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FOURTH-ORDER ELASTIC BEAM EQUATIONS WITH VARIABLE COEFFICIENTS HAVING COMBINED EFFECTS OF CONCAVE AND CONVEX NONLINEARITIES

GABRIELE BONANNO [a*], BEATRICE DI BELLA [a] AND DONAL O'REGAN [b]

ABSTRACT. The existence of two non trivial solutions for a nonlinear fourth-order boundary value problem is established. The coefficients of the equations are possibly variable and the nonlinearity includes, in particular, the combined effects of concavity and convexity. The approach is based on variational methods.

1. Introduction

Nonlinear differential equations of high order, like those of fourth and sixth order, arise from several real phenomena as for instance when an interface between two phases is examined because they help to reveal a more detailed structure of the interface and a description of the behaviour of phase fronts in materials that are undergoing a transition between the liquid and solid state (see, for example, Caginalp and Fife 1986; Gardner and Jones 1990; Peletier, Troy, and van der Vorst 1995; Bates *et al.* 1997). Moreover, it is well known that the deformations of an elastic beam in an equilibrium state, whose two ends are simply supported, can be described by fourth-order boundary value problems (see, for instance, Peletier, Troy, and van der Vorst 1995; Chaparova, Peletier, and Tersian 2003; Webb and Zima 2012, and the references therein) and, in addition, these type of equations, when the coefficients are variable, describe many types of physical, power engineering and chemical events, for example the Boyd equation about eddies in the atmosphere (Boyd 1980), gas dynamics in a fuel cell (Bailey *et al.* 2003), the analysis of the dynamical performance of a class of power converter (Amoroso *et al.* 2022, 2023), the Laplace tidal wave equation (Homer 1990) and the Meissner equation which arises in a model of a one-dimensional crystal (Meissner 1918). For these above reasons, fourth-order boundary value problems, in particular, have been extensively studied (see, for example, Tersian and Chaparova 2001; Chaparova, Peletier, and Tersian 2003; do Rosário Grossinho, Sanchez, and Tersian 2005; Cabada, Cid, and Sanchez 2007; Agarwal, Kovacs, and O'Regan 2014; Bonanno, Chinnì, and Tersian 2015) and various mathematical aspects have been examined such as the existence and multiplicity of solutions (see Bonanno and Di Bella 2008, 2010, 2011; Bonanno, Di Bella, and O'Regan 2011). It is worth noticing that the equation in such problems is usually expressed in a not complete form, that is without derivatives of odd order, and with coefficients which are constant (see, for instance, Tersian and Chaparova 2001; do Rosário Grossinho, Sanchez, and Tersian 2005; Bonanno, Di Bella, and O'Regan 2011; D'Aguì, Di Bella, and Winkert 2019, and references therein).

The aim of this paper is to investigate an elastic beam equation with variable coefficients and, as a consequence, to be able to study some cases also in the presence of all derivatives up to order four. Precisely, the existence of non trivial solutions under a suitable behaviour of the nonlinear term is established. Indeed, when the nonlinearity is, in particular, sublinear at zero, we establish the existence of at least one non-zero solution and if, in addition, it satisfies the Ambrosetti-Rabinowitz condition (which implies that the nonlinearity is superlinear at infinity), then we obtain the existence of a second non-zero solution. In particular, when one has combined effects of concave and convex nonlinearities, as that introduced by Ambrosetti, Brézis, and Cerami (1994) in their powerful paper, the existence of two non-zero solutions is established. As an example we report here one among possible applications of our results (see Theorem **3.3**).

Example 1.1. *The problem*

$$\begin{cases} \sqrt{1+t}\ \ u^{(iv)} + \frac{1}{\sqrt{1+t}}\ \ u''' - \frac{1+4\sqrt{(1+t)^3(1+t^2)}}{4\sqrt{(1+t)^3}}\ \ u'' - \frac{t}{\sqrt{1+t^2}}u' + \sqrt{t}u = \\ \qquad\qquad\qquad\qquad = u\left(\dfrac{1}{\sqrt{|u|}} + \sqrt{|u|}\right) \quad in \quad [0,1]\times\mathbb{R} \\ u(0) = u(1) = 0; \\ u''(0) = u''(1) = 0, \end{cases}$$

admits at least two non-zero classical solutions.

Our main tools are the local minimum theorem established by Bonanno, Chinnì, and Tersian (2015) and a theorem due to Bonanno and D'Aguì (2016), which is based on the same local minimum theorem and the classical Ambrosetti-Rabinowitz mountain pass theorem (Ambrosetti and Rabinowitz 1973), as given by Ghoussoub-Preiss (Brézis and Nirenberg 1991; Ghoussoub 1993; Bonanno and Livrea 2021a).

