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STAR COVERING PROPERTIES AND NEIGHBORHOOD ASSIGNMENTS

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ABSTRACT. Following the idea of van Mill, Tkachuk, and Wilson (2007), we define a space X to be *neighborhood assignment* $\mathscr{P}$ (briefly NA-$\mathscr{P}$, where $\mathscr{P}$ is a class of topological spaces) if for any neighbourhood assignment $\{O_x : x \in X\}$ there is a subspace $Y \subset X$ such that Y has the property $\mathscr{P}$ and $\bigcup_{x\in Y} O_x = X$. In this paper we investigate the classes of NA-$\mathscr{P}$ spaces and compare them with the star covering properties for some class of topological spaces $\mathscr{P}$.

1. Introduction

In 2007 van Mill, Tkachuk, and Wilson gave the following definition which represents a development of an idea of van Douwen (1977) used to define D-spaces. A class $\mathscr{P}^*$ is *dual to a class* $\mathscr{P}$ (with respect to neighbourhood assignments) if a space X belongs to $\mathscr{P}^*$ if and only if X is NA-$\mathscr{P}$; if X is a member of the class $\mathscr{P}^*$, then X is called *dually* $\mathscr{P}$. In this paper we express the idea of van Mill, Tkachuk, and Wilson (2007) in the following way.

Definition 1. A space X is called *neighborhood assignment* $\mathscr{P}$ (briefly NA-$\mathscr{P}$) if for any neighbourhood assignment $\{O_x : x \in X\}$ there is a subspace $Y \subset X$ such that Y has the property $\mathscr{P}$ and $\bigcup_{x\in Y} O_x = X$.

Recall the following. Let $\mathscr{U}$ be a cover of a space X and let M be a subset of X; the star of M with respect to $\mathscr{U}$ is the set $St(M,\mathscr{U}) = \bigcup\{U : U \in \mathscr{U} \text{ and } U \cap M \neq \emptyset\}$. The star of a one-point set $\{x\}$ with respect to a cover $\mathscr{U}$ is called the star of the point x with respect to $\mathscr{U}$ and it is denoted by $St(x,\mathscr{U})$.

Definition 2. (Ikenaga 1990) *A space X has the star-$\mathscr{P}$ property (briefly St-$\mathscr{P}$) if for every open cover $\mathscr{U}$ of the space X, there exists a subset Y of X with the property $\mathscr{P}$ such that $St(Y,\mathscr{U}) = X$.*

It is obvious that if a space X has a dense subspace with the property $\mathscr{P}$, then it is star-$\mathscr{P}$. Star covering properties, relative versions and selective versions on them have been widely studied in literature (see, for example, Bonanzinga 1998; Bonanzinga, Giacopello, and Maesano 2022; Bonanzinga and Maesano 2021, 2022; Bonanzinga and Matveev 2000, 2009; van Douwen *et al.* 1991; Ikenaga 1983, 1990; Matveev 1998).

In this paper we study the previous classes of spaces for some covering property $\mathscr{P}$. In Section **2**, we give a diagram (Diagram 1) summarizing the main relations between the considered properties and present several examples distinguishing almost all of them. In Section **3**, we give a description of the previous properties in terms of cardinal functions and generalize known results. In Section **4** we present some open problems.

1.1. Notation and terminology. Our terminology is standard and follows Engelking (1989) and Hart *et al.* (2003). Recall that a space X is: *compact* (resp. *Lindelöf*) (briefly C, resp. L) if for every open cover $\mathscr{U}$ of X there exists a finite (countable) subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $X = \bigcup \mathscr{V}$; *countably compact* (briefly CC) if for every countable open cover $\mathscr{U}$ of X there exists a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $X = \bigcup \mathscr{V}$; σ*-compact* (briefly σC) if it is the union of countably many compact subsets; *paracompact* (briefly PC) if every open cover has a locally finite open refinement; *metacompact* (briefly MC) if every open cover has a point-finite open refinement; *metaLindelöf* (briefly ML) if every open cover has a point-countable open refinement; *Linearly Lindelöf* (briefly LL) if for every linearly ordered open cover $\mathscr{U}$ of X there exists a countable subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $X = \bigcup \mathscr{V}$; *Menger* (briefly M) if for every sequence of open cover $(\mathscr{U}_n : n \in \omega)$ of X there exists a finite subfamily $\mathscr{V}_n$ of $\mathscr{U}_n$ for every $n \in \omega$ such that $X = \bigcup_{n\in\omega} \bigcup \mathscr{V}_n$.

2. Neighbourhood assignments and expansion operators

For a set X we use $P(X)$ to denote the family of all subsets of X.

Definition 3. Let X be a topological space.

(i) A *neighbourhood assignment of* X is a map $\mathscr{N} : X \to P(X)$ such that $\mathscr{N}(x)$ is an open neighbourhood of x in X for every x.

(ii) For a topological space X, we denote by $\mathtt{NA}(X)$ the family of all open neighbourhood assignments of X.

Definition 4. Let X be a topological space. A map $\Phi : P(X) \times \mathtt{NA}(X) \to P(X)$ satisfying properties (i) and (ii) below will be called an *expansion operator on neighbourhood assignments of* X, or shortly, an *expansion operator on* X.

(i) $Y \subseteq \Phi(Y, \mathscr{N})$ for every $(Y, \mathscr{N}) \in P(X) \times \mathtt{NA}(X)$;

(ii) $Z \subseteq Y \subseteq X$ implies $\Phi(Z, \mathscr{N}) \subseteq \Phi(Y, \mathscr{N})$ for every $\mathscr{N} \in \mathtt{NA}(X)$.

