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μ-α-LINDELÖF SETS IN μ-SPACES

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ABSTRACT. We introduce and study μ-α-Lindelöf sets in μ-spaces as a subclass of μ-Lindelöf sets and as a superclass of both μ-semi-Lindelöf sets and strongly μ-Lindelöf sets. Several properties and mapping properties of μ-α-Lindelöf sets are studied extensively.

1. Introduction

A generalized topology (GT) (Császár 2002) μ on a non-empty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under non-empty arbitrary unions. For a GT μ on X, the pair (X,μ) will be called a generalized topological space (GTS), elements of μ will be called μ-open sets, and a subset A of (X,μ) will be called μ-closed if $X\backslash A$ is μ-open. A space (X,μ) will always mean a GTS. If A is a subset of a space (X,μ), then the μ-closure of A (Császár 2005), $c_\mu(A)$, is the intersection of all μ-closed sets containing A, and the μ-interior of A (Császár 2005), $i_\mu(A)$, is the union of all μ-open sets contained in A. A GT μ on X is called a strong GT (Császár 2004) if $X \in \mu$. A space (X,μ) is called a μ-space (Noiri 2006) if $X \in \mu$, that is μ is a strong GT.

Recently, various types of Lindelöf properties in GTSs have been considered by several mathematicians. The primary purpose of this article is to introduce and study μ-α-Lindelöf sets in μ-spaces. We also introduce and study a new type of generalized open sets in GTSs, called ω_μ-α-open sets, and invest them to obtain more properties of μ-α-Lindelöf sets.

If (X,τ) is a topological space and $A \subset X$, then $\overline{A}$ and $\mathrm{Int}A$ will stand, respectively, for the closure of A in X and the interior of A in X. A subset A of a topological space (X,τ) is called semi-open (Levine 1963) (resp. preopen (Mashhour *et al.* 1982), β-open (Abd El-Monsef *et al.* 1983), α-open (Njåstad 1965)) if $A \subset \overline{\mathrm{Int}A}$ (resp. $A \subset \mathrm{Int}\overline{A}$, $A \subset \overline{\mathrm{Int}\overline{A}}$, $A \subset \mathrm{Int}\overline{\mathrm{Int}A}$). The families of semi-open (resp. preopen, β-open, α-open) subsets of a topological space (X,τ) will be denoted by $SO(X)$ (resp. $PO(X)$, $\beta(X)$, $\alpha(X)$). It is known that if $\mu \in \{SO(X), PO(X), \beta(X), \alpha(X)\}$, then (X,μ) is a μ-space. Njåstad (1965) pointed out that if $\mu = \alpha(X)$, then μ is a topology on X finer than τ.

2. Preliminaries

A subset A of a space (X,μ) is called (Császár 2005) μ-semi-open (resp. μ-preopen, μ-β-open, μ-α-open) if $A \subset c_\mu\left(i_\mu(A)\right)$ (resp. $A \subset i_\mu\left(c_\mu(A)\right)$, $A \subset c_\mu\left(i_\mu\left(c_\mu(A)\right)\right)$, $A \subset i_\mu\left(c_\mu\left(i_\mu(A)\right)\right)$). We will denote the class of μ-semi-open sets by $\sigma(\mu)$ or σ, the class of μ-preopen sets by $\pi(\mu)$ or π, the class of μ-β-open sets by $\beta(\mu)$ or β, and the class of μ-α-open sets by $\alpha(\mu)$ or α. It was pointed out by Császár (2005) that each of σ, π, β, and α is a GT. However, it is easy to see that each of σ and β is a strong GT, while π and α need not.

We recall the following definitions and facts for their importance in the material of our article.

Proposition 2.1. ***(Császár 2005)*** *Let (X,μ) be a space. Then*
(i) $\mu \subset \alpha = \sigma \cap \pi$;
(ii) $\sigma \cup \pi \subset \beta$;
(iii) $\alpha(\zeta) = \zeta, \forall \zeta \in \{\sigma,\pi,\beta,\alpha\}$.

Definition 2.2. ***(Sarsak 2013)*** *(i) A subset A of a μ-space (X,μ) is called μ-Lindelöf if any cover of A by μ-open subsets of X has a countable subcover.*
(ii) A μ-space (X,μ) is called μ-Lindelöf if any cover of X by μ-open sets has a countable subcover.