For an overview on nonlinear differential problems we refer the reader to the monographs authored by O'Regan (1994, 1997), Agarwal, O'Regan, and Wong (1999), and Agarwal, Meehan, and O'Regan (2001), while for critical point theory and mountain pass theorems we cite the seminal books by Rabinowitz (1986), Ghoussoub (1993), Ambrosetti and Malchiodi (2007), and Brézis (2010).

This paper is organized as follows. In Section 2, we give some auxiliary results to insert the problem in a variational framework and we recall the critical point theorems which are our main tools. In Section 3, we present our main results which guarantee the existence of non trivial solutions to fourth-order boundary value problems with variable coefficients for which, as a consequence, also some cases of equations in a complete form can be investigated (see Examples 1.1 and 3.1).

2. Basic notations and auxiliary results

Here and in the sequel, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $p \in C^2[0,1]$, $q \in C^1[0,1]$, $r \in C[0,1]$ are regular functions with $p^- = \operatorname{ess\,inf}_{[0,1]} p > 0$, and λ is a real positive parameter. Consider the problem

$$\begin{cases} (p(t)u'')'' - (q(t)u')' + r(t)u = \lambda f(t,u) \quad \text{in} \quad [0,1] \times \mathbb{R} \\ u(0) = u(1) = 0; \\ u''(0) = u''(1) = 0. \end{cases} \tag{P_λ}$$

In this section, we give basic notations and auxiliary results for inserting the (P_λ) into a variational framework. Let X be the Sobolev space $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ endowed with the norm

$$\|u\| = \left(\|u''\|_2^2 + \|u'\|_2^2 + \|u\|_2^2)\right)^{1/2} \quad \forall\, u \in X,$$

where $\|\cdot\|_2$ is the usual norm in $L^2(a,b)$. It is well known that $\|\cdot\|$ is induced by the inner product

$$\langle u, v \rangle = \int_a^b (u''(x)v''(x) + u'(x)v'(x) + u(x)v(x))\, dx \quad \forall\, u,\ v \in X.$$

We point out the following Poincaré type inequalities (see Peletier, Troy, and van der Vorst 1995; Bonanno and Livrea 2021b).

Proposition 2.1. *For every $u \in X$, if $k = 1/\pi^2$, one has*

$$\|u^{(i)}\|_2^2 \le k^{j-i} \|u^{(j)}\|_2^2, \quad i = 0,1 \ j = 1,2 \text{ with } i < j. \tag{0}$$

Now, defining p^- as above, $q^- = \operatorname{ess\,inf}_{[0,1]} q$ and $r^- = \operatorname{ess\,inf}_{[0,1]} r$, consider the following set of conditions according to their signs:

$(H)_1$ $p^- > 0$, $q^- \ge 0$, $r^- \ge 0$;
$(H)_2$ $p^- > 0$, $q^- < 0$, $r^- \ge 0$ *and* $p^- + q^- k > 0$;
$(H)_3$ $p^- > 0$, $q^- \ge 0$, $r^- < 0$ *and* $p^- + r^- k^2 > 0$.
$(H)_4$ $p^- > 0$, $q^- < 0$, $r^- < 0$ *and* $p^- + q^- k + r^- k^2 > 0$.

Moreover, consider the following condition:

(H) $\min\{p^- + q^- k,\ p^- + r^- k^2,\ p^- + q^- k + r^- k^2\} > 0$.

Put

$$\sigma := \min\{p^-, p^- + q^- k,\ p^- + r^- k^2,\ p^- + q^- k + r^- k^2\}. \tag{1}$$

Clearly, assuming the condition (H) means that $\sigma > 0$. Moreover, an easy computation shows the following result.

Proposition 2.2. *Condition (H) holds if and only if one of conditions $(H)_1 - (H)_4$ holds.*

We now will introduce a convenient norm, equivalent to $\|\cdot\|$, that still makes X a Hilbert space. For this reason, for p, q, r as fixed above, let us define the function $N : X \to \mathbb{R}$ by putting

$$N(u) = \int_0^1 (p(t)|u''(t)|^2 + q(t)|u'(t)|^2 + r(t)|u|^2)dt$$

for every $u \in X$. We have the following proposition which will useful to verify that $\sqrt{N(\cdot)}$ is a norm equivalent to the usual one.