Typical expansion operators we shall consider are the *neighbourhood assignement opeartor NA* defined by

$$NA(Y, \mathscr{N}) = \bigcup \{\mathscr{N}(y) : y \in Y\} \text{ for every } (Y, \mathscr{N}) \in P(X) \times \mathtt{NA}(X). \tag{1}$$

and the *star operator St* defined by

$$St(Y, \mathscr{N}) = \bigcup \{\mathscr{N}(x) : x \in X \text{ and } \mathscr{N}(x) \cap Y \neq \emptyset\} \text{ for every } (Y, \mathscr{N}) \in P(X) \times \mathtt{NA}(X). \tag{2}$$

Definition 5. Given two expansion operators $\Phi : P(X) \times \mathtt{NA}(X) \to P(X)$ and $\Psi : P(X) \times \mathtt{NA}(X) \to P(X)$, we write $\Phi \le \Psi$ provided that

$$\Phi(Y,\mathscr{N}) \subseteq \Psi(Y,\mathscr{N}) \text{ for every } (Y,\mathscr{N}) \in P(X) \times \mathtt{NA}(X). \tag{3}$$

Lemma 6. *$NA \le St$.*

Proof. Let X be a space and $(Y,\mathscr{N}) \in P(X) \times \mathtt{NA}(X)$. Fix an arbitrary $x \in NA(Y,\mathscr{N})$. It follows from (1) that $x \in \mathscr{N}(y)$ for some $y \in Y$. Since $y \in \mathscr{N}(y)$ by Definition 3 (i), we have $\mathscr{N}(y) \cap Y \neq \emptyset$. Therefore, $\mathscr{N}(y) \subseteq St(Y,\mathscr{N})$ by (2). Since $x \in \mathscr{N}(y)$, we conclude that $x \in St(Y,\mathscr{N})$. Since this inclusion holds for every $x \in NA(Y,\mathscr{N})$, this shows that $NA(Y,\mathscr{N}) \subseteq St(Y,\mathscr{N})$. Since this inclusion holds for every $(Y,\mathscr{N}) \in P(X) \times \mathtt{NA}(X)$, we have $NA \le St$ by Definition 5. □

Definition 7. If X is a space, Φ is an expansion operator on X and $\mathscr{N} \in \mathtt{NA}(X)$, then a subset Y of X is called a *Φ-core of $\mathscr{N}$* provided that $X = \Phi(Y,\mathscr{N})$.

Definition 8. Given a class $\mathscr{P}$ of topological spaces and an expansion operator Φ on a space X, we shall say that X is a *Φ-$\mathscr{P}$ space* provided that every $\mathscr{N} \in \mathtt{NA}(X)$ has a Φ-core which belongs to the class $\mathscr{P}$.

The proof of the following proposition is straightforward.

Proposition 9. *Let Φ be an expansion operator on a space X. If $\mathscr{P}$, $\mathscr{Q}$ are classes of spaces such that $\mathscr{P} \subseteq \mathscr{Q}$, and X is a Φ-$\mathscr{P}$ space, then X is an Φ-$\mathscr{Q}$ space.*

Proposition 10. *Let $\mathscr{P}$ be a class of topological spaces. Suppose that Φ and Ψ are expansion opearators satisfying $\Psi \le \Phi$. Then every Ψ-$\mathscr{P}$ space is also a Φ-$\mathscr{P}$ space.*

Proof. Let X be a Ψ-$\mathscr{P}$ space. Fix $\mathscr{N} \in \mathtt{NA}(X)$. By Definition 8, X has a Ψ-core Y which belongs to the class $\mathscr{P}$. By Definition 7, this means that $X = \Psi(Y,\mathscr{N})$. Since $\Psi \le \Phi$ by our assumption, we have $\Psi(Y,\mathscr{N}) \subseteq \Phi(Y,\mathscr{N})$ by Defnition 5. Since $\Phi(Y,\mathscr{N}) \in P(X)$, we get $\Phi(Y,\mathscr{N}) \subseteq X$. Combining the above inclusions, we obtain $X = \Phi(Y,\mathscr{N})$. By Definition 7, this means that Y is Φ-core for $\mathscr{N}$. We have proved that every $\mathscr{N} \in \mathtt{NA}(X)$ has a Φ-core Y which belongs to $\mathscr{P}$. By Definition 8, X is a Φ-$\mathscr{P}$ space. □

Corollary 11. *For every class $\mathscr{P}$ of topological spaces, each NA-$\mathscr{P}$ space is a St-$\mathscr{P}$ space.*

Proof. Indeed, $NA \le St$ by Lemma 6. Now the conclusion follows from Proposition 10. □

The implications of the following diagram are obvious.

$$\begin{array}{ccccccccccccc}
C & \longrightarrow & \sigma C & \longrightarrow & M & \longrightarrow & L & \xrightarrow{reg.} & PC & \longrightarrow & MC & \longrightarrow & ML \\
\updownarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
NA\text{-}C & \longrightarrow & NA\text{-}\sigma C & \longrightarrow & NA\text{-}M & \longrightarrow & NA\text{-}L & \xrightarrow{reg.} & NA\text{-}PC & \longrightarrow & NA\text{-}MC & \longrightarrow & NA\text{-}ML \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
NA\text{-}C & \longrightarrow & St\text{-}\sigma C & \longrightarrow & St\text{-}M & \longrightarrow & St\text{-}L & \xrightarrow{reg.} & St\text{-}PC & \longrightarrow & St\text{-}MC & \longrightarrow & St\text{-}ML \\
\downarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \downarrow & & \downarrow & & \downarrow \\
St_\omega\text{-}C & \longrightarrow & St_\omega\text{-}\sigma C & \longrightarrow & St_\omega\text{-}M & \longrightarrow & St_\omega\text{-}L & \xrightarrow{reg.} & St_\omega\text{-}PC & \longrightarrow & St_\omega\text{-}MC & \longrightarrow & St_\omega\text{-}ML
\end{array}$$

