimization problems. A simple "constraint qualification" condition is introduced, (34). We also underline the Fréchet differentiability of the operators that we use and the fact that such smooth operations can even generate or close holes, see Rem. 3.1 and its example.

The plan of the paper is as follows. In the next section, we discuss some preliminaries and the optimal control formulation. In Sect. 3, we obtain abstract optimality conditions via a Lagrange multiplier rule, under a simple constraint qualification condition. These optimality conditions are detailed in the last section. Among the advantages of our methodology, we mention its purely analytic character, the generality of the employed variations and the intimate relationship with optimal control theory. We also underline its applicability at the computational level (Neittaanmäki and Tiba 2012; Murea and Tiba 2022b,c). As drawbacks, we indicate the dimension two setting and certain regularity hypotheses, both due to the application of the Poincaré-Bendixson theory.

2. Preliminaries and equivalence

Let $g \in C(\overline{D})$ be some given function. To it, we associate the open set $\Omega_g \subset D$:

$$\Omega_g = \text{int}\{x \in D;\ g(x) \le 0\} \tag{4}$$

(assumed non void), that may be not connected and its components may be not simply connected. Notice as well that the set

$$G = \{x \in D;\ g(x) = 0\} \tag{5}$$

may be of positive Lebesgue measure, in this very general framework. In order to select some connected (not necessarily simply connected) component of Ω_g, we fix a point $\bar{x} \in D$ and define the *domain* Ω_g to be the component that contains $\bar{x}$ in its closure (it may be void). If it is nonvoid, then $g \in C(\overline{D})$ satisfies

$$g(\bar{x}) \le 0. \tag{6}$$

In fact, in many examples of shape optimization problems, the following supplementary geometric constraint is imposed on the admissible domains Ω_g:

$$E \subset \overline{\Omega}_g \Rightarrow g(x) \le 0 \text{ on } E,$$

where $E \subset D$ is some given set. Therefore, the condition (6) and the definition of the domain Ω_g are natural. In the sequel, Ω_g is always the connected component of the open set defined in (4), satisfying $\bar{x} \in \overline{\Omega}_g$ and (6). It is obvious that more regularity of Ω_g is necessary in order that (1)-(3) make sense. We assume that $g \in C^1(\overline{D})$ and (see (5)):

$$|\nabla g(x)| > 0, \ \forall x \in G. \tag{7}$$

We also impose the condition

$$g(x) > 0, \ \forall x \in \partial D \tag{8}$$

which ensures that $\partial\Omega_g \cap \partial D = \emptyset$. We denote by $\mathscr{F} \subset C^1(\overline{D})$ the family of functions satisfying (6), (7), (8). It is a cone in $C^1(\overline{D})$. By the implicit functions theorem, we get that $\partial\Omega_g$ is of class C^1 and

$$\Omega_g = \{x \in D, \ g(x) < 0\}, \tag{9}$$

$\partial\Omega_g = G$ and has null Lebesgue measure. Relation (9) assumes that $g > 0$ outside the domain Ω_g, that can be obtained by adding to g the squared distance function to Ω_g multiplied by a convenient constant. Obviously, an infinity of $g \in \mathscr{F}$ yield the same Ω_g.

If more regularity is imposed on $\mathscr{F}$, then $\partial\Omega_g$ becomes more regular. In general, ∂D is assumed to have the same regularity. The family $\mathscr{O}$ of all admissible domains for the minimization problem (1)-(3) is generated by the above procedure starting from the level functions $g \in \mathscr{F}$. These domains are not necessarily simply connected and that's why our approach allows topology optimization in combination with shape optimization. We also underline that $\mathscr{O}$ is a rich family of admissible domains, that is the problem (1)-(3) is meaningful (see Delfour and Zolesio 2011, Ch.2).

Denote by $V_\varepsilon = \{x \in D; \ d(x,G) < \varepsilon\} \subset D$ an ε-neighbourhood of G; and $G_\lambda = \{x \in D, \ (g+\lambda h)(x) = 0\}$, where $\varepsilon > 0$, $h \in \mathscr{F}$, $\lambda \in R$.

Proposition 2.1. *Let $\mathscr{F} \subset C(\overline{D})$. There is $\lambda(\varepsilon) > 0$ such that, for $\lambda \in R$, $|\lambda| < \lambda(\varepsilon)$, we have $G_\lambda \subset V_\varepsilon$.*

Notice that G_λ may be not connected (as a compact subset). Under the subsequent hypotheses (13) and (23), $\bar{x} \in G$ and G_λ has a component that passes through $\bar{x}$ that may be considered the boundary variation of G. The other components of G_λ may generate topological perturbations. As usual, $\Omega_{g+\lambda h}$ is the connected component satisfying (6). In general, it may not be diffeomorphic with Ω_g. Some components of G_λ may be outside $\Omega_{g+\lambda h}$ and play no role.

Proposition 2.2. *Let $\mathscr{F} \subset C^2(\overline{D})$. Under conditions (6), (7), (8), G is a finite union of closed curves, without self intersections and disjoint from ∂D, globally parametrized by the solution of the Hamiltonian system:*

$$x_1'(t) = -\partial_2 g(x_1(t), x_2(t)), \ t \in I, \tag{10}$$

$$x_2'(t) = \partial_1 g(x_1(t), x_2(t)), \ t \in I, \tag{11}$$

$$x(0) = (x_1(0), x_2(0)) = x_0, \tag{12}$$

where x_0 is a point on each component of G and I is an interval around the origin, depending on the respective component. Moreover, the solution of (10)-(12) is unique.

Remark 2.1. *Prop. 2.1 and Prop. 2.2 are very similar to results proved by Tiba (2018a,b). Tiba (2018b) also reports on a partial (local) extension to arbitrary dimension and the uniqueness result. In particular, $\mathscr{F}$ is a cone and, for any $g, r \in \mathscr{F}$, we get $g + \lambda r \in \mathscr{F}$ for λ small, under the conditions of Prop. 2.2 with (6) replaced by: there is $\bar{x} \in D$ such that*

$$g(\bar{x}) = 0, \ \forall g \in \mathscr{F}. \tag{13}$$

The trajectories of the Hamiltonian system (10)-(12) are closed, that is periodic and I may be choosen $I = [0, T_g]$, the main period interval, depending on each component of G (Tiba 2018a). Condition (7) gives the Poincaré-Bendixson hypothesis (the absence of equilibrium points on the trajectories) for this system. The Hamiltonian structure has, in fact, the property not to allow the presence of a limit cycle and the solutions are periodic (see Pontriaguine 1968, Ch. 5, Sect. 28).