Proposition 2.3. *Assume (H). Then, there exits $m > 0$ such that*

$$N(u) \geq m\|u\|^2 \quad \forall\, u \in X, \tag{2}$$

with $m = \frac{\sigma}{1+k+k^2}$. Moreover, one has

$$N(u) \geq \sigma\|u''\|_2^2 \quad \forall\, u \in X. \tag{3}$$

Proof. From Proposition **2.1** one has $\|u\|_2^2 \leq k^2\|u''\|_2^2$ and $\|u'\|_2^2 \leq k\|u''\|_2^2$ for which $\|u\|_2^2 + \|u'\|_2^2 + \|u''\|_2^2 \leq k^2\|u''\|_2^2 + k\|u''\|_2^2 + \|u''\|_2^2 \leq (1+k+k^2)\|u''\|_2^2$ and the following inequality

$$\|u''\|_2^2 \geq \frac{1}{1+k+k^2}\|u\|^2 \tag{4}$$

follows. Moreover, in this proof, we take Proposition **2.2** into account.

Assume $(H)_1$.

One has $N(u) = \int_0^1 (p(t)|u''(t)|^2 + q(t)|u'(t)|^2 + r(t)|u|^2)dt \geq p^-\|u''\|_2^2$. So, from (4) follows that $N(u) \geq p^-\|u''\|_2^2 \geq \frac{p^-}{1+k+k^2}\|u\|^2$. Therefore, one has

$$N(u) \geq p^-\|u''\|_2^2 \qquad \text{and} \qquad N(u) \geq \frac{p^-}{1+k+k^2}\|u\|^2.$$

Hence, since in this case $\sigma = p^-$ being $q^-, r^- \geq 0$, the conclusion is achieved.

Assume $(H)_2$.

One has $N(u) = \int_0^1 (p(t)|u''(t)|^2 + q(t)|u'(t)|^2 + r(t)|u|^2)dt \geq p^-\|u''\|_2^2 + q^-\|u'\|_2^2$. So, from Proposition **2.1** it follows that $N(u) \geq p^-\|u''\|_2^2 + q^-k\|u''\|_2^2 = (p^- + q^-k)\|u''\|_2^2$ for which, arguing again from (4), one has $N(u) \geq \frac{p^-+q^-k}{1+k+k^2}\|u\|^2$. Therefore, one has

$$N(u) \geq (p^- + q^-k)\|u''\|_2^2 \qquad \text{and} \qquad N(u) \geq \frac{p^- + q^-k}{1+k+k^2}\|u\|^2.$$

Hence, since in this case $\sigma = p^- + q^-k$ being $q^- < 0$, $r^- \geq 0$, the conclusion is achieved.

Assume $(H)_3$.

One has $N(u) = \int_0^1 (p(t)|u''(t)|^2 + q(t)|u'(t)|^2 + r(t)|u|^2)dt \geq p^-\|u''\|_2^2 + r^-\|u\|_2^2$. So, from Proposition **2.1** it follows that $N(u) \geq p^-\|u''\|_2^2 + r^-k^2\|u''\|_2^2 = (p^- + r^-k^2)\|u''\|_2^2$ for which, again from (4), one has $N(u) \geq \frac{p^-+r^-k^2}{1+k+k^2}\|u\|^2$. Therefore, one has

$$N(u) \geq (p^- + r^-k^2)\|u''\|_2^2 \qquad \text{and} \qquad N(u) \geq \frac{p^- + r^-k^2}{1+k+k^2}\|u\|^2.$$

Hence, since in this case $\sigma = p^- + r^-k^2$ being $q^- \geq 0$, $r^- < 0$, the conclusion is achieved.

Assume $(H)_4$.
One has $N(u)=\int_0^1(p(t)|u''(t)|^2+q(t)|u'(t)|^2+r(t)|u|^2)dt \geq p^-\|u''\|_2^2+q^-\|u'\|_2^2+r^-\|u\|_2^2$. So, from Proposition **2.1** it follows that $N(u)\geq p^-\|u''\|_2^2+q^-k\|u'\|_2^2+r^-k^2\|u''\|_2^2=(p^-+q^-k+r^-k^2)\|u''\|_2^2$ for which, again from (4), one has $N(u)\geq \frac{p^-+q^-k+r^-k^2}{1+k+k^2}\|u\|^2$. Therefore, one has

$$N(u)\geq (p^-+q^-k+r^-k^2)\|u''\|_2^2 \qquad \text{and} \qquad N(u)\geq \frac{p^-+q^-k+r^-k^2}{1+k+k^2}\|u\|^2.$$

Hence, since in this case $\sigma=p^-+q^-k+r^-k^2$ being $q^-<0$, $r^-<0$, the conclusion is achieved. □

Proposition 2.4. *Assume that (H) holds and put*

$$\|u\|_X=\sqrt{N(u)}, \quad \forall\, u\in X. \tag{5}$$

Then, $\|\cdot\|_X$ is a norm equivalent to the usual one defined in (0) and $(X,\|\cdot\|_X)$ is a Hilbert space.