Diagram 1

Recall that (see also Kočinac 1999a,b, where a different terminology is used) a space X has the *Star*$_\omega$-$\mathscr{P}$ property (briefly St$_\omega$-$\mathscr{P}$) if for every sequence $(\mathscr{U}_n : n \in \omega)$ of open covers of the space X, there exist subsets Y_n of X, for every $n \in \omega$, having property $\mathscr{P}$ such that $\{\bigcup St(Y_n, \mathscr{U}_n) : n \in \omega\}$ is a cover of X. It is obvious that if the property $\mathscr{P}$ is closed under countable unions, then a space is St-$\mathscr{P}$ if and only if the space is St$_\omega$-$\mathscr{P}$. Matveev (1997) noted that in every T_1-space X for every open cover $\mathscr{U}$ of the space X, there exists a closed and discrete subset Y of X such that $St(Y, \mathscr{U}) = X$. Then, every T_1-space is St-PC (hence St$_\omega$-PC), St-MC (hence St$_\omega$-MC) and St-ML (hence St$_\omega$-ML).

We will denote by D the class of all discrete spaces. With an abuse of terminology we denote by CD the topological property to be closed and discrete (of course, being closed and discrete is technically not a topological property but one relative to the space in which the discrete space lies). Then we have the following diagram.

$$\text{NA-C} \overset{CC}{\Longleftrightarrow} \text{NA-CD} \Longrightarrow \text{NA-D} \Longrightarrow \text{NA-PC}.$$

Note that NA-D was introduced by Alas, Junqueira, and Wilson (2008) with a different terminology and NA-CD spaces are exactly D-spaces defined by van Douwen and Pfeffer (1997).

Now we give examples showing that some of the arrows of Diagram 1 can not be reversed. Before doing it we prove some useful results.

Proposition 12. *Let X be a NA-$\mathscr{P}$ space and C a closed and discrete subset of X. Then there exists a subset Y of X having the property $\mathscr{P}$ such that $Y \supset C$.*

Proof. We can consider the following neighborhood assignment. For every $x \in C$ we choose an open subset O_x such that $O_x \cap C = \{x\}$ and we consider $O_x = X \setminus C$ for every $x \in X \setminus C$. We put $\mathscr{O} = \{O_x : x \in X\}$. Since the space is NA-$\mathscr{P}$, there exists $Y \subset X$ having the property $\mathscr{P}$ such that $\bigcup_{x \in Y} O_x = X$. We want to show that $Y \supset C$. Let $x \in C$, then $\exists! O_x \in \mathscr{O}$ such that $x \in O_x$ therefore $x \in Y$. □

In particular by the proposition above we can prove the existence of spaces which do not have the NA-$\mathscr{P}$ property.

Corollary 13. *Let X be a NA-$\mathscr{P}$ space. If for every subset Y of X having the property $\mathscr{P}$ we have $e(Y) = \aleph_0$, then $e(X) = \aleph_0$.*

Corollary 14. *If X is NA-L, then $e(X) = \aleph_0$.*

Alas, Tkachuk, and Wilson (2009, Theorem 3.1) proved that every scattered space with finite height is NA-CD. Then, using Corollary 14, we obtain the following example.

Example 15. *Every Isbell-Mrowka space is a NA-CD but not NA-L space.*

Following step by step the proof of Alas, Tkachuk, and Wilson (2009, Theorem 2.4), we can give the following improvement of it.

Theorem 16. *Let $\mathscr{P}$ be a property that is hereditary with respect to closed subsets and preserved under the union of two disjoint subsets. If a space X is the union of two subspaces Y and Z, where Y is NA-$\mathscr{P}$ and Z is a closed subset of X such that for every open subset U of X containing Z, $X \setminus U$ is NA-$\mathscr{P}$. Then X is NA-$\mathscr{P}$.*

2.1. Examples.

Lemma 17. *The following hold for any topological space.*

(i) *An NA-ML, hereditarily separable space is NA-L.*
(ii) *A locally countable, NA-L space is countable.*
(iii) *A locally countable, hereditarily separable, NA-ML space is countable.*

Proof. (i) Let X be an NA-ML, hereditarily separable space, and let $\{O_x : x \in X\}$ be a neighbourhood assignment for X. Since X is NA-ML, we can find a metaLindelöf subspace Y of X such that

$$X = \bigcup\{O_y : y \in Y\}. \tag{4}$$

Since X is hereditarily separable, and every separable metaLindelöf space is Lindelöf, Y must be Lindelöf.

(ii) Let X be a locally countable, NA-L space. Since X is locally countable, we can fix a neighbourhood assignment $\{O_x : x \in X\}$ such that each O_x is countable. Since X is NA-L, there exists a Lindelöf subspace Y of X satisfying (4). Since Y is Lindelöf, the cover $\{O_y : y \in Y\}$ of Y has a countable subcover; that is,

$$Y \subseteq \bigcup\{O_y : y \in Z\} \tag{5}$$

for a countable subset Z of Y. Since each O_y is countable, it follows from (5) that Y is countable. By the same reason, it follows from (4) that X is countable.

(iii) Let X be a locally countable, hereditarily separable, NA-ML space. By item (i), X is NA-L. By item (ii), X is countable. □

Corollary 18. ω_1 *with the order topology is not NA-L.*

Proof. ω_1 is locally countable but not countable, so the conclusion follows from item (ii) of Lemma 17. □

Example 19. *A St-C space which is not NA-L.*

Indeed, ω_1 with the order topology is countably compact, hence since Hausdorff it is St-C. On the other hand, ω_1 is not NA-L by Corollary 18.

Under Jensen's Axiom $\diamond$, we have even an example with stronger properties.

Example 20. *(Under $\diamond$). A St-C space which is not NA-ML.*

Ostaszewski (1976) gives a Hausdorff, countably compact (hence St-C), hereditary separable space having cardinality ω_1. Therefore, X is not NA-ML by item (iii) of Lemma 17. A ZFC example of a non NA-ML space is constructed in Proposition 2.1(2) by Buzyakova, Tkachuk, and Wilson (2007).