Definition 2.3. *(i) A subset A of a topological space (X,τ) is called strongly Lindelöf (Hdeib and Sarsak 2000) relative to (X,τ) if any cover of A by preopen subsets of X has a countable subcover.*
(ii) A topological space (X,τ) is called strongly Lindelöf (Mashhour et al. *1984) if any cover of X by preopen subsets of X has a countable subcover.*

Definition 2.4. ***(Sarsak 2019)*** *(i) A subset A of a μ-space (X,μ) is called strongly μ-Lindelöf if any cover of A by μ-preopen subsets of X has a countable subcover, that is A is $\pi(\mu)$-Lindelöf.*
(ii) A μ-space (X,μ) is called strongly μ-Lindelöf if any cover of X by μ-preopen sets has a countable subcover, that is $(X,\pi(\mu))$ is $\pi(\mu)$-Lindelöf.

Definition 2.5. *(i) A subset A of a topological space (X,τ) is called semi-Lindelöf in X (Sarsak 2009) if any cover of A by semi-open subsets of X has a countable subcover. We will use the term "relative to X" to mean "in X".*
(ii) A topological space (X,τ) is called semi-Lindelöf (Ganster 1990) if any cover of X by semi-open subsets of X has a countable subcover.

Definition 2.6. *(i) A subset A of a space (X,μ) is called μ-semi-Lindelöf relative to X (Mustafa 2012) if any cover of A by μ-semi-open subsets of X has a countable subcover, or equivalently (Sarsak 2021), A is $\sigma(\mu)$-Lindelöf. We will say A is μ-semi Lindelöf to mean A is μ-semi Lindelöf relative to X.*
(ii) A space (X,μ) is called μ-semi-Lindelöf (Mustafa 2012) if any cover of X by μ-semi-open sets has a countable subcover, or equivalently (Sarsak 2021), $(X,\sigma(\mu))$ is $\sigma(\mu)$-Lindelöf.

Definition 2.7. ***(Sarsak 2019)*** *(i) A subset A of a topological space (X,τ) is called β-Lindelöf relative to (X,τ) if any cover of A by β-open subsets of X has a countable*

subcover.
(ii) A topological space (X,τ) is called β-Lindelöf if any cover of X by β-open subsets of X has a countable subcover.

Definition 2.8. ***(Sarsak 2023)*** *(i) A subset A of a space (X,μ) is called μ-β-Lindelöf if any cover of A by μ-β-open subsets of X has a countable subcover, that is A is $\beta(\mu)$-Lindelöf.*
(ii) A space (X,μ) is called μ-β-Lindelöf if any cover of X by μ-β-open sets has a countable subcover, that is $(X,\beta(\mu))$ is $\beta(\mu)$-Lindelöf.

3. μ-α-Lindelöf sets

This section is mainly devoted to introducing and studying μ-α-Lindelöf sets in μ-spaces.

Definition 3.1. *(i) A subset A of a topological space (X,τ) is called α-Lindelöf relative to (X,τ) if any cover of A by α-open subsets of X has a countable subcover.*
(ii) (Ganster 1990) A topological space (X,τ) is called α-Lindelöf if any cover of X by α-open subsets of X has a countable subcover.

Definition 3.2. *(i) A subset A of a μ-space (X,μ) is called μ-α-Lindelöf if any cover of A by μ-α-open subsets of X has a countable subcover, that is A is $\alpha(\mu)$-Lindelöf.*
(ii) A μ-space (X,μ) is called μ-α-Lindelöf if any cover of X by μ-α-open sets has a countable subcover, that is $(X,\alpha(\mu))$ is $\alpha(\mu)$-Lindelöf.

Remark 3.3. *It is easy to see that the countable union of μ-α-Lindelöf subsets of a μ-space (X,μ) is μ-α-Lindelöf.*

The next six remarks follow from Proposition **2.1**(iii).

Remark 3.4. *Let A be a subset of a space (X,μ). Then it is clear that*
(i) A is μ-semi-Lindelöf if and only if A is $\sigma(\mu)$-α-Lindelöf;
(ii) A is μ-β-Lindelöf if and only if A is $\beta(\mu)$-α-Lindelöf.

Remark 3.5. *Let A be a subset of a μ-space (X,μ). Then it is clear that*
(i) A is μ-α-Lindelöf if and only if A is $\alpha(\mu)$-α-Lindelöf;
(ii) A is strongly μ-Lindelöf if and only if A is $\pi(\mu)$-α-Lindelöf.