Proof. The definition of N and Proposition **2.3** assure that $\|\cdot\|_X$ is the norm induced by the inner product

$$\langle\cdot,\cdot\rangle_X=\int_0^1(p(x)u''(x)v''(x)+q(x)u'(x)v'(x)+r(x)u(x)v(x))\,dx \quad \forall\, u,\, v\in X.$$

It is simple to observe that there exists $M>0$ such that

$$\|u\|_X^2\leq M\|u\|^2 \tag{6}$$

for every $u\in X$. Hence, the equivalence is an immediate consequence of (2) and (6) and the proof is complete. □

Now, assuming again (H), put

$$\delta=\sqrt{\sigma}=\Big(\min\{p^-,p^-+q^-k,\ p^-+r^-k^2,\ p^-+q^-k+r^-k^2\}\Big)^{1/2}. \tag{7}$$

The constant δ is well-defined being $\sigma>0$ since (H) holds.
The following proposition will be useful in next section.

Proposition 2.5. *Assume that (H) holds. One has*

$$\|u\|_\infty\leq \frac{1}{2\pi\delta}\|u\|_X, \quad \forall\, u\in X. \tag{8}$$

Proof. It follows from Proposition **2.1** and inequality (3), taking into account of the well-known inequality $\|u\|_\infty\leq\frac{1}{2}\|u'\|_2$ for all $u\in X$. Indeed, one has $\|u\|_\infty\leq\frac{1}{2}\|u'\|_2\leq\frac{1}{2}\sqrt{k}\|u''\|_2\leq\frac{1}{2}\frac{1}{\pi}\frac{1}{\sqrt{\sigma}}\|u\|_X$, that is, the conclusion.

□

In order to clarify the variational structure of problem (P_λ), we introduce the functionals Φ, $\Psi : X\to\mathbb{R}$ defined by putting

$$\Phi(u)=\frac{1}{2}\|u\|_X^2, \quad \Psi(u)=\int_0^1 F(x,u(x))\,dx \quad \forall\, u\in X, \tag{9}$$

where $F(x,t) = \int_0^t f(x,s)\, ds$ for every $(x,t) \in [0,1] \times \mathbb{R}$. With standard arguments one can verify that Φ and Ψ are continuously Gâteaux differentiable, being in particular

$$\Phi'(u)(v) = \int_0^1 \big(p(x)u''(x)v''(x) + q(x)u'(x)v'(x) + r(x)u(x)v(x)\big)\, dx$$

and

$$\Psi'(u)(v) = \int_0^1 f(x,u(x))v(x)\, dx$$

for every $u,\ v \in X$. Recall that a weak solution of problem (P_λ) is any $u \in X$ such that

$$\int_0^1 \big(p(x)u''(x)v''(x) + q(x)u'(x)v'(x) + r(x)u(x)v(x)\big)\, dx = \lambda \int_0^1 f(x,u(x))v(x)\, dx, \forall v \in X.$$

Hence, the weak solutions of (P_λ) are exactly the critical points of the functional $\Phi - \lambda\Psi$.

Finally, we point out the following proposition.

Proposition 2.6. *Every critical point of $I_\lambda = \Phi - \lambda\Psi$, where Φ, Ψ are defined by* (9)*, is also a classical solution of* (P_λ).

Proof. By standard computations it is proved that every weak solution is also classical (see, for instance, Bonanno and Livrea 2021b, Proposition 2.7) and therefore, as observed above, we obtain the conclusion. ☐

The main tools in our approach are critical point theorems. First, we recall some general definitions. Let $(X, \|\cdot\|)$ be a Banach space, its dual space is X^* and the corresponding duality pairing is denoted by $\langle\cdot,\cdot\rangle$. Let $I : X \to \mathbb{R}$ be a Gâteaux differentiable functional. We say that I satisfies the Palais-Smale condition, (in short $(PS)-$condition), if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

(P_1) $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded,
(P_2) $\{I'(u_n)\}_{n\in\mathbb{N}}$ converges to 0 in X^*,

admits a convergent subsequence in X.

Moreover, fixed $r \in \mathbb{R}$, we say that $I = \Phi - \Psi$ satisfies the $(PS)^{[r]}$-condition if any sequence $\{u_n\}$ such that

(P_1) $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded,
(P_2) $\{I'(u_n)\}_{n\in\mathbb{N}}$ converges to 0 in X^*,
(P_3) $\Phi(u_n) < r \quad \forall\, n \in \mathbb{N}$

has a convergent subsequence. Now, put

$$\underline{\varphi}(r) := \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r}; \qquad \overline{\varphi}(r) := \sup_{u\in\Phi^{-1}(]0,r[)} \frac{\Psi(u)}{\Phi(u)}$$

The following theorem obtained by Bonanno (2012b, Theorem 2.3), as a consequence of a theorem proved by Bonanno (2012a, Theorem 4.1), guarantees the existence of at least one non-zero local minimum.