Example 21. *A St-PC not St-L space.*

(This is the space of Example 2.2 given by Song and Zheng (2014), when $\mathfrak{c}$ is taken instead of ω_1) Let $A(\omega_1)$ be the Alexandroff (one-point) compactification of the discrete space ω_1. We may assume that ω_1 is the only non-isolated point of $A(\omega_1)$. Define $X = [0, \omega_1] \times A(\omega_1) \setminus \{(\omega_1, \omega_1)\}$.

(i) *X contains a dense paracompact subspace, so X is* St-PC. Indeed, the subspace $Y = [0, \omega_1] \times \omega_1$ of X is dense in X and homeomorphic to a disjoint sum of ω_1-many copies of the compact space $[0, \omega_1]$, so Y is paracompact.

(ii) *X is not* St-*L.*[1] Before proving this, we shall prove the following.

Claim 1. *For every Lindelöf subspace Y of X, there exists $\gamma \in \omega_1$ such that*

$$(0, \omega_1) \notin \{0\} \times \pi((X \setminus ([0, \gamma) \times A(\omega_1))) \cap Y), \tag{6}$$

where $\pi : [0, \omega_1] \times A(\omega_1) \to A(\omega_1)$ is the projection on the second coordinate.

Proof. Since Y is Lindelöf and $Z = [0, \omega_1) \times \{\omega_1\}$ is a closed subspace of X, $Y \cap Z$ is Lindelöf as well. Since Z is homeomorphic to the ordinal space $[0, \omega_1)$, the Lindelöf subspace $Y \cap Z$ of Z must be countable. Therefore, there exists $\gamma \in \omega_1$ such that $Y \cap Z \subseteq [0, \gamma) \times \{\omega_1\}$. Finally, note that this γ satisfies (6). □

For every $\alpha \in \omega_1$, the set $U_\alpha = (\alpha, \omega_1] \times \{\alpha\}$ is open in X. Furthermore, $V = [0, \omega_1) \times A(\omega_1)$ is an open subset of X. Therefore, $\mathscr{U} = \{U_\alpha : \alpha \in \omega_1\} \cup \{V\}$ is an open cover of X. Suppose that Y is a Lindelöf subspace of X. We are going to show that $\mathrm{St}(Y, \mathscr{U}) \neq X$. Let $\gamma \in \omega_1$ be the ordinal as in the conclusion of Claim 1. $Y' = (X \setminus ([0, \gamma) \times A(\omega_1))) \cap Y$ is a closed subset of Y, so it is Lindelöf. Since π is a continuous mapping, $\pi(Y')$ is Lindelöf subspace of $A(\omega_1)$. By the conclusion of Claim 1, $(0, \omega_1) \notin \{0\} \times \pi(Y')$. Therefore, $\{0\} \times \pi(Y')$ is a Lindelöf subspace of the discrete space $\{0\} \times \omega_1$, so $\{0\} \times \pi(Y')$ is countable. Therefore, we can find $\beta \in \omega_1$ such that $\{0\} \times \pi(Y') \subseteq \{0\} \times [0, \beta)$. Now let $\alpha = \max\{\beta, \gamma\}$. Note that $U_\alpha \cap Y = \emptyset$ by our construction.

We claim that $(\omega_1, \alpha) \in X \setminus \mathrm{St}(Y, \mathscr{U})$. To see this, it is sufficient to realize that U_α is the only element of $\mathscr{U}$ containing $(\omega_1, \alpha) \in X$. Since $U_\alpha \cap Y = \emptyset$, this means that $(\omega_1, \alpha) \notin \mathrm{St}(Y, \mathscr{U})$. □

Example 22. *A NA-D not St-L space.*

Let S be the set of isolated points in ω_1. Consider the set $X = (\omega_1 \times \omega) \cup (S \times \{\omega\})$ with the subspace topology inherited from the product $\omega_1 \times (\omega + 1)$ of two cardinals ω_1 and $\omega + 1$. This example is included by Alas, Junqueira, and Wilson (2011) as item (5) at page 623 and is attributed to an anonymous referee.

(i) *X is not St-L.* This is proved by Alas, Junqueira, and Wilson (2011).

(ii) *X is NA-D, and so NA-PC*. Indeed, the subset $Y = S \times \{\omega\}$ of X is discrete, so NA-D, and its complement $X \setminus Y = \omega_1 \times \omega$ is a disjoint sum of countably many copies of ω_1. Since the latter space is NA-D (see van Mill, Tkachuk, and Wilson 2007, Example 2.3), so is $X \setminus Y$. Now the conclusion of item (ii) follows from Theorem 16.

Example 23. *In some model of ZFC there exists a Urysohn space X which has a dense subspace homeomorphic to the space of irrational numbers (so X is a St-L not St-M space).*

Let P be the space of irrational numbers in its usual topology. Define $X = P \times (\omega + 1)$. For $(p, n) \in P \times \omega$, we declare

$$\{U \times \{n\} : p \in U \text{ and } U \text{ is open in } P\}$$

[1] Song and Zheng (2014) only proved that X is not St_ω-C.

to be the neighbourhood base of a point (p,n). Therefore, $Z = P\times\omega$ is a disjoint sum of countably many copes of P, so it is homeomorphic to P itself. For every $p\in P$, a basic open neighbourhood of (p,ω) in X is of the form

$$O(p,U,n,M) = \{(p,\omega)\}\cup((U\times(n,\omega))\setminus M), \tag{7}$$

where U is a clopen subset of P containing p, $n\in\omega$ and M is a Menger subset of $Z = P\times\omega$.