It turns out that the shape optimization problem (1)-(3) is equivalent with the following constrained optimal control problem, defined in D:

$$\underset{g,u}{\operatorname{Min}} \int_0^{T_g} j(z_g(t), y_g(z_g(t)))|z_g'(t)|dt \tag{14}$$

$$-\Delta y_g + y_g = f + g_+^2 u \quad \text{in } D, \tag{15}$$

$$y_g = 0 \quad \text{on } \partial D, \tag{16}$$

$$\int_0^{T_g} |\nabla y_g(z_g(t)) \cdot \nabla g(z_g(t))|^2 dt = 0. \tag{17}$$

In (14)-(17), $z_g(t) = (z_g^1(t), z_g^2(t))$ denotes the unique solution of the Hamiltonian system (10)-(12), g, u are control functions in D, $g \in \mathscr{F}$ and u satisfies the conditions in Thm. 2.1 below. Due to (7), constraint (17) is a simplified equivalent form of (3):

$$0 = \int_{\partial\Omega_g} \left| \frac{\partial y_{\Omega_g}}{\partial n} \right|^2 d\sigma = \int_G \left| \frac{\partial y_{\Omega_g}}{\partial n} \right|^2 d\sigma = \int_0^{T_g} \frac{|\nabla y_g(z_g(t)) \cdot \nabla g(z_g(t))|^2}{|\nabla g(z_g(t))|^2} |z_g'(t)|dt. \tag{18}$$

If G has several connected components, then (18) and (17) are finite sums (see Prop. 2.2) of such integrals and an initial condition should be fixed on each component of G, for the Hamiltonian system. The same is valid for the cost index (14) which is an equivalent form of (1) and may take as well the form of a finite sum if $G = \partial\Omega_g$ has several connected components as it happens in topology optimization.

Theorem 2.1. *For any $g \in \mathscr{F} \subset C^2(\overline{D})$, $f \in L^p(D)$, $p > 2$, there is u_g measurable in D, not unique, such that $g_+^2 u_g \in L^p(D)$ and the solution of (15), (16) coincides in Ω_g with the solution of (2), (3) and satisfies (17). The associated costs (1), (14) coincide as well.*

Proof

Since $\partial\Omega_g$ is in C^2 under conditions $\mathscr{F} \subset C^2(\overline{D})$ and (7), then the unique solution y of (2), (3) satisfies $y \in W^{2,p}(\Omega_g) \subset C^1(\overline{\Omega}_g)$. Let $\widetilde{y} \in W^{2,p}(D \setminus \overline{\Omega}_g)$ be such that $\widetilde{y} = y$, $\frac{\partial \widetilde{y}}{\partial n} = 0$ on $\partial\Omega_g$ and $\widetilde{y} = 0$ on ∂D. Here $D \setminus \overline{\Omega}_g$ may not be connected. This is possible due to the trace theorem and $\widetilde{y}$ is not unique.

We denote

$$u_g = \frac{-\Delta\widetilde{y} - f}{g_+^2} \quad \text{in } D \setminus \overline{\Omega}_g$$

and 0 in Ω_g. Here, we may assume $g > 0$ in $D \setminus \overline{\Omega}_g$ without loss of generality. The concatenation of y and $\widetilde{y}$ satisfies (15), (16), (17) in D and $g_+^2 u_g \in L^p(D)$ by construction. The equality of the costs (1), (14) is obvious. $\square$

Remark 2.2. *This statement is a variant of the similar results used in other cases by Murea and Tiba (2019, 2022a,c). It proves the equivalence of the shape optimization problem (1)-(3) with the constrained optimal control problem (14)-(17) defined in D. Notice as well that, although in (3) the Neumann condition is considered, the equivalent formulation in D uses the Dirichlet condition on ∂D in (16).*

The existence of some (local) optimal domain Ω^* is assumed in this paper, as it is usual in the discussion of optimality conditions (Barbu and Precupanu 2012). For general existence results, under minimal geometric regularity assumptions (the segment property), we quote Tiba (2003) and Neittaanmäki, Sprekels, and Tiba (2006).

3. Differentiability and Lagrange multipliers

In the sequel, we shall use the so called "reduced" problem for the optimal control problem, instead of the formulation given by (14)-(17) and (10)-(12). We denote by $A : L^p(D) \to W^{2,p}(D) \cap H_0^1(D)$ the isomorphism defined by the Dirichlet problem in D and (15), (16) can be written as $y_g = A(f + g_+^2 u)$. Then, the constrained optimal control problem is of the form:

$$\operatorname*{Min}_{g,u} \int_0^{T_g} j(z_g(t), A(f + g_+^2 u)(z_g(t)))|z_g'(t)|dt, \tag{19}$$

$$\int_0^{T_g} \left|\nabla A(f + g_+^2 u)(z_g(t)) \cdot \nabla g(z_g(t))\right|^2 dt = 0. \tag{20}$$

The notations T_g, z_g are explained in §2 and relations (19), (20) define the reduced problem. Remark that the relations (19), (20) are in fact incomplete since $\partial\Omega_g$ may have several components and on each one, other T_g, z_g values are valid, that is more terms should appear in (19), (20).

To simplify the writing and without losing generality, we assume now that Ω^* (a "local optimal" domain) is double connected (it has one hole and its boundary $\partial\Omega^*$ has exactly two components). We also assume that the cost functional (19) is defined just on the "exterior" component. The constraint (20) has to be satisfied on all the components of $\partial\Omega^*$. We denote by $\bar{z}_g, \widehat{z}_g$ the solutions of the Hamiltonian system (10)-(12) corresponding to some given initial conditions $\bar{x},\hat{x}$, on these two components and the trajectory passing through $\hat{x}$ is the exterior one. Then, relations (19)-(20) become (21)-(22). In case $\partial\Omega^*$ has more components (their number is finite), the terms in relations (21)-(22) should be replaced by certain sums of similar quantities, as explained in Section 2. Moreover, the arguments developed below remain valid under such a general formulation.

$$\underset{g,u}{\mathrm{Min}} \int_0^{\widehat{T}_g} j(\widehat{z}_g(t), A(f+g_+^2 u)(\widehat{z}_g(t)))|\widehat{z}_g'(t)|dt, \tag{21}$$

$$\int_0^{\widehat{T}_g} \left|\nabla A(f+g_+^2 u)(\widehat{z}_g(t)) \cdot \nabla g(\widehat{z}_g(t))\right|^2 dt + \int_0^{\overline{T}_g} \left|\nabla A(f+g_+^2 u)(\bar{z}_g(t)) \cdot \nabla g(\bar{z}_g(t))\right|^2 dt = 0. \tag{22}$$

In (21), (22), the notations $\widehat{T}_g, \overline{T}_g$ are, respectively, the main periods associated to the two Hamiltonian systems used here. All the above considerations from this section are based on Prop. 2.1, Prop. 2.2, Thm. 2.1. In fact, we discuss about the optimal domain Ω^*, just for intuition. The problem (21), (22) has no reference to the geometry and we denote by $[g^*,u^*] \in \mathscr{F} \times L^p(D)$ some (local) optimal pair for (21) and satisfying (22). As $\mathscr{F} \subset C^1(\overline{D})$, we get $(g_+^*)^2 \in C(\overline{D})$ and $(g_+^*)^2 u^*$ is in $L^p(D)$ (the assumption on u^* is here slightly stronger than in §2).