Remark 3.6. *If A is a subset of a topological space (X,τ) and $\mu \in \{\tau, \alpha(X)\}$, then*

$$A \text{ is } \mu\text{-}\alpha\text{-Lindelöf} \Leftrightarrow A \text{ is } \alpha\text{-Lindelöf relative to } (X,\tau).$$

In particular,

$$(X,\mu) \text{ is } \mu\text{-}\alpha\text{-Lindelöf} \Leftrightarrow (X,\tau) \text{ is } \alpha\text{-Lindelöf}.$$

Remark 3.7. *If A is a subset of a topological space (X,τ) and $\mu = SO(X)$, then*

$$A \text{ is } \mu\text{-}\alpha\text{-Lindelöf} \Leftrightarrow A \text{ is semi-Lindelöf relative to } (X,\tau).$$

In particular,

$$(X,\mu) \text{ is } \mu\text{-}\alpha\text{-Lindelöf} \Leftrightarrow (X,\tau) \text{ is semi-Lindelöf}.$$

Remark 3.8. *If A is a subset of a topological space (X, τ) and $\mu = PO(X)$, then*

$$A \text{ is } \mu\text{-}\alpha\text{-Lindelöf} \Leftrightarrow A \text{ is strongly Lindelöf relative to } (X, \tau).$$

In particular,

$$(X, \mu) \text{ is } \mu\text{-}\alpha\text{-Lindelöf} \Leftrightarrow (X, \tau) \text{ is strongly Lindelöf.}$$

Remark 3.9. *If A is a subset of a topological space (X, τ) and $\mu = \beta(X)$, then*

$$A \text{ is } \mu\text{-}\alpha\text{-Lindelöf} \Leftrightarrow A \text{ is } \beta\text{-Lindelöf relative to } (X, \tau).$$

In particular,

$$(X, \mu) \text{ is } \mu\text{-}\alpha\text{-Lindelöf} \Leftrightarrow (X, \tau) \text{ is } \beta\text{-Lindelöf.}$$

Remark 3.10. *Clearly, if a subset A of a μ-space (X, μ) is μ-semi-Lindelöf or strongly μ-Lindelöf, then A is μ-α-Lindelöf, and if A is μ-α-Lindelöf, then A is μ-Lindelöf; in particular, if a μ-space (X, μ) is μ-semi-Lindelöf or strongly μ-Lindelöf, then (X, μ) is μ-α-Lindelöf, and if (X, μ) is μ-α-Lindelöf, then (X, μ) is μ-Lindelöf. However, the converses need not be true even for topological spaces as it is well known.*

Recall that a subset A of a space (X, μ) is called μ-α-closed (Sarsak 2012) if $X \backslash A$ is μ-α-open. The following proposition characterizes μ-α-Lindelöf sets in terms of μ-α-closed sets; the straightforward proof is omitted.

Proposition 3.11. *A non-empty subset A of a μ-space (X, μ) is μ-α-Lindelöf if and only if for every family $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ of μ-α-closed sets having the property that for every non-empty countable subfamily $\mathscr{F}^i$ of $\mathscr{F}$, $\left(\bigcap \mathscr{F}^i\right) \cap A \neq \emptyset$, then $\left(\bigcap \mathscr{F}\right) \cap A \neq \emptyset$.*

Next, we characterize μ-α-Lindelöf sets using filter bases; to proceed, we recall the following.

Definition 3.12. ***(Sarsak 2019)*** *(i) A filter base $\mathscr{F}$ on a non-empty set X is called a strong filter base on X if whenever $\mathscr{F}^i$ is a non-empty countable subcollection of $\mathscr{F}$, there exists $F \in \mathscr{F}$ such that $F \subset \bigcap \mathscr{F}^i$;*
(ii) A strong filter base $\mathscr{F}$ on a non-empty set X is called a maximal strong filter base on X if whenever $\mathscr{H}$ is a strong filter base on X with $\mathscr{F} \subset \mathscr{H}$, then $\mathscr{F} = \mathscr{H}$.

Proposition 3.13. ***(Sarsak 2019)*** *Every strong filter base $\mathscr{F}$ on a non-empty set X is contained in a maximal strong filter base on X.*

Definition 3.14. *A filter base $\mathscr{F}$ on a space (X, μ) is said to $\alpha(\mu)$-converge to a point $x \in X$ if for each μ-α-open subset U of X such that $x \in U$, there exists $F \in \mathscr{F}$ such that $F \subset U$. $\mathscr{F}$ is said to $\alpha(\mu)$-accumulate at $x \in X$ if $U \cap F \neq \emptyset$ for every $F \in \mathscr{F}$ and for every μ-α-open subset U of X such that $x \in U$.*

Remark 3.15. *Let $\mathscr{F}$ be a filter base on a space (X, μ) and $x \in X$. Then it is easy to see that*
(i) If $\mathscr{F}$ $\alpha(\mu)$-converges to x, then $\mathscr{F}$ $\alpha(\mu)$-accumulates at x;
(ii) If $\mathscr{F}$ is a maximal filter base (maximal strong filter base), then $\mathscr{F}$ $\alpha(\mu)$-converges to x if and only if $\mathscr{F}$ $\alpha(\mu)$-accumulates at x.