Claim 2. *Z is dense in X, so X is St-L.*

Proof. Let $p\in P$ be arbitrary and let $O(p,U,n,M)$ be a basic open neighbourhood of (p,ω) as in (7). Note that $U\times(n,\omega)$ is a non-empty clopen subset of Z. Since Z is homeomorphic to P, this implies that $U\times(n,\omega)$ is homeomorphic to P. In particular, $U\times(n,\omega)$ is not Menger. This implies that $(U\times(n,\omega))\setminus M\neq\emptyset$. (Indeed, otherwise $(U\times(n,\omega))$ be a closed subset of the Menger space M, so would be Menger.) Since $(U\times(n,\omega))\setminus M\subseteq Z$, it follows from (7) that $O(p,U,n,M)\cap Z\neq\emptyset$. ☐

Claim 3. *X is not St-M.*

Proof. It was mentioned by Repovš and Zdomskyy (2017) that in the Sacks' model constructed by Sacks (1971), the family of all Menger subsets of the real line has cardinality of the continuum (this result is contained in an unpublished paper by Gartside, Medini, and Zdomskyy 2023). Since P is a subset of the reals, the number of Menger subsets of P is at most continuum. Since $|P| = \mathfrak{c}$, we can fix an enumeration $\{M_p : p\in P\}$ of all Menger subsets of Z such that the set $P_M = \{p\in P : M_p = M\}$ has size $\mathfrak{c}$ for every Menger subspace of Z.

Consider the following assignment $\mathscr{N}\in \mathtt{NA}(X)$. For every $p\in P$ define $\mathscr{N}(p,\omega) = O(p,P,0,M_p)$ and $\mathscr{N}(p,n) = Z$ for every $n\in\omega$. Suppose that Y is a Menger subset of X. Note that $D = P\times\{\omega\}$ is a closed subset of X, so $Y\cap D$ is a Menger subspace of D, so it is Lindelöf. Since D is a discrete subset of X, this means that $|Y\cap D|\leq\omega$. Since Z is an F_σ-subset of X, $Y\cap Z$ is an F_σ-subset of Y. Since Y is Menger, so is $Y\cap Z$. Clearly, $Y\cap Z$ is a Menger subset of Z. By the property of our enumeration, the set $P_{Y\cap Z}$ has cardinality $\mathfrak{c}$. Since $|Y\cap D|\leq\omega$, there exists $p\in P_{Y\cap Z}$ such that $(p,\omega)\in D\setminus Y$. Note that $M_p = Y\cap Z$, as $p\in P_{Y\cap Z}$. It now follows from (7) that $O(p,P,0,M_p)\cap Y = \emptyset$. Thus, $\mathscr{N}(p,\omega)\cap Y = \emptyset$ by the definition of $\mathscr{N}$. Finally, note that $\mathscr{N}(p,\omega)$ is the only member of the family $\{\mathscr{N}(x) : x\in X\}$ containing (p,ω). From this fact and (2), we conclude that $(p,\omega)\notin St(Y,\mathscr{N})$. Therefore, Y is not a core for $\mathscr{N}$.

We have shown that no Menger subset of X can be a St-core for $\mathscr{N}$, this means that X is not St-M. ☐

Remark 24. The space of Example 23 is constructed in the Sacks' model, where the family of all Menger subsets of the real line has the cardinality of the continuum. The authors do not know if in ZFC there is a family of continuum many Menger sets such that any Menger set is a subset of one of them.

A St-M not St-σC space is given by Example 30 in Section **3**.

3. Cardinal invariants associated with an expansion operator on neighbourhood assignments

Definition 25. Having a cardinal function φ on a class of topological spaces, a space X, an expansion operator $\Phi : P(X) \times \mathtt{NA}(X) \to P(X)$ and $\mathscr{N} \in \mathtt{NA}(X)$, we define

$$\Phi\text{-}\varphi_{\mathscr{N}}(X) = \min\{\varphi(Y) : Y \text{ is a } \Phi\text{-core for } \mathscr{N}\}, \tag{8}$$

and we let

$$\Phi\text{-}\varphi(X) = \sup\{\Phi\text{-}\varphi_{\mathscr{N}}(X) : \mathscr{N} \in \mathtt{NA}(X)\}. \tag{9}$$

Proposition 26. *If φ is a cardinal function on a class of topological spaces, X is a space and $\Phi : P(X) \times \mathtt{NA}(X) \to P(X)$ is an expansion operator, then $\Phi\text{-}\varphi(X) \leq \varphi(X)$.*

Proof. Let $\mathscr{N} \in \mathtt{NA}(X)$ be arbitrary. Note that $X \subseteq \Phi(X, \mathscr{N})$ by Definition 4 (i) and $\Phi(X, \mathscr{N}) \subseteq X$ by the definition of Φ, so $X = \Phi(X, \mathscr{N})$. Therefore, X is Φ-core of $\mathscr{N}$ by Definition 7. This shows that the minimum in (8) is well defined and $\Phi\text{-}\varphi_{\mathscr{N}}(X) \leq \varphi(X)$. Since this holds for every $\mathscr{N} \in \mathtt{NA}(X)$, from (9) we obtain that $\Phi\text{-}\varphi(X) \leq \varphi(X)$. ☐

Lemma 27. *If $\Phi \leq \Psi$, then $\Psi\text{-}\varphi(X) \leq \Phi\text{-}\varphi(X)$ for every cardinal function φ.*

Proof. Let $\mathscr{N} \in \mathtt{NA}(X)$ be arbitrary. Since $\Phi \leq \Psi$, the inequality (3) holds. Therefore, if $X = \Phi(Y, \mathscr{N})$ for some $Y \subseteq X$, then $X = \Psi(Y, \mathscr{N})$ holds as well. This means that every Φ-core for $\mathscr{N}$ is also a Ψ-core.