In this analytic setting, we can define functional variations (Neittaanmäki, Pennanen, and Tiba 2009; Neittaanmäki and Tiba 2012), around $[g^*,u^*]$, $g^*+\lambda h$, $u^*+\lambda v$, where $\lambda \in R$, $h \in \mathscr{F}$ and $v \in L^p(D)$. We underline that the condition $u^* \in L^p(D)$ allows to consider variations $v \in L^p(D)$, independently of g^*, ensuring that the right-hand side in (15) is in $L^p(D)$. If $\mathscr{F}$ is given by (7), (8) and (13) (in the form (23) below), then $g^*+\lambda h \in \mathscr{F}$ for $|\lambda|$ small, due to the compactness of ∂D, G and the Weierstrass theorem. That is, the above variations are “admissible” (the question of the constraint (22) will be handled via the Lagrange multipliers).

Example 3.1. *The functional variations have a linear character. If g^* and $\Omega^* = \Omega_{g^*}$ are given, by imposing $h > 0$ in $D \setminus [V_\varepsilon \cap \Omega^*]$ and $h < 0$ in $V_\varepsilon \cap \Omega^*$, then the variations $g^*+\lambda h$, $\lambda > 0$, leave the boundary $\partial\Omega^*$ unchanged and can generate just topological variations in $\Omega^* \setminus V_\varepsilon$ (but not necessarily all the possible topological variations in Ω^*). Here, V_ε is chosen as in Proposition 2.1 and $\varepsilon > 0$ is “small”.*

If $h+g^* < 0$ in Ω^*, then no topological variations can be generated in Ω^* by $g^*+\lambda h$, $\lambda \in [0,1]$. Clarifying the relationship of functional variations with other types of variations is suitable both for theoretical and numerical applications.

We notice however that in typical minimization algorithms, the perturbation h is obtained from the gradient of the cost and conditions as above (for h) are, in general, not fulfilled. In practice, for instance in mechanics, one is interested both in the optimal shape of an object and in the crucial question where to place the holes (Bendsøe and Sigmund 2004). That is, the combined shape and topology variations are natural and important.

The above geometric structure (21), (22) of the cost and of the constraint (one interior and one exterior component of the boundary) remains the same for the considered perturbed controls $g^* + \lambda h$, $u^* + \lambda v$, for $|\lambda|$ small. The next proposition strengthens the conclusion of Prop. 2.1 and gives more clarifications in this respect (it may be easily extended to the general case of a finite number of holes).

Proposition 3.1. *Assume that Ω_{g^*} is double connected and (23) is valid. We have:*
i) There is $\lambda(g^*,h) > 0$ such that $\Omega_{g^*+\lambda h} \cap \Omega_{g^*}$ is double connected as well, for $\lambda(g^*,h) > |\lambda| > 0$;
ii) There is $\hat{\lambda}(g^*,h) > 0$, such that any existing hole in $\Omega_{g^*+\lambda h} \setminus \Omega_{g^*}$, disappears for $\hat{\lambda}(g^*,h) > |\lambda| > 0$.
iii) The same is valid for the holes that touch/intersect G^*, contained in $\Omega_{g^*+\lambda h}$.

Proof
i) By Prop. 2.1, the number of holes in $\Omega_{g^*+\lambda h} \cap \Omega_{g^*}$ is at least equal to the number of holes in Ω_{g^*} for $|\lambda| < \lambda(\varepsilon)$. In $V_\varepsilon \cap \Omega_{g^*+\lambda h} \cap \Omega_{g^*}$, it is possible to have some supplementary holes and let x be a point from such a hole. We have $g^*(x) < 0$ and $g^*(x) + \lambda h(x) > 0$. As g^*, h are bounded in D, then these two inequalities contradict each other for $|\lambda|$ small. That is the holes decrease as $\lambda \to 0$. Their number is bounded and, by the Weierstrass theorem, there is a constant such that $g^*(x) < c < 0$ on the supplementary holes. Since $g^*, h \in C(\overline{D})$, then the above contradiction is valid for $\lambda(g^*,h) > |\lambda| > 0$ and the supplementary holes disappear. Moreover, in any point from $\Omega_{g^*+\lambda h} \cap \Omega_{g^*}$, we have $g^*(x) < 0$ and $g^*(x) + \lambda h(x) < 0$ and these inequalities are preserved for $\lambda \to 0$, that is new holes cannot appear for $\lambda \to 0$.

ii) In this case, for any x from such a hole, we have $g^*(x) > 0$, $g^*(x) + \lambda h(x) > 0$ and these inequalities remain valid for $\lambda \to 0$. That is, the holes will increase as $\lambda \to 0$ and this contradicts Prop. 2.1. In fact, the components of $\partial\Omega_{g^*+\lambda h}$ passing through $\bar{x}$, $\widehat{x}$ will touch the boundaries of such small holes, for $\lambda \to 0$ and this means that these interior holes will disappear by unifying with $D \setminus \Omega_{g^*+\lambda h}$.

We examine now the possible formation of new small holes, in $\Omega_{g^*+\lambda h} \setminus \Omega_{g^*}$ and approaching G^*, as $\lambda \to 0$. Due to the constant null value of $g^* + \lambda h$ on the boundary of such holes , there is at least one extremal point x_λ inside the hole, such that $\nabla(g^* + \lambda h)(x_\lambda) = 0$ and, moreover, $x_\lambda \to \tilde{x} \in G^*$. This contradicts assumption (7) in $\tilde{x}$. That is, there is $\hat{\varepsilon}$ such that, for smaller ε, no holes exist in $V_\varepsilon \cap [\Omega_{g^*+\lambda h} \setminus \Omega_{g^*}]$. Consequently, starting with some $\hat{\lambda}(g^*,h)$, there is no hole in $\Omega_{g^*+\lambda h} \setminus \Omega_{g^*}$.

iii) This situation combines the previous two points: the supplementary holes may have a part inside Ω_{g^*} and a part outside. They behave as in points i), respectively ii) as $\lambda \to 0$ and the argument ends as in point ii). $\square$

We know that $g^*(\bar{x}) = g^*(\widehat{x}) = 0$ by the above choice of the initial conditions for (10)-(12). We have taken the variations $h \in \mathscr{F}$, around g^*, satisfying as well

$$h(\bar{x}) = h(\widehat{x}) = 0. \tag{23}$$

In fact, (23) should be understood in the sense that any points $\bar{x}, \widehat{x} \in G^* = \partial\Omega^*$ (one on each of the two components of G^*) such that (23) is fulfilled, may be used. That is the class of variations $h \in \mathscr{F}$, taken into account, is rich.

Remark 3.1. *Such variations are also used in the numerical experiments by Murea and Tiba (2019, 2022a,c) and the condition (23) is easy to handle at the numerical level, to* *obtain descent directions. The number of holes in such examples, can change during the iterations, both by adding new holes or by closing existing holes. Although functional variations are used both here (for optimality conditions) and in computational examples, one should distinguish between the two situations. Here,* $\lambda \to 0$ *and we have stability properties of the boundaries, by Prop. 2.1, Prop. 3.1. In the case of experiments,* λ *is chosen according to the descent property of the cost functional as large as given by the line search. Then, topological changes are possible. For instance, if* $g = x_1^2 + x_2^2 - 1$, *and* Ω_g *is the unit disk, take* $h = \delta_1 + \delta_2$, *the sum of two positive regularized Dirac distributions, supported by disks of radius* 0.1 *centered in the origin, respectively in* $(0.5, 0)$ *and with height* 1, *respectively* 2. *The perturbed domain* $\Omega_{g+\lambda h}$ *has one hole for* $\lambda > 0.375$ *and two holes for* $\lambda > 1$, *but we even have* $\Omega_{g+\lambda h} = \Omega_g$, *for small* λ.