Proposition 3.16. *For a non-empty subset A of a μ-space (X,μ), the following are equivalent:*
(i) A is μ-α-Lindelöf;
(ii) Every maximal strong filter base on X, each of whose members meets A, $\alpha(\mu)$-converges to some point of A;
(iii) Every strong filter base on X, each of whose members meets A, $\alpha(\mu)$-accumulates at some point of A.

Proof. (i)⇒(ii): Let $\mathscr{F}$ be a maximal strong filter base on X, each of whose members meets A, such that $\mathscr{F}$ does not $\alpha(\mu)$-converge to any point of A. Since $\mathscr{F}$ is maximal, it follows from Remark **3.15**(ii) that $\mathscr{F}$ does not $\alpha(\mu)$-accumulate at any point of A. Thus, for each $x \in A$ there exist $F_x \in \mathscr{F}$ and a μ-α-open subset U_x of X such that $x \in U_x$ and $U_x \cap F_x = \emptyset$. But A is μ-α-Lindelöf, so there exist $x_1, x_2, x_3, ... \in X$ such that $A \subset \bigcup_{i=1}^{\infty} U_{x_i}$. Since $\mathscr{F}$ is a strong filter base on X, there exists $F \in \mathscr{F}$ such that $F \subset \bigcap_{i=1}^{\infty} F_{x_i}$, but $U_{x_i} \cap F_{x_i} = \emptyset$ for each $i \in \{1,2,3,...\}$, so $U_{x_i} \cap F = \emptyset$ for each $i \in \{1,2,3,...\}$, i.e. $\emptyset = (\bigcup_{i=1}^{\infty} U_{x_i}) \cap F \supset A \cap F$, a contradiction.

(ii)⇒(iii): Let $\mathscr{F}$ be a strong filter base on X, each of whose members meets A. Then $\mathscr{F}^{\mathscr{A}} = \{F \cap A : F \in \mathscr{F}\}$ is a strong filter base on X. Thus by Proposition **3.13**, $\mathscr{F}^{\mathscr{A}}$ is contained in a maximal strong filter base $\mathscr{H}$ on X, each of whose members meets A. By (ii), $\mathscr{H}$ $\alpha(\mu)$-converges to some point x of A, thus by Remark **3.15**(i), $\mathscr{H}$ $\alpha(\mu)$-accumulates at x, but $\mathscr{F}^{\mathscr{A}} \subset \mathscr{H}$, so $\mathscr{F}^{\mathscr{A}}$ $\alpha(\mu)$-accumulates at x. Hence, $\mathscr{F}$ $\alpha(\mu)$-accumulates at x.

(iii)⇒(i): Suppose that A is not μ-α-Lindelöf. Then by Proposition **3.11**, there exists a cover $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ of A by μ-α-open subsets of X such that for any non-empty countable subset Λ_0 of Λ,

$$\left[\bigcap\{(X \backslash U_\alpha) : \alpha \in \Lambda_0\}\right] \cap A \neq \emptyset.$$

For each non-empty countable subset Λ_0 of Λ, let

$$F_{\Lambda_0} = \left[\bigcap\{(X \backslash U_\alpha) : \alpha \in \Lambda_0\}\right] \cap A.$$

Then $\mathscr{F} = \left\{F_{\Lambda_0} : \Lambda_0 \text{ is a non-empty countable subset of } \Lambda\right\}$ is a strong filter base on X, each of whose members meets A. Thus by (iii), $\mathscr{F}$ $\alpha(\mu)$-accumulates at some point x of A. Since $\mathscr{U}$ is a cover of A, there exists $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$, but $\mathscr{F}$ $\alpha(\mu)$-accumulates at x and U_{α_0} is μ-α-open, so $U_{\alpha_0} \cap F \neq \emptyset$ for every $F \in \mathscr{F}$. Let $F = \left(X \backslash U_{\alpha_0}\right) \cap A$. Then $F \in \mathscr{F}$ and thus $U_{\alpha_0} \cap \left(X \backslash U_{\alpha_0}\right) \cap A \neq \emptyset$, a contradiction. □

At the end of this section, we study the behaviour of μ-α-Lindelöf sets in a subspace; to proceed, we recall the following.