Applying (8), we conclude that $\Psi\text{-}\varphi_{\mathscr{N}}(X) \leq \Phi\text{-}\varphi_{\mathscr{N}}(X)$. Since this inequality holds for every $\mathscr{N} \in \mathtt{NA}(X)$, from (9) we conclude that $\Psi\text{-}\varphi(X) \leq \Phi\text{-}\varphi(X)$. ☐

Since $NA \leq St$ by Lemma 6, we obtain the following

Lemma 28. *$St\text{-}\varphi(X) \leq NA\text{-}\varphi(X) \leq \varphi(X)$ holds for every space X and each cardinal function φ.*

Now we can give an example of St-L not NA-L space.

Definition 29. (Baloglou and Comfort 1988) *Let X be a space. The* compact covering number $\varkappa(X)$ of *X is the least cardinal number τ such that X can be covered by τ-many compact subsets. A space X is σ-compact if and only if $\varkappa(X) \leq \aleph_0$.*

Example 30. *For every uncountable cardinal κ, there exists a space X having the following properties:*

(i) *$e(X) = \kappa$;*
(ii) *X is NA-D, so also NA-PC;*
(iii) *NA-L$(X) = \kappa$; in particular, X is not NA-L;*
(iv) *X is St-M, so also St-L;*
(v) *NA-$\varkappa(X) = \kappa$; in particular, X is not St-σC.*

Indeed, fix an uncountable cardinal κ. Let D be a discrete space satisfying $|D| = \kappa$. Let L be the one-point Lindelöfication of D, let p be the unique non-isolated point of L, so that $L \setminus \{p\} = D$. Define $X = L \times [0, \omega] \setminus \{(p, \omega)\}$.

(i) Note that $C = D \times \{\omega\}$ is a closed discrete subspace of X. Since $|C| = |D| = \kappa = |X|$, this shows that the extent of X is equal to κ.

(ii) The closed subspace C of X defined in (i) is discrete, so it is trivially NA-D. Moreover, its completement $X \setminus C = L \times [0, \omega)$ is homeomorphic to a disjount sum of countably many copes of L. Since L is obviously NA-D, so is $X \setminus C$. By Theorem 16, X is NA-D.

(iii) Let us check that $NA\text{-}L(X) = \kappa$. For $x = (l, \omega) \in D \times \{\omega\}$, define $O_x = \{l\} \times [0, \omega]$. For $x = (p, \alpha) \in \{p\} \times [0, \omega)$, define $O_x = L \times \{\alpha\}$. Finally, for $x = (l, \alpha) \in D \times [0, \omega)$, define $O_x = \{x\}$. Then $\{O_x : x \in X\}$ is a neighbourhood assignment for X.

Let Y be a subspace of X such that $X = \bigcup_{y \in Y} O_y$. Let $x \in C$ be arbitrary. There exists $y_x \in Y$ such that $x \in O_{y_x}$. On the other hand, by the choice of our assignment, O_x is the only elelement of the assignment $\{O_x : x \in X\}$ which contains x. Therefore, $x = y_x \in Y$. This shows that $C \subseteq Y$. Since C is a closed subset of X, it is also closed in Y, which implies $L(C) \leq L(Y)$. Since C is discrete, we have $L(C) = |C| = \kappa$. Thus, $L(Y) \geq L(C) = \kappa$. This means that $NA\text{-}L(X) \geq \kappa$. On the other hand, $NA\text{-}L(X) \leq L(X) \leq |X| = \kappa$.

(iv) X has a dense Menger subspace $L \times [0, \omega)$, so X is St-M.

(v) First, we show that *for every compact subset K of X, there exists a finite set $S_K \subseteq L$ such that $K \subseteq S_K \times [0, \omega]$*. Indeed, if not, there would exist a set $\{(d_i, x_i) : i < \omega\} \subseteq K$ where the d_i's are pairwise distinct. It is easy to see that this set has no limit point, contradicting the fact that K is compact. Let $\mathscr{U} = \{O_x : x \in X\}$ be the cover of X defined in item (iii). Let $Y = \bigcup\{K_\alpha : \alpha < \tau\}$, where τ is a cardinal and each K_α is a compact subset of X. Suppose that $\tau < \kappa$. Note that by the property of our cover $\mathscr{U}$,

$$C \cap St(K_\alpha, \mathscr{U}) \subseteq S_{K_\alpha} \times \{\omega\},$$

so

$$C \cap St(Y, \mathscr{U}) \subseteq \bigcup\{S_{K_\alpha} \times \{\omega\} : \alpha < \tau\},$$

which implies $|C \cap St(Y, \mathscr{U})| \leq \max\{\omega, \tau\} < \kappa$. Since $|C| = \kappa$, we have $C \setminus St(Y, \mathscr{U}) \neq \emptyset$, and so $X \neq St(Y, \mathscr{U})$. This establishes the inequality $St\text{-}\varkappa(X) \geq \kappa$. The converse inequality follows from $St\text{-}\varkappa(X) \leq \varkappa(X) \leq |X| = \kappa$. □

Remark 31. *The space X from Example 30 is not normal.*

Remark 32. *When $\kappa = \omega_1$, the space X from Example 30 was considered by Ikenaga (1990) who showed that it is not St-σC. When $\kappa = \mathfrak{c}$, this space was considered by Song and Zheng (2014, Example 2.3).*

Recall that the *Lindelöf number* of a space X is the smallest cardinal κ such that for every open cover $\mathscr{U}$ of X there exists $\mathscr{V} \in [\mathscr{U}]^{\leq \kappa}$ such that $X = \bigcup \mathscr{V}$ (Engelking 1989). In the following we focus on the cardinal functions St-L and NA-L. For a space X:

$$St\text{-}L(X) = min\{\kappa : \text{for every open cover } \mathscr{U} \text{ of } X, \text{ there is } F \subset X \text{ such that } L(F) \leq \kappa \text{ and } St(F, \mathscr{U}) = X\} \text{(see also Cao } et\ al. \text{ 2002)}$$

and

$$NA\text{-}L(X) = \min\{\tau : \text{for every neighbourhood assignment } \{O_x : x \in X\} \text{ of } X, \exists \text{ a subspace } Y \text{ of } X \text{ such that } X = \bigcup_{y \in Y} O_y \text{ and } L(Y) \leq \tau\}.$$

Note that a space X is St-L if and only if $St\text{-}L(X) \leq \omega$ and it is NA-L if and only if $NA\text{-}L(X) \leq \omega$. By Lemma 28 we have that $St\text{-}L(X) \leq NA\text{-}L(X) \leq L(X)$.