Theorem 2.1. *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exists $r > 0$ such that*

$$\underline{\varphi}(r) < \overline{\varphi}(r), \tag{10}$$

and for each $\lambda \in \Lambda_r := \left]\frac{1}{\overline{\varphi}(r)}, \frac{1}{\underline{\varphi}(r)}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition. Then, for each $\lambda \in \Lambda_r$ there is $u_\lambda \in \Phi^{-1}(]0,r[)$ (hence, $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0,r[)$ and $I'_\lambda(u_\lambda) = 0$.

The following theorem established by Bonanno and D'Aguì (2016) guarantees two non-zero critical points. It is a consequence (not immediate) of the previous non-zero local minimum theorem and of the classical Ambrosetti-Rabinowitz theorem established by Ambrosetti and Rabinowitz (1973) (see also: Rabinowitz 1986; Bonanno and Livrea 2021a).

Theorem 2.2. *Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two functions of class C^1 such that $\inf_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there is $r \in \mathbb{R}$ such that* (10) *holds and for each $\lambda \in \Lambda_r := \left]\frac{1}{\overline{\varphi}(r)}, \frac{1}{\underline{\varphi}(r)}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)-$condition and it is unbounded from below. Then, for each $\lambda \in \Lambda_r$, the function I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2} \in X$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.*

3. Main results

In this section, we establish the main results of the paper. First, put

$$K = \frac{2\delta^2\pi^2}{\frac{2048}{27}p^+ + \frac{32}{9}q^+ + \frac{13}{40}r^+},$$

where δ is given by (7) and p^+, q^+, r^+ are the $\operatorname{ess\,sup}$ in $[0,1]$ of the functions p, q, r respectively. The constant K plays an important role in the following statements since it regulates the growth less than quadratic of the nonlinearity in a suitable range (see (11) below), which is a fundamental assumption of our results. Moreover, put

$$A(c) = \int_0^1 \max_{|s| \leq c} F(t,s)dt,$$

with $c > 0$, and

$$B(d) = \int_{3/8}^{5/8} F(t,d)dt + \inf_{s \in [0,d]} \left(\int_0^{3/8} F(t,s)dt + \int_{5/8}^1 F(t,s)dt \right),$$

with $d > 0$.

Our first result is the following.

Theorem 3.1. *Assume that there exist two positive constants c, d, with $d < c$ such that*

$$\frac{A(c)}{c^2} < K\frac{B(d)}{d^2}. \tag{11}$$

Then, for each $\lambda \in \left]\frac{2\delta^2\pi^2}{K}\frac{d^2}{B(d)}, 2\delta^2\pi^2\frac{c^2}{A(c)}\right[$, problem (P_λ) admits at least one non-zero solution $\bar{u} \in C^4([0,1])$ such that $\|\bar{u}\|_\infty < c$.

Proof. Fix λ as in the conclusion. Put $X = W^{2,2}(0,1) \cap W_0^{1,2}(0,1)$ endowed with the norm $\|\cdot\|_X$ as given in Proposition **2.4**, Φ, Ψ as defined in (9) and $I_\lambda = \Phi - \lambda\Psi$. Our aim is to apply Theorem **2.1**. Clearly, Φ, Ψ satisfy all regularity assumptions. Moreover, by standard computations, I_λ satisfies the $(PS)^{[r]}$–condition for all $r > 0$ (see Bonanno 2012a, Remark 2.1). Now, we prove (10). To this end, put $r = 2\delta^2\pi^2c^2$. On the one hand, one has

$$\underline{\varphi}(r) = \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \leq \frac{1}{2\delta^2\pi^2}\frac{A(c)}{c^2}. \tag{12}$$

Indeed, one has $\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \leq \sup\limits_{\|u\|_X\leq\sqrt{2r}} \int_0^1 F(t,u(t))dt \leq \int_0^1 \max\limits_{|s|\leq c} F(t,s)dt = A(c)$, bearing in mind that, owing to Proposition **2.5**, one has $\|u\|_\infty \leq \frac{1}{2\pi\delta}\|u\|_X \leq \frac{1}{2\pi\delta}\sqrt{2r} = c$. On the other hand, we claim that

$$\overline{\varphi}(r) = \sup_{u\in\Phi^{-1}(]0,r[)} \frac{\Psi(u)}{\Phi(u)} \geq \frac{1}{2\delta^2\pi^2}K\frac{B(d)}{d^2}. \tag{13}$$