It is possible, in principle, to remove condition (23) by defining $\overline{x}_\lambda = \mathrm{proj}_{G_\lambda^*}(\overline{x})$, $\widehat{x}_\lambda = \mathrm{proj}_{G_\lambda^*}(\widehat{x})$ and, automatically, we get $(g+\lambda h)(\overline{x}_\lambda) = (g+\lambda h)(\widehat{x}_\lambda) = 0$. But, the aim of this section is to prove differentiability properties, for instance of the mapping $z_g(\cdot)$ depending on g (and on the initial conditions $\overline{x}, \widehat{x}, \overline{x}_\lambda, \widehat{x}_\lambda$). It is clear that differentiability properties of the $\mathrm{proj}_{G_\lambda^*}(\cdot)$ or of the corresponding $\mathrm{dist}(\cdot)$ function have to be valid in this setting. However, as shown, for instance, by Delfour and Zolesio (2011, p. 169), this may not be true. In order to avoid such technicalities, we have imposed (23), which still ensures a large class of variations.

We denote by $x_\lambda = (x_{1\lambda}, x_{2\lambda})$ the solution of the perturbed Hamiltonian system

$$x_{1\lambda}' = -\partial_2 g(x_{1\lambda}, x_{2\lambda}) - \lambda \partial_2 h(x_{1\lambda}, x_{2\lambda}), \tag{24}$$

$$x_{2\lambda}' = \partial_1 g(x_{1\lambda}, x_{2\lambda}) + \lambda \partial_1 h(x_{1\lambda}, x_{2\lambda}), \tag{25}$$

$$(x_{1\lambda}(0), x_{2\lambda}(0)) = x_0. \tag{26}$$

In (24)-(26) and in (10)-(12) we assume x_0 to be either $\overline{x}$ or $\widehat{x}$ and (23) is fulfilled. Let $w_\lambda = \frac{x_\lambda - x}{\lambda}$.

Proposition 3.2. *Let* $g, h \in C^2(\overline{D})$. *Then,* $x_\lambda \to x$ *in* $C^1(0,T)^2$ *and* $w_\lambda \to w = (w_1, w_2)$ *in* $C^1(0,T)^2$. *The limit* w *satisfies the system in variations:*

$$w_1' = -\nabla \partial_2 g(x) \cdot w - \partial_2 h(x) \quad \textit{in } [0,T], \tag{27}$$

$$w_2' = \nabla \partial_1 g(x) \cdot w + \partial_1 h(x) \quad \textit{in } [0,T], \tag{28}$$

$$w(0) = (w_1(0), w_2(0)) = (0,0). \tag{29}$$

Proof

Both (24)-(26) and (10)-(12) have periodic solutions, by the above arguments. Due to this, the stability property $x_\lambda \to x$ in $C^1(0,T)^2$, for some arbitrary $T > 0$ is standard (Pontriaguine 1968; Barbu 1985). Subtracting (24)-(26) and (10)-(12) we obtain

$$w_{1\lambda}' = -\frac{1}{\lambda}[\partial_2 g(x_\lambda) - \partial_2 g(x)] - \partial_2 h(x_\lambda) \quad \text{in } [0,T], \tag{30}$$

$$w_{2\lambda}' = \frac{1}{\lambda}[\partial_1 g(x_\lambda) - \partial_1 g(x)] + \partial_1 h(x_\lambda) \quad \text{in } [0,T], \tag{31}$$

$$w_\lambda(0) = (w_{1\lambda}(0), w_{2\lambda}(0)) = (0,0). \tag{32}$$

The mean value theorem allows to replace the parentheses in (30), (31), respectively by $\nabla\partial_2 g(\theta_\lambda)\cdot w_\lambda$, $\nabla\partial_1 g(\eta_\lambda)\cdot w_\lambda$, where $\theta_\lambda(t), \eta_\lambda(t)$ are some points on the segment between $x_\lambda(t)$ and $x(t)$ and $\theta_\lambda(t) \to x(t)$, $\eta_\lambda(t) \to x(t)$ as $\lambda \to 0$. Moreover, $\nabla\partial_2 g(\theta_\lambda)$, $\nabla\partial_1 g(\eta_\lambda)$ have respectively the limits $\nabla\partial_1 g(x)$, $\nabla\partial_2 g(x)$ and $\partial_2 h(x_\lambda) \to \partial_2 h(x)$, $\partial_1 h(x_\lambda) \to \partial_1 h(x)$ under our assumptions, as $\lambda \to 0$.

From (30)-(32) and their equivalent formulation using $\theta_\lambda, \eta_\lambda$, we obtain that $\{w_\lambda\}$ is bounded in $C(0,T)^2$ via the Gronwall lemma. It is also bounded in $C^1(0,T)^2$ due to the above convergences. The Arzela-Ascoli theorem gives $w_\lambda \to w$ in $C(0,T)^2$, on a subsequence. One can pass to the limit $w_\lambda \to w$ in $C^1(0,T)^2$, in fact, and obtain (27)-(29). Since the solution of the linear system (27)-(29) is unique, the limit is valid without taking subsequences. $\square$

Remark 3.2. *This is a modification of Prop. 6 formulated by Tiba (2013). If $g,h \in \mathscr{F}$, then $T > 0$ is arbitrary due to Prop. 2.2. Moreover, the nonlinear operator $z_g : C^2(\overline{D}) \to C^1(0,T)^2$ is Gâteaux differentiable and its Gâteaux differential is linear bounded, due to (27)-(29) (Tiba 2013). This also gives continuity with respect to $g \in C^2(\overline{D})$, that is the nonlinear operator $z_g(\cdot)$ is Fréchet differentiable in the points of $\mathscr{F} \subset C^2(\overline{D})$.*

Proposition 3.3. *We have:*

$$A(f + g_+^2 u) : C^2(\overline{D}) \times L^p(D) \to W^{2,p}(D) \cap H_0^1(D)$$

$$\nabla A(f + g_+^2 u) : C^2(\overline{D}) \times L^p(D) \to W^{1,p}(D)^2$$

are Fréchet differentiable.