In order to prove that if X is a paracompact space and A is a dense subset of X, then $L(X) \leq St\text{-}L(A)$ (Theorem 36 below), we need the following. Recall that a space X is paracompact if and only if every open cover of X has a star refinement (Engelking 1989). We introduce the following definition.

Definition 33. Let $n \in \mathbb{N}$, $n \geq 2$, X be a space, $\mathscr{U}$ and $\mathscr{V}$ two families of subsets of X. We say that $\mathscr{V}$ is a *n-star-refinement* of $\mathscr{U}$ and we write $\mathscr{V} \prec_n^* \mathscr{U}$ if for every $V \in \mathscr{V}$ there exists $U \in \mathscr{U}$ such that $St^n(V, \mathscr{U}) \subseteq U$. If $n = 1$ we have the notion of star-refinement.

Lemma 34. *Let X be a topological space and $\mathscr{U}$ a family of sets of X. If $\mathscr{W} \prec_{n-1}^* \mathscr{V} \prec_1^* \mathscr{U}$, then $\mathscr{W} \prec_n^* \mathscr{U}$.*

Proof. We prove the statement for $n = 2$. Take $W \in \mathscr{W}$. Since $\mathscr{W} \prec_1^* \mathscr{V}$, there exists $V \in \mathscr{V}$ such that $St(W, \mathscr{W}) \subseteq V$. We have to show that for every $W \in \mathscr{W}$, there exists $U \in \mathscr{U}$ such that $St^2(W, \mathscr{W}) \subseteq U$. Let $y \in St^2(W, \mathscr{W})$. Then there exists $W' \in \mathscr{W}$ such that $y \in W'$ and $W' \cap St(W, \mathscr{W}) \neq \emptyset$. Since $\mathscr{W} \prec_1^* \mathscr{V}$, there exists $V' \in \mathscr{V}$ such that $St(W', \mathscr{W}) \subseteq V'$. We can notice that $W' \subseteq St(W', \mathscr{W}) \subseteq V'$ and $St(W, \mathscr{W}) \subseteq V$. So, $\emptyset \neq St(W, \mathscr{W}) \cap W' \subseteq V \cap V'$, then $V \cap V' \neq \emptyset$. This means $V' \subseteq St(V, \mathscr{V})$. We also have $y \in W' \subseteq V' \subseteq St(V, \mathscr{V})$. Therefore, $St^2(W, \mathscr{W}) \subseteq St(V, \mathscr{V})$. Since $\mathscr{V} \prec_1^* \mathscr{U}$, there exists $U \in \mathscr{U}$ such that $St(V, \mathscr{V}) \subseteq U$. Thus, $St^2(W, \mathscr{W}) \subseteq U$ This means $\mathscr{W} \prec_2^* \mathscr{U}$.

□

Theorem 35. *A space X is paracompact if and only if every open cover of X has a n-star refinement.*

Now we can prove the following.

Theorem 36. *Let X be a paracompact regular space and let A be a dense subset of X. Then $L(X) \leq St\text{-}L(A)$.*

Proof. Let A be a dense subset of X and let $St\text{-}L(A) \leq \kappa$. Let $\mathscr{U}$ be an open cover of X, $\mathscr{W}$ a closed locally finite refinement of $\mathscr{U}$ and $\mathscr{V}$ a 2-star refinement of $\mathscr{W}$. We consider $\mathscr{V}_A = \{V \in \mathscr{V} : V \cap A \neq \emptyset\}$ that is a open cover of A. Since $St\text{-}L(A) \leq \kappa$, there exists $Y \subset A$ such that $L(Y) \leq \kappa$ and $A = St(Y, \mathscr{V}_A) = \bigcup\{V \in \mathscr{V}_A : V \cap Y \neq \emptyset\} = \bigcup_{y \in Y}\{V \in \mathscr{V}_A : y \in V\} = \bigcup_{y \in Y} St(y, \mathscr{V}_A)$. Since $Y \subset A$ and $L(Y) \leq \kappa$, there exixts $Z \in [Y]^{\leq \kappa}$ such that $Y \subseteq \bigcup_{y \in Z} St(y, \mathscr{V}_A)$(*).

Claim 4. $A \subseteq \bigcup_{y \in Z} St^2(y, \mathscr{V}_A)$.

Proof. Take $a \in A$. Since $A = \bigcup_{y \in Y} St(y, \mathscr{V}_A)$, there exists $y_0 \in Y$ such that $a \in St(y_0, \mathscr{V}_A)$, then there exists $V_0 \in \mathscr{V}_A$ such that $a, y_0 \in V_0$. Since $y_0 \in Y$, by (*), there exists $y_1 \in Z$ such that $y_0 \in St(y_1, \mathscr{V}_A)$, then there exists $V_1 \in \mathscr{V}_A$ such that $y_0, y_1 \in V_1$. This means $a \in St^2(y_1, \mathscr{V}_A)$. □