For our goal, put

$$v_d(t) = \begin{cases} -\frac{64}{9}d(t^2 - \frac{3}{4}t) & t \in [0, \frac{3}{8}[\\ d & t \in [\frac{3}{8}, \frac{5}{8}] \\ -\frac{64}{9}d(t^2 - \frac{5}{4}t + \frac{1}{4}) & t \in]\frac{5}{8}, 1]. \end{cases}$$

Clearly, $v_d \in X$ and $\|v_d\|_X^2 \leq \left(\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+\right)d^2 = \frac{4\delta^2\pi^2}{K}d^2$. Now, taking into accont that from $d < c$ one has $d < \sqrt{K}c$ owing to (11), it follows that $\Phi(v_d) < r$. Moreover, one has

$$\Psi(v_d) = \int_0^{3/8} F(t,v_d(t))dt + \int_{3/8}^{5/8} F(t,d)dt + \int_{5/8}^1 F(t,v_d(t))dt$$

$$\geq \int_{3/8}^{5/8} F(t,d)dt + \inf_{s\in[0,d]}\left(\int_0^{3/8} F(t,s)dt + \int_{5/8}^1 F(t,s)dt\right) = B(d),$$ so that

$\dfrac{\Psi(v_d)}{\Phi(v_d)} \geq \dfrac{1}{2\delta^2\pi^2}K\dfrac{B(d)}{d^2}$ and our claim is proved.

At this point, from (12), (13) and from the choice of λ, one has

$$\underline{\varphi}(r) = \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \leq \frac{1}{2\delta^2\pi^2}\frac{A(c)}{c^2} < \frac{1}{\lambda} < \frac{1}{2\delta^2\pi^2}K\frac{B(d)}{d^2} \leq \overline{\varphi}(r)$$

and (10) is achieved.

Hence, Theorem **2.1** ensures that I_λ admits a local minimum $\bar{u}$ which, owing to Proposition **2.6** is a classical solution of problem (P_λ). Finally, from Proposition **2.5** one has $\|\bar{u}\|_\infty < c$ and the proof is completed.

□

Remark 3.1. Condition (11) expresses a suitable growth of the primitive F (see (14) in Remark **3.3**) which can be read as a growth less than quadratic of the primitive in a suitable real interval $[d,c]$ with respect to second variable (see also (15) in Remark **3.4**). Roughly speaking, it can be translated as a growth less than linear of the nonlinearity f in $[d,c]$. In particular, that is by picking $d \to 0^+$, it is satisfied when f is sublinear at 0 in the second variable uniformly respect to the first (see (17) in Remark **3.4**).

Remark 3.2. The conclusion of Theorem **3.1** can be further specified. In fact, since $\bar{u}$ is a local minimum for the function I_λ, we have in addition that

$$\frac{1}{2}\int_0^1 (p(t)|\bar{u}''(t)|^2 + q(t)|\bar{u}'(t)|^2 + r(t)|\bar{u}|^2)dt - \lambda\int_0^1 F(t,\bar{u}(t))dt \leq$$

$$\leq \frac{1}{2}\int_0^1 (p(t)|u''(t)|^2 + q(t)|u'(t)|^2 + r(t)|u|^2)dt - \lambda\int_0^1 F(t,u(t))dt$$

for all $u \in X$ such that $\|u\|_X < (2\delta\pi)c$.

Our second result is the following.

Theorem 3.2. *Assume that there exist two positive constants c,d, with $d < c$ such that* (11) *holds and there are $\eta > 2$, $T > 0$ such that*

$$0 < \eta F(t,\xi) \leq \xi f(t,\xi) \qquad \forall t \in [0,1], \forall|\xi| > T. \tag{AR}$$

Then, for each $\lambda \in \left]\frac{2\delta^2\pi^2}{K}\frac{d^2}{B(d)}, 2\delta^2\pi^2\frac{c^2}{A(c)}\right[$, problem (P_λ) admits at least two non-zero solutions u_1, u_2 such that

$$\frac{1}{2}\int_0^1 (p(t)|u_1''(t)|^2 + q(t)|u_1'(t)|^2 + r(t)|u_1|^2)dt - \lambda\int_0^1 F(t,u_1(t))dt < 0$$

and

$$\frac{1}{2}\int_0^1 (p(t)|u_2''(t)|^2 + q(t)|u_2'(t)|^2 + r(t)|u_2|^2)dt - \lambda\int_0^1 F(t,u_2(t))dt > 0.$$

Proof. Our aim is to apply Theorem **2.2**. To this end, we fix λ as in the conclusion and we define X, Φ, Ψ and I_λ as done in the proof of Theorem **3.1**. Therefore, the same computations of the previous proof ensure that (10) is verified. Further, condition (AR), by standard computations, implies that I_λ satisfies the $(PS)-$ condition and it is unbounded from below (see, for instance, Rabinowitz 1986). Hence, Theorem **2.2** establish the existence of two non-zero critical points of I_λ which, owing to Proposition **2.6**, are classical solutions of problem (P_λ) and the conclusion is obtained.