This is obvious due to the properties of the operator A and of the positive part. We denote by $\tilde{C}^2(\overline{D}) = \{g \in C^2(\overline{D});\ g(\hat{x}) = g(\overline{x}) = 0\}$, a Banach space under the same norm as in $C^2(\overline{D})$. It is clear that $\mathscr{F} \subset \tilde{C}^2(\overline{D})$ is an open cone, due to (7) and (8). Notice that (7) is equivalent with $|g(x)| + |\nabla g(x)| > 0$ in $\overline{D}$ and Weierstrass theorem can be applied here too. By Prop. 3.2 and Prop. 3.3, it yields that the nonlinear trace operator, in $H^{3/2}(G)$, of $A(f + g_+^2 u)(z_g(\cdot)) : [\mathscr{F} \subset \tilde{C}^2(\overline{D})] \times L^p(D) \to C(0,T)$, given by $g, u \to y_g(z_g(\cdot))$, is Fréchet differentiable. Similarly $z_g' : \mathscr{F} \subset \tilde{C}^2(\overline{D}) \to C(0,T)^2$, $\nabla g(z_g(\cdot)) : \mathscr{F} \subset \tilde{C}^2(\overline{D}) \to C^1(0,T)^2$ (trace of ∇g on G), $\nabla A(f + g_+^2 u)(z_g(\cdot)) : [\mathscr{F} \subset \tilde{C}^2(\overline{D})] \times L^p(D) \to C(0,T)^2$ (trace of ∇y_g in $H^{1/2}(G)^2$) are Fréchet differentiable. In case $G^* = \partial\Omega^*$ has more connected components, then in (23) more initial conditions have to be considered and the definition of $\tilde{C}^2(\overline{D})$ has to be adapted accordingly.

Proposition 3.4. (Murea and Tiba 2022c) *Under condition (7), the functional $T_g : \mathscr{F} \subset \tilde{C}^2(\overline{D}) \to R$ (the main period) is Fréchet differentiable.*

Murea and Tiba (2022c) proved the Gâteaux differentiability of the period T_g in Prop. 4.2. The Fréchet differentiability follows by arguments as in Rem. 3.2.

We denote by $J : \mathscr{F} \times L^p(D) \subset \tilde{C}^2(\overline{D}) \times L^p(D) \to R$ and $S : \mathscr{F} \times L^p(D) \subset \tilde{C}^2(\overline{D}) \times L^p(D) \to R$, the integral functionals appearing respectively in (21), (22) (the functional S is the sum of the two integral functionals appearing in (22)).

Corollary 3.1. *Assume that $j(\cdot,\cdot) \in C^1(R^2 \times R)$. Then $J, S : \mathscr{F} \times L^p(D) \subset \tilde{C}^2(\overline{D}) \times L^p(D) \to R$ are Fréchet differentiable.*

We denote by S', J' their Fréchet differentials, respectively. More details and arguments, concerning the above properties, will be discussed in the next section. We consider now the shape optimization problem (1)-(3) (in its equivalent control formulation (10)-(12), (14)-(17)), in the following abstract form:

$$\underset{g\in\mathscr{F}, u\in L^p(D)}{\operatorname{Min}} \{J(g,u);\ S(g,u) = 0\}, \tag{33}$$

where $\mathscr{F} \subset \tilde{C}^2(\overline{D})$ satisfies the assumptions (7), (8) and (23) follows automatically by the definition of $\tilde{C}^2(\overline{D})$. We have slightly restricted the set of admissible auxiliary controls $u \in L^p(D)$ with respect to the assumptions of Thm. 2.1. But we underline that the class of admissible geometries, described by $\mathscr{F} \subset \tilde{C}^2(\overline{D})$, is not modified. However, the constraint (17) may restrict the set of admissible controls quite severely. We recall that $[g^*, u^*]$ is some (local) optimal control pair (assumed to exist) and we denote by W^* the closed linear one dimensional subspace in $\tilde{C}^2(\overline{D}) \times L^p(D)$ generated by $[g^*, u^*]$.

It is known that a one dimensional subspace in a Banach space admits a complementary subspace. This is due to the Hahn-Banach theorem (Wikipedia 2023). In our case $W^* = \{\lambda[g^*, u^*];\ \lambda \in R\}$ and we define $\varphi_0 : W^* \to R$, $\varphi_0(\lambda[g^*, u^*]) = \lambda$ a linear continuous functional. It may be extended to a linear continuous functional $\varphi : \tilde{C}^2(\overline{D}) \times L^p(D) \to R$ such that $\varphi([g^*, u^*]) = 1$. Then $P : \tilde{C}^2(\overline{D}) \times L^p(D) \to W^*$, $P([g,u]) = \varphi([g,u])[g^*, u^*]$ is a projection operator and the complementary subspace is $V^* = (I-P)(\tilde{C}^2(\overline{D}) \times L^p(D))$, *i.e.*, $\tilde{C}^2(\overline{D}) \times L^p(D) = V^* \oplus W^*$. We assume that $S : \tilde{C}^2(\overline{D}) \times L^pD) = V^* \oplus W^* \to R$ has its partial derivative with respect to W^*, $\partial_{W^*} S(g^*, u^*) : W^* \to R$, an isomorphism (see Ciarlet 1989, Ch. 7). As W^* has dimension one, this is equivalent to the directional derivative in $[g^*, u^*]$ along $\lambda[g^*, u^*]$ to satisfy:

$$S'(g^*, u^*)(g^*, u^*) \neq 0. \tag{34}$$

We have proved:

Theorem 3.1. *Under condition (34), if $[g^*, u^*]$ is a (local) minimizer of (33), there is $k \in R$, $k \neq 0$, such that*

$$J'(g^*, u^*) + kS'(g^*, u^*) = 0. \tag{35}$$

This is the classical Lagrange multipliers approach, under the constraint qualification (34), for equality constrained minimization problems (Ciarlet 1989, Ch.7). Important here is that $\mathscr{F} \times L^p(D)$ is an open subset of $\tilde{C}^2(\overline{D}) \times L^p(D)$.

Remark 3.3. *There are other techniques based on regularization or penalization procedures and involving interiority hypotheses (Bergounioux and Tiba 1996; Amstutz and Ciligot-Travain 2010; Barbu and Precupanu 2012) in various forms, but not possible to be applied here. Moreover, the one dimensional character of (34) provides simplicity, as much as possible in this context. Another advantageous point is that, although obtaining (34) uses essentially the decomposition $V^* \oplus W^*$, the result of Thm. 3.1 is expressed in terms of the original formulation (33). The fact that the optimal pair is involved in (34) is standard in the literature (Ciarlet 1989; Barbu and Precupanu 2012).*

In the next section, we shall explain (35) in full details.