Since $\mathscr{V}_A \prec \mathscr{V}$, then $\{St^2(z, \mathscr{V}_A) : z \in Z\} \prec \{St^2(x, \mathscr{V}) : x \in X\} \prec \mathscr{W} \prec \mathscr{U}$. For every $z \in Z$, take W_z and U_z such that $St^2(z, \mathscr{V}_A) \subseteq W_z$ and $W_z \subseteq U_z$. By the previous Claim, $A \subseteq \bigcup\{St^2(z, \mathscr{V}_A) : z \in Z\} \subseteq \bigcup\{W_z : z \in Z\}$. Now $A = \bigcup\{A \cap W_z : z \in Z\}$ and $\{A \cap W_z : z \in Z\}$ is locally finite. Then $X = \overline{A} = \overline{\bigcup\{A \cap W_z : z \in Z\}} = \bigcup\{\overline{A \cap W_z} : z \in Z\} \subseteq \bigcup\{\overline{W_z} : z \in Z\} \subseteq \bigcup\{U_z : z \in Z\}$ that is $L(X) \leq \kappa$. □

Corollary 37. *Let X be a paracompact regular space with a dense St-L subspace, then it is Lindelöf.*

Corollary 38. (Engelking 1989) *Let X be a paracompact regular space with a dense Lindelöf subspace, then it is Lindelöf.*

Remark 39. *Theorem 36 does not hold replacing St-L with St-M spaces and L with M. In fact, the space of irrational numbers is Lindelöf and separable so it contains a countable dense subspace that is also Menger but the space is not Menger.*

In order to obtain a generalization of Proposition 3.8 given by Alas, Tkachuk, and Wilson (2006) and of Theorem 2.10 discussed by Alas, Tkachuk, and Wilson (2000),we need to define the following cardinal invariants.

Definition 40. The *metacompact number* of a space X is

$$MC(X) = min\{\kappa : \text{every open cover } \mathscr{U} \text{ of } X \text{ such that } |\mathscr{U}| \leq \kappa \text{ has a point-finite open refinement}\}.$$

Definition 41. The *linearly Lindelöf number* of a space X is

$$LL(X) = min\{\kappa : \text{for every linearly ordered open cover } \mathscr{U} \text{ of } X, \exists \mathscr{V} \in [\mathscr{U}]^{\leq\kappa} : X = \bigcup \mathscr{V}\}.$$

We have the following relation:

$$NA\text{-}LL(X) \leq NA\text{-}L(X) \leq L(X).$$

It was proved by van Mill, Tkachuk, and Wilson (2007) that the properties LL and NA-LL are equivalent. More in general, we can prove, following essentially the same proof of Proposition 2.7 given by van Mill, Tkachuk, and Wilson (2007), that $NA\text{-}LL(X) = LL(X)$, for every space X.

To prove the following results, we follow step by step, respectively, the proofs of Theorem 2.10 given by Alas, Tkachuk, and Wilson (2000), and of Proposition 3.8 given by Alas, Tkachuk, and Wilson (2006) using cardinal functions.

Theorem 42. *Let X be a space, then* $L(X) \leq MC(X)LL(X)$.

Proof. Let $\tau = MC(X)LL(X)$ and $\kappa = min\{\mu : \mu$ is a cardinal such that there exists an open cover of cardinality μ that does not have subcovers of cardinality $\tau\}$.

For every closed subset F of X and for every family $\mathscr{U}$ of open subsets of X with $|\mathscr{U}| < \kappa$ such that $\bigcup \mathscr{U} \supseteq F$, there is $\mathscr{U}' \in [\mathscr{U}]^{\leq\tau}$ such that $F \subset \bigcup \mathscr{U}'$. Let $\mathscr{V} = \{V_\alpha : \alpha \in \kappa\}$ an open cover of X of size κ. For every $\beta < \kappa$, let $W_\beta = \bigcup_{\alpha<\beta} V_\alpha$. The family of W_β for every $\beta \in \kappa$ is a linearly ordered open cover of X. Since $LL(X) \leq \tau$, there is $\{\beta_\alpha : \alpha \in \tau\}$ such that $\bigcup\{W_{\beta_\alpha} : \alpha \in \tau\} = X$. Considering that $MC(X) \leq \tau$, there exists closed subsets F_α of W_{β_α} such that $X = \bigcup_{\gamma\in\tau} F_\gamma$. The family $\mathscr{V}_\gamma = \{V_\alpha : \alpha < \beta_\gamma\}$ is an open cover of F_γ having cardinality strictly less than κ, so there is $\mathscr{V}'_\gamma \in [\mathscr{V}_\gamma]^{\leq\tau}$ such that $F_\gamma \subset \bigcup \mathscr{V}_\gamma{}'$. The family $\mathscr{V}' = \{\bigcup \mathscr{V}'_\gamma : \gamma \in \tau\}$ has cardinality at most τ and it is a subcover of $\mathscr{V}$, that is a contradiction. □

Corollary 43. (Alas, Tkachuk, and Wilson 2009, Theorem 2.10) *A linearly Lindelöf, countably metacompact space is Lindelöf.*

Proposition 44. *Let X be a space, then* $L(X) \leq NA\text{-}L(X)MC(X)$.

Proof. Let $NA\text{-}L(X)MC(X) = \kappa$, then $NA\text{-}LL(X) = LL(X) \leq \kappa$. Using Theorem 42 we have $L(X) \leq MC(X)LL(X) \leq \kappa$. □

Corollary 45. (Alas, Tkachuk, and Wilson 2000, Proposition 3.8) *A NA-L, countably metacompact space is Lindelöf.*

4. Open questions

We present the following list of open problems.

Question 46. *Does there exist a NA-M not NA-*σ*C space?*

Question 47. *Does there exist a ZFC or Tychonoff St-L not St-M space?*

Question 48. *Does there exist a NA-L not NA-M space?*

Question 49. *Does there exist a NA-MC not NA-PC space?*

Question 50. *Does there exist a NA-ML not NA-MC space?*

Question 51. *Does there exist a regular non-paracompact space X such that L(X)* $>$ *St-L(A) for all dense subsets A of X?*

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