□

Remark 3.3. If the function f is nonnegative in $[0,1]\times[0,+\infty[$, then one has

$$\inf_{s\in[0,d]}\left(\int_0^{3/8} F(t,s)dt+\int_{5/8}^1 F(t,s)dt\right)\geq 0.$$

Therefore, in this case, to verify (11) it is enough to assume that the following inequality

$$\frac{\int_0^1 F(t,c)dt}{c^2}<K\frac{\int_{3/8}^{5/8} F(t,d)dt}{d^2} \tag{14}$$

holds.

Remark 3.4. If the function f is with separable variables, that is,

$$f(t,s)=h(t)g(s),$$

where $h:[0,1]\to\mathbb{R}$, $g:[0,+\infty[\to\mathbb{R}$ are two nonnegative continuous functions, then to verify (11) it is enough to prove that the following inequality

$$\frac{G(c)}{c^2}<H\frac{G(d)}{d^2} \tag{15}$$

holds, where $G(s)=\int_0^s g(\xi)d\xi$ and H is the constant $\dfrac{\int_{3/8}^{5/8} h(t)dt}{\|h\|_1}K$.

Further notice that the previous condition is more general than the following

$$\lim_{s\to 0^+}\frac{g(s)}{s}=+\infty \tag{16}$$

and, in this case the conclusions of the previous theorems hold for each

$$\lambda\in\left]0,2\delta^2\pi^2\frac{c^2}{\|h\|_1 G(c)}\right[$$

for all $c>0$. Indeed, fix $c>0$ and $\lambda>0$ such that $\dfrac{1}{\lambda}>\|h\|_1\dfrac{1}{2\delta^2\pi^2}\dfrac{G(c)}{c^2}$, from $\displaystyle\lim_{s\to 0^+}\frac{G(s)}{s^2}=\frac{1}{2}\lim_{s\to 0^+}\frac{g(s)}{s}=+\infty$ there is $\bar{d}>0$ such that $\dfrac{1}{2\delta^2\pi^2}K\int_{3/8}^{5/8} h(t)dt\dfrac{G(s)}{s^2}>\dfrac{1}{\lambda}$ for all $s<\bar{d}$ for which there is $d<\min\{\bar{d},c\}$ such that $\dfrac{1}{2\delta^2\pi^2}K\int_{3/8}^{5/8} h(t)dt\dfrac{G(d)}{d^2}>\dfrac{1}{\lambda}>\|h\|_1\dfrac{1}{2\delta^2\pi^2}\dfrac{G(c)}{c^2}$

and our claim is proved. Hence, when the nonlinearity g is sublinear at zero, then problem (P_λ) admits at least one non-zero solution.

Finally, we emphasize that, arguing as above, the condition

$$\lim_{s\to 0^+} \frac{f(t,s)}{s} = +\infty \quad \textit{uniformly with respect to } t \in [0,1] \tag{17}$$

implies that assumption (14) is verified.

Remark 3.5. We observe that the previous theorems are true also for a Carathéodory function f and $p \in W^{2,1}([0,1])$, $q \in W^{1,1}([0,1])$, $r \in L^\infty([0,1])$. Clearly, in this case, the solutions are generalized (see Bonanno and Di Bella 2008).

Example 3.1. Consider the following problem where the equation is expressed in a complete form

$$\begin{cases} (1+t^2)u^{(iv)} + 4tu''' + (2-t)u'' - u' + t^3 u = t^2 \max\{\sqrt{|u|}, u^2\} & \text{in} \quad [0,1]\times\mathbb{R} \\ u(0) = u(1) = 0; \\ u''(0) = u''(1) = 0. \end{cases}$$

Owing to Theorem **3.1** (see Remark **3.4**), it admits at least one non-zero classical solution. Indeed, it is enough to pick $p(t) = (1+t^2)$; $q(t) = t$; $r(t) = t^3$; $h(t) = t^2$ for all $t \in [0,1]$ and

$$g(t) = \begin{cases} \sqrt{|u|} & u \in [-1,1] \\ u^2 & \text{otherwise.} \end{cases}$$

It follows that $\delta = 1$ and, by choosing $c = \frac{\sqrt[3]{4}}{2}$, one has $2\delta^2\pi^2 \dfrac{c^2}{\|h\|_1 G(c)} = 2^{5/3}\pi^2 > 1$ for which we can apply our theorem for $\lambda = 1$. We explicitly observe that Theorem 3.1 of Bonanno, Di Bella, and O'Regan (2011) cannot be applied to such a problem, since the coefficients are not constant.