4. Optimality conditions

In this section we maintain the hypothesis that $\mathscr{F} \subset \tilde{C}^2(\overline{D})$ and (7), (8), (34) are satisfied. Moreover, to simplify the setting and to fix the ideas, we assume that the boundary of the optimal domain $\partial\Omega^* = G^*$ has just two components. The functional variations $g^* + \lambda h$, $h \in \mathscr{F}$ are satisfying (23), for some points $\widehat{x}$, $\overline{x}$ on these components. In fact, for any $g \in \mathscr{F}$ and $h \in \tilde{C}^2(\overline{D})$, we have $g + \lambda h \in \mathscr{F}$ for λ sufficiently small, depending on g, h. The variations $u^* + \lambda v$ are supposed to satisfy just $v \in L^p(D)$. We denote by (H) all these conditions. It is a standard result (Murea and Tiba 2019) that the corresponding equation in variations associated to (15), (16) around $[g^*, u^*]$ is

Proposition 4.1. *The limit of $q_\lambda = \frac{1}{\lambda}(y_{[g^*,u^*]+\lambda[h,v]} - y_{[g^*,u^*]})$ exists in $W^{2,p}(D)$ and satisfies*

$$-\Delta q + q = (g_+^*)^2 v + 2 g_+^* u^* h \quad \text{in } D, \tag{36}$$

$$q = 0 \quad \text{on } \partial D. \tag{37}$$

Prop. 4.1 together with Prop. 3.2 give the system in variations of the state system (10)-(12), (15), (16), on each component of $\partial\Omega^*$. Due to the periodicity of the Hamiltonian system, the equation (27), (28) are valid on any interval $[0, T]$. Following Murea and Tiba (2022c), we also have the formulas for the differentials of the main periods $\widehat{T}_{g^*}$, $\overline{T}_{g^*}$ (on each component of G^*), with respect to functional variations, completing Prop. 3.4. Below, we use the simplified notations T_{g^*}, z_{g^*}.

Proposition 4.2. *We have*

$$\lim_{\lambda \to 0} \frac{T_{g^*+\lambda h} - T_{g^*}}{\lambda} = -\frac{w_2(T_{g^*})}{(z_{g^*}^2)'(T_{g^*})}, \tag{38}$$

if $(z_{g^*}^2)'(T_{g^*}) \neq 0$.

Notice that (7) yields either this condition or $(z_{g^*}^1)'(T_{g^*}) \neq 0$ and in the latter case, relation (38) has to be replaced by

$$\lim_{\lambda \to 0} \frac{T_{g^*+\lambda h} - T_{g^*}}{\lambda} = -\frac{w_1(T_{g^*})}{(z_{g^*}^1)'(T_{g^*})}. \tag{39}$$

We denote by $\theta(g^*, h)$ this limit, in general. We give now a detailed formulation of Cor. 3.1.

Proposition 4.3. *Under the assumptions (H) and $y_{[g,u]} \in C^2(\overline{D})$, consider the directional derivative of $S : C^2(\overline{D}) \times L^p(D) \to R$ for perturbations $g^* + \lambda h$, $u^* + \lambda v$, $\lambda \in R$, $v \in L^p(D)$, $h \in \tilde{C}^2(\overline{D})$ satisfying $h(\overline{x}) = h(\widehat{x}) = 0$ for some points $\overline{x}, \widehat{x} \in G^*$ (situated on each of the two components of G^*, corresponding to g^*). It is given by:*

$$
\begin{aligned}
&\hat{\theta}(g^*,h)|\nabla A(f+(g^*_+)^2u^*)(\widehat{x})\cdot\nabla g^*(\widehat{x})|^2 + 2\int_0^{T_{g^*}} \nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\cdot\nabla g^*(\widehat{z}_{g^*}(t)) \\
&\qquad\Big[\nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\cdot\nabla h(\widehat{z}_{g^*}(t)) + \nabla q(\widehat{z}_{g^*}(t))\cdot\nabla g^*(\widehat{z}_{g^*}(t)) + \\
&\qquad\qquad + H\left[A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\right]\hat{w}(t)\cdot\nabla g^*(\widehat{z}_{g^*}(t)) + \\
&\qquad\qquad\qquad + \nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\cdot H\left[g^*(\widehat{z}_{g^*}(t))\right]\hat{w}(t)\Big]dt + \qquad (40) \\
&\overline{\theta}(g^*,h)|\nabla A(f+(g^*_+)^2u^*)(\overline{x})\cdot\nabla g^*(\overline{x})|^2 + 2\int_0^{T_{g^*}} \nabla A(f+(g^*_+)^2u^*)(\overline{z}_{g^*}(t))\cdot\nabla g^*(\overline{z}_{g^*}(t)) \\
&\qquad\Big[\nabla A(f+(g^*_+)^2u^*)(\overline{z}_{g^*}(t))\cdot\nabla h(\overline{z}_{g^*}(t)) + \nabla q(\overline{z}_{g^*}(t))\cdot\nabla g^*(\overline{z}_{g^*}(t)) + \\
&\qquad\qquad + H\left[A(f+(g^*_+)^2u^*)(\overline{z}_{g^*}(t))\right]\overline{w}(t)\cdot\nabla g^*(\overline{z}_{g^*}(t)) + \\
&\qquad\qquad\qquad + \nabla A(f+(g^*_+)^2u^*)(\overline{z}_{g^*}(t))\cdot H\left[g^*(\overline{z}_{g^*}(t))\right]\overline{w}(t)\Big]dt.
\end{aligned}
$$

The notation $H[\cdot]$ is the Hessian matrix and $\hat{\theta}, \overline{\theta}$ are associated to (38) or (39) in the points $\hat{x}, \overline{x}$ respectively . Similarly, $\hat{w}, \overline{w}$ refer to the vector solutions of (27)-(29) associated respectively to the two components of G^*.

Proof

It is enough to examine the first term in the definition (22) of S:

$$
\begin{aligned}
&\lim_{\lambda\to 0}\frac{1}{\lambda}\Bigg[\int_0^{T_{g^*+\lambda h}} \left|\nabla A(f+(g^*+\lambda h)^2_+(u^*+\lambda v))(\widehat{z}_{g^*+\lambda h}(t))\cdot\nabla(g^*+\lambda h)(\widehat{z}_{g^*+\lambda h}(t))\right|^2 dt - \\
&\qquad - \int_0^{T_{g^*}} \left|\nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\cdot\nabla g^*(\widehat{z}_{g^*}(t))\right|^2 dt\Bigg].
\end{aligned}
$$

Here $\widehat{z}_{g^*}$, $\widehat{z}_{g^*+\lambda h}$ denote the solution of the Hamiltonian system (10), (11), respectively of the corresponding perturbed Hamiltonian system, with initial condition $\widehat{x} \in G^* \cap G^*_\lambda$. We notice first that

$$
\begin{aligned}
&\frac{1}{\lambda}\int_{T_{g^*}}^{T_{g^*}+\lambda h} \left|\nabla A(f+(g^*+\lambda h)^2_+(u^*+\lambda v))(\widehat{z}_{g^*+\lambda h}(t))\cdot\nabla(g^*+\lambda h)(\widehat{z}_{g^*+\lambda h}(t))\right|^2 dt \to \\
&\qquad \to \hat{\theta}(g^*,h)\left|\nabla A(f+(g^*_+)^2u^*)(\widehat{x})\cdot\nabla g^*(\widehat{x})\right|^2,
\end{aligned}
$$

due to Prop. 4.2, the regularity assumptions and the mean value theorem for integrals.

Concerning the integrals over $[0, T_{g^*}]$, we have (we indicate one case):

$$\frac{1}{\lambda}\left\{\int_0^{T_{g^*}} \left|\nabla A(f+(g^*+\lambda h)_+^2(u^*+\lambda v))(\widehat{z}_{g^*+\lambda h}(t))\cdot\nabla(g^*+\lambda h)(\widehat{z}_{g^*+\lambda h}(t))\right|^2 dt - \right.$$

$$\left. -\int_0^{T_{g^*}} \left|\nabla A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t))\cdot\nabla g^*(\widehat{z}_{g^*}(t))\right|^2 dt\right\} \to$$

$$\to 2\int_0^{T_{g^*}} \nabla A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t))\cdot\nabla g^*(\widehat{z}_{g^*}(t))\Big[\nabla A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t))\cdot$$

$$\cdot\nabla h(\widehat{z}_{g^*}(t)) + \nabla q(\widehat{z}_{g^*}(t))\cdot\nabla g^*(\widehat{z}_{g^*}(t)) +$$

$$+H\Big[A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t))\Big]\hat{w}(t)\cdot\nabla g^*(\widehat{z}_{g^*}(t)) +$$

$$+\nabla A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t))\cdot Hg^*(\widehat{z}_{g^*}(t))\hat{w}(t)\Big]dt.$$

This ends the proof. □

In a similar manner, we establish

Proposition 4.4. *Assume that $j \in C^1(R^3)$ and the conditions (H), $y_{[g,u]} \in C^2(\overline{D})$ hold. Then, the directional derivative of J at $[g^*, u^*]$ in the direction $[h, v] \in \mathscr{F} \times L^p(D) \subset \tilde{C}^2(\overline{D}) \times L^p(D)$ is given by*

$$\int_0^{T_{g^*}} \nabla_1 j(\widehat{z}_{g^*}(t), A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t)))\cdot\hat{w}(t)\left|\widehat{z}'_{g^*}(t)\right|dt +$$

$$+\int_0^{T_{g^*}} \partial_2 j(\widehat{z}_{g^*}(t), A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t)))\Big[\nabla A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t))\cdot$$

$$\cdot\hat{w}(t) + q(\widehat{z}_{g^*}(t))\Big]\left|\widehat{z}'_{g^*}(t)\right|dt +$$

$$+\int_0^{T_{g^*}} j(\widehat{z}_{g^*}(t), A(f+(g_+^*)^2u^*)(\widehat{z}_{g^*}(t)))\frac{\widehat{z}'_{g^*}(t)\cdot\hat{w}'(t)}{\left|\widehat{z}'_{g^*}(t)\right|}dt +$$

$$+\hat{\theta}(g^*, h)\, j(\widehat{x}, A(f+(g_+^*)^2u^*)(\widehat{x})). \quad (41)$$

In (41) $\nabla_1 j(\cdot,\cdot)$ denotes the gradient of j with respect to the first two arguments and $\partial_2 j(\cdot,\cdot)$ denotes the derivative of j with respect to the last argument. We have also assumed that the cost J is defined just on the "exterior" component of G^*, that contains $\widehat{x}$. The proof follows the same lines as for (40). The condition $y_{[g,u]} \in C^2(\overline{D})$ is ensured, for instance, if $f, u, v \in W^{1,p}(D)$. More details can be found in the paper by Murea and Tiba (2022c), in a general context. We can now formulate the adjoint system associated to the constrained control problem (10)-(12), (14)-(17). We also use Thm. 3.1, that is the existence of the

Lagrange multiplier $k \in R$ such that (35) is satisfied. We first define the adjoint system associated to the elliptic equation (15), (16) and to the terms containing q from (40), (41):

$$\begin{aligned}
2k \int_0^{T_{g^*}} \nabla A(f + (g_+^*)^2 u^*)(\widehat{z}_{g^*}(t)) \cdot \nabla g^*(\widehat{z}_{g^*}(t)) \nabla q(\widehat{z}_{g^*}(t)) \cdot \nabla g^*(\widehat{z}_{g^*}(t)) dt + \\
+ 2k \int_0^{T_{g^*}} \nabla A(f + (g_+^*)^2 u^*)(\bar{z}_{g^*}(t)) \cdot \nabla g^*(\bar{z}_{g^*}(t)) \nabla q(\bar{z}_{g^*}(t)) \cdot \nabla g^*(\bar{z}_{g^*}(t)) dt + \\
+ \int_0^{T_{g^*}} \partial_2 j(\widehat{z}_{g^*}(t), A(f + (g_+^*)^2 u^*))(\widehat{z}_{g^*}(t)) q(\widehat{z}_{g^*}(t)) \left| \widehat{z}'_{g^*}(t) \right| dt = \\
= \int_D p(x)(-\Delta q(x) + q(x)) dx, \quad (42)
\end{aligned}$$

for any $q \in H^2(D) \cap H_0^1(D)$. Notice that, in (42), q plays the role of an arbitrary test function and the last term in the left-hand side in (42) can be rewritten as

$$\int_{\widehat{G}^*} \partial_2 j(\sigma, y^*(\sigma)) q(\sigma) d\sigma, \quad (43)$$

where $\widehat{G}^*$ is the component of G^* containing $\widehat{x}$ and y^* is the solution of (15), (16) associated to g^*, u^*. This is also valid for the other terms there, by multiplying and dividing by $|\widehat{z}'_{g^*}(t)| = |\nabla g^*(\widehat{z}_{g^*}(t))|$. Then, (42) can be rewritten as:

$$\begin{aligned}
2k \int_{G^*} \nabla y^*(\sigma) \cdot \nabla g^*(\sigma) \frac{\partial q}{\partial n} d\sigma + \int_{\widehat{G}^*} \partial_2 j(\sigma, y^*(\sigma)) q(\sigma) d\sigma = \\
= \int_D p(x)(-\Delta q(x) + q(x)) dx, \quad \forall q \in H^2(D) \cap H_0^1(D). \quad (44)
\end{aligned}$$

In (44), we also use that $\frac{\nabla g^*(\cdot)}{|\nabla g^*(\cdot)|}$ is the unit normal vector to G^*, that is we replace $\frac{\partial q}{\partial n} = \nabla q(\sigma) \cdot \nabla g^*(\sigma) \frac{1}{|\nabla g^*(\sigma)|}$. The first two terms in (42) are written as one term since G^* has two components. If the right-hand side in (36) is denoted by $\mu \in L^2(D)$, the correspondence $\mu \to q$ defined by (36), (37) is an isomorphism between $L^2(D)$ and $H^2(D) \cap H_0^1(D)$. Consequently, the left-hand side in (44) is a linear continuous functional of $\mu \in L^2(D)$ and there is a unique $p \in L^2(D)$ such that (44) is satisfied, by the Riesz theorem. The function $p \in L^2(D)$ is called a very weak solution (solution by transposition) of the elliptic equation

$$-\Delta p + p = \xi \quad \text{in } D, \quad (45)$$

where ξ is a functional in the dual of $H^2(D) \cap H_0^1(D)$, expressed by the sum of the boundary integrals in (44). To (45), the null condition on ∂D is formally added.