We conclude this section with the renowned case of combined effects of concave and convex nonlinearities, introduced for the first time by Ambrosetti, Brézis, and Cerami (1994) for second order elliptic equations. Consider the problem

$$\begin{cases} (p(t)u'')'' - (q(t)u')' + r(t)u = \mu|u|^{s-1}u + |u|^{v-1}u & \text{in} \quad [0,1]\times\mathbb{R} \\ u(0) = u(1) = 0; \\ u''(0) = u''(1) = 0, \end{cases} \tag{ABC}$$

with $0 < s < 1 < v$.

We have the following result.

Theorem 3.3. *There is $\mu^* > 0$ such that for each $\mu \in]0, \mu^*[$ the problem (ABC) admits two non-zero classical solutions.*

Proof. Given the problem

$$\begin{cases} (p(t)u'')'' - (q(t)u')' + r(t)u = \lambda\left(\mu|u|^{s-1}u + |u|^{\nu-1}u\right) \quad \text{in} \quad [0,1]\times\mathbb{R} \\ u(0) = u(1) = 0; \\ u''(0) = u''(1) = 0, \end{cases} \tag{ABC_λ}$$

we apply Theorem **3.2** by choosing $g(r) = \mu|r|^{s-1}r + |r|^{\nu-1}r$ for all $r \in \mathbb{R}$ (see Remark **3.4**). The assumption (16) is verified since

$$\lim_{r\to 0^+} \frac{g(r)}{r} = \lim_{r\to 0^+} \frac{\mu r^s + r^\nu}{r} = +\infty.$$

Moreover, by a simple computation (AR) is satisfied (see also Ambrosetti, Brézis, and Cerami 1994). Therefore, Theorem **3.2** ensures two non-zero classical solutions to problem (ABC_λ) for all $\lambda \in]0, \lambda^*[$, where

$$\lambda^* = \left(\frac{1}{\mu}\right)^{\frac{\nu-1}{\nu-s}} 2\delta^2\pi^2 \frac{\left[(s+1)^{(\nu-1)}(\nu+1)^{(1-s)}(1-s)^{(1-s)}(\nu-1)^{(\nu-1)}\right]^{\frac{1}{\nu-s}}}{\nu-s}.$$

Indeed, $\lambda^* = 2\delta^2\pi^2 \dfrac{1}{\|h\|_1} \sup\limits_{c>0} \dfrac{c^2}{G(c)} = 2\delta^2\pi^2 \max\limits_{c>0} \dfrac{c^2}{\mu\frac{c^{s+1}}{s+1} + \frac{c^{\nu+1}}{\nu+1}} = 2\delta^2\pi^2 \dfrac{\bar{c}^2}{\mu\frac{\bar{c}^{s+1}}{s+1} + \frac{\bar{c}^{\nu+1}}{\nu+1}}$, where $\bar{c} = \left(\mu\dfrac{(\nu+1)(1-s)}{(s+1)(\nu-1)}\right)^{\frac{1}{\nu-s}}$. Therefore, taking into account that $\lambda^* > 1$ if and only if $\mu < \mu^*$, where

$$\mu^* = (2\delta^2\pi^2)^{\frac{\nu-s}{\nu-1}} \frac{(s+1)(\nu+1)^{\frac{1-s}{\nu-1}}(1-s)^{\frac{1-s}{\nu-1}}(\nu-1)}{(\nu-s)^{\frac{\nu-s}{\nu-1}}},$$

problem (ABC_λ) admits two non-zero classical solutions for $\lambda = 1$, provided that $\mu < \mu^*$. Hence, the conclusion is achieved. □

Remark 3.6. We observe that the proof, in addition to guaranteeing the existence of μ^*, provides its precise numerical value.

Remark 3.7. The Example in the Introduction is obtained by Theorem **3.3** by choosing $p(t) = \sqrt{1+t}$, $q(t) = \sqrt{1+t^2}$, $r(t) = \sqrt{t}$ for all $t \in [0,1]$. Indeed, in this case, one has $\delta = 1$ and $\mu^* = \frac{15}{4}\pi^4$.

Remark 3.8. We observe that Theorem 3.1 of D'Aguì, Di Bella, and Winkert (2019) cannot be applied to study problems as (ABC_λ) which are in presence of non-constant coefficients and with odd derivatives.

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[a] Università degli Studi di Messina,
Dipartimento di Ingegneria,
C/da Di Dio, Sant'Agata, 98166 Messina, Italy

[b] National University of Ireland,
School of Mathematical and Statistical Sciences,
Galway, Ireland

[*] To whom correspondence should be addressed | email: gabriele.bonanno@unime.it

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