We formulate now the adjoint system that takes into account the terms containing w, w', $\hat{\theta}(g^*,h)$, $\overline{\theta}(g^*,h)$ from (40), (41). Notice that the term w' may be replaced via (27), (28) by quantities including just w and ∇h. In fact, we define the adjoint systems corresponding to (27)-(29). It is to be underlined that in (27)-(29), the state x is to be replaced by $\widehat{z}_{g^*}$, respectively $\overline{z}_{g^*}$ (corresponding to the respective component of G^*, taken into account). Moreover, in relations (40), (41), we shall not use the terms containing ∇h (including the ones appearing from the rewriting of w'). Finally, we write the adjoint system corresponding to the component of G^* containing $\widehat{x}$. The adjoint state here is $\widehat{m} = [\widehat{m}_1, \widehat{m}_2]$:

$$
\begin{aligned}
-\widehat{m}'(t) = {} & \widehat{M}^*(t)\widehat{m}(t) + \nabla_1 j(\widehat{z}_{g^*}(t), A(f+(g^*_+)^2u)(\widehat{z}_{g^*}(t)))\left|\widehat{z}'_{g^*}(t)\right| + \\
& + \partial_2 j(\widehat{z}_{g^*}(t), A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t)))\nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\left|\widehat{z}'_{g^*}(t)\right| + \\
& + j(\widehat{z}_{g^*}(t), A(f+(g^*_+)^2u^*))(\widehat{z}_{g^*}(t))\widehat{M}^*(t)\frac{\widehat{z}'_{g^*}(t)}{\left|\widehat{z}'_{g^*}(t)\right|} + \\
& + 2k\nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\cdot\nabla g^*(\widehat{z}_{g^*}(t))\Big\{H^*\left[A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\right]\nabla g^*(\widehat{z}_{g^*}(t)) + \\
& + H^* g^*(\widehat{z}_{g^*}(t))\nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t))\Big\}, \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{m}_1(T_{g^*}) = 0,\ \widehat{m}_2(T_{g^*}) = {} & -\frac{1}{(\widehat{z}^2_{g^*})'(T_{g^*})}\Big[j(\widehat{x}, A(f+(g^*_+)^2u^*))(\widehat{x}) + \\
& + \left|\nabla A(f+(g^*_+)^2u^*)(\widehat{x})\cdot\nabla g^*(\widehat{x})\right|^2\Big]. \qquad (47)
\end{aligned}
$$

In (46), $\widehat{M}$ is the matrix appearing implicitly in (27), (28) for $x(t) = \widehat{z}_{g^*}(t)$ and $\widehat{M}^*$ is its adjoint (and H^* is the adjoint as well). On the component of G^* containing $\overline{x}$, the adjoint system is simpler since (41) is not defined on it. The notations use the natural modification of the notations from (46), (47):

$$
\begin{aligned}
-\overline{m}'(t) = {} & \overline{M}^*(t)\overline{m}(t) + 2k\nabla A(f+(g^*_+)^2u^*)(\overline{z}_{g^*}(t))\cdot \\
& \cdot\nabla g^*(\overline{z}_{g^*}(t))\Big\{H^*\Big[A(f+(g^*_+)^2u^*)(\overline{z}_{g^*}(t))\Big]\nabla g^*(\overline{z}_{g^*}(t)) + \\
& + H^* g^*(\overline{z}_{g^*}(t))\nabla A(f+(g^*_+)^2u^*)(\overline{z}_{g^*}(t))\Big\}, \qquad (48)
\end{aligned}
$$

$$
\overline{m}_1(T_{g^*}) = 0,\quad \overline{m}_2(T_{g^*}) = -\frac{1}{(\overline{z}^2_{g^*})'(T_{g^*})}\left|\nabla A(f+(g^*_+)^2u^*)(\overline{x})\cdot\nabla g^*(\overline{x})\right|^2. \qquad (49)
$$

The adjoint system consists of the equations (44)-(49). In case G^* has more components, more equations have to be added in (27), (28) and in the adjoint system.

Theorem 4.1. *Under the assumptions (H) and $f, u^* \in W^{1,p}(D)$, there is $k \neq 0$ such that*

$$
\begin{aligned}
&\int_0^{T_{g^*}} j(\widehat{z}_{g^*}(t), A(f+(g^*_+)^2u^*))(\widehat{z}_{g^*}(t)) \frac{\widehat{z}'_{g^*}(t)}{\left|\widehat{z}'_{g^*}(t)\right|} \cdot (-\partial_2 h(\widehat{z}_{g^*}(t)), \partial_1 h(\widehat{z}_{g^*}(t)))+\\
&+2k\int_0^{T_{g^*}} \nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t)) \cdot \nabla g^*(\widehat{z}_{g^*}(t)) \nabla A(f+(g^*_+)^2u^*)(\widehat{z}_{g^*}(t)) \cdot \nabla h(\widehat{z}_{g^*}(t))dt+\\
&+2k\int_0^{T_{g^*}} \nabla A(f+(g^*_+)^2u^*)(\bar{z}_{g^*}(t)) \cdot \nabla g^*(\bar{z}_{g^*}(t)) \nabla A(f+(g^*_+)^2u^*)(\bar{z}_{g^*}(t)) \cdot \nabla h(\bar{z}_{g^*}(t))dt+\\
&\qquad+\int_D p(x)\left[(g^*_+)^2 v + 2g^*_+ u^* h\right]dx + \int_0^{T_{g^*}} \overline{m}(t) \cdot (-\partial_2 h(\bar{z}_{g^*}(t)), \partial_1 h(\bar{z}_{g^*}(t)))dt+\\
&\qquad\qquad+\int_0^{T_{g^*}} \widehat{m}(t) \cdot (-\partial_2 h(\widehat{z}_{g^*}(t)), \partial_1 h(\widehat{z}_{g^*}(t)))dt = 0. \qquad (50)
\end{aligned}
$$

for any $v \in W^{1,p}(D)$, $h \in \mathscr{F} \subset \tilde{C}^2(\overline{D})$ such that there are $\widehat{x}, \bar{x} \in G^*$ situated respectively on each component of G^*, with $h(\widehat{x}) = h(\bar{x}) = 0$.

This follows by Thm. 3.1, Prop. 4.3, Prop. 4.4 and the definitions of the adjoint systems (44)-(49) and the systems in variations (27)-(29), respectively (36), (37). The relation (50) expresses the so-called maximum principle for the problem (21), (22), which is a slight modification of the original shape optimization problem (1)-(3), in the sense that in its equivalent form (14)-(17) we restrict the class of admissible controls to be $u \in L^p(D)$. Relation (50) together with the adjoint systems (44)-(49) and the state system (15), (16) give the optimality conditions for this problem.

Remark 4.1. *We use functional variations and optimal control methods and we don't impose explicitly classical boundary variations or topological variations (Sokołowski and Zolesio 1992; Novotny and Sokołowski 2013). In the statement of Thm. 4.1, such variations are intimately combined and the formulation has a purely analytic character, i.e. no geometric condition is involved in it. Moreover, in the recent article by Murea and Tiba (2022c), it is shown that the gradient behind our approach can be effectively used in numerical experiments, including both topology and shape optimization simultaneously. The corresponding algorithm chooses automatically the type of variation (it is not prescribed) in each iteration.*

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