Definition 3.17. ***(Sarsak 2013)*** *Let A be a non-empty subset of a space (X,μ). The generalized subspace topology on A is the collection $\{U \cap A : U \in \mu\}$, which will be denoted by μ_A. The generalized subspace A is the GTS (A,μ_A).*

Remark 3.18. ***(Sarsak 2013)*** *Let A be a non-empty subset of a μ-space (X,μ). Then it is easy that (A,μ_A) is a μ_A-space.*

Proposition 3.19. *Let B be a non-empty subset of a μ-space (X,μ) and $A \subset B$. Then A is μ-α-Lindelöf if and only if A is $(\alpha(\mu))_B$-Lindelöf.*

Proof. **Necessity.** Observe first that $(X,\alpha(\mu))$ is an $\alpha(\mu)$-space, thus by Remark **3.18**, $(B,(\alpha(\mu))_B)$ is an $(\alpha(\mu))_B$-space. Suppose that $\mathscr{A} = \{A_\alpha : \alpha \in \Lambda\}$ is a cover of A by $(\alpha(\mu))_B$-open sets. Then $A_\alpha = S_\alpha \cap B$, where S_α is $\alpha(\mu)$-open for each $\alpha \in \Lambda$. Thus $\mathscr{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of A by μ-α-open sets, but A is μ-α-Lindelöf, so there exist $\alpha_1, \alpha_2, \alpha_3, ... \in \Lambda$ such that $A \subset \bigcup_{i=1}^{\infty} S_{\alpha_i}$, and thus, $A \subset \bigcup_{i=1}^{\infty} (S_{\alpha_i} \cap B) = \bigcup_{i=1}^{\infty} A_{\alpha_i}$. Hence, A is $(\alpha(\mu))_B$-Lindelöf.

Sufficiency. Suppose that $\mathscr{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of A by μ-α-open sets. Then $\mathscr{A} = \{S_\alpha \cap B : \alpha \in \Lambda\}$ is an $(\alpha(\mu))_B$-open cover of A, but A is $(\alpha(\mu))_B$-Lindelöf, so there exist $\alpha_1, \alpha_2, \alpha_3, ... \in \Lambda$ such that $A \subset \bigcup_{i=1}^{\infty} (S_{\alpha_i} \cap B) \subset \bigcup_{i=1}^{\infty} S_{\alpha_i}$. Hence, A is μ-α-Lindelöf. □

Corollary 3.20. *Let A be a non-empty subset of a μ-space (X,μ). Then A is μ-α-Lindelöf if and only if A is $(\alpha(\mu))_A$-Lindelöf.*

4. ω_μ-α-open sets

In this section, we introduce and study a generalized form of μ-α-open sets in GTSs, called ω_μ-α-open sets.

Definition 4.1. *Let (X,μ) be a space and A be a subset of X. Then A is called ω_μ-α-open if whenever $x \in A$, there exists a μ-α-open set U_x containing x such that $U_x \backslash A$ is countable. A is called ω_μ-α-closed if $X \backslash A$ is ω_μ-α-open. The collection of all ω_μ-α-open subsets of X will be denoted by $\omega_{\alpha(\mu)}$.*

Proposition 4.2. *Let (X,μ) be a space and A be a subset of X. Then A is ω_μ-α-open if and only if whenever $x \in A$, there exists a μ-α-open set U_x containing x and a countable subset C_x of X such that $U_x \backslash C_x \subset A$.*

Proof. **Necessity.** Let A be ω_μ-α-open and $x \in A$. Then there exists a μ-α-open set U_x containing x such that $U_x \backslash A$ is countable. Let $C_x = U_x \backslash A$. Then $U_x \backslash C_x \subset A$.

Sufficiency. Let $x \in A$. Then by assumption, there exists a μ-α-open set U_x containing x and a countable subset C_x of X such that $U_x \backslash C_x \subset A$. Thus, $U_x \backslash A \subset C_x$, and therefore, $U_x \backslash A$ is countable. Hence, A is ω_μ-α-open. □

Corollary 4.3. *Let (X,μ) be a space and A be an ω_μ-α-closed subset of X. Then $A \subset B \cup C$ for some μ-α-closed subset B of X and some countable subset C of X.*

Proof. Let A be an ω_μ-α-closed subset of X. Then $X \backslash A$ is ω_μ-α-open. If $X \backslash A = \emptyset$, choose $B = X$ and $C = \emptyset$. If $X \backslash A \neq \emptyset$, choose $x \in X \backslash A$. Thus by Proposition **4.2**, there exists a μ-α-open set U_x containing x and a countable subset C_x of X such that $U_x \backslash C_x \subset X \backslash A$. Therefore, $A \subset (X \backslash U_x) \cup C_x$. Let $B = X \backslash U_x$ and $C = C_x$. Then B is μ-α-closed, and $A \subset B \cup C$. □

Proposition 4.4. *(i) If (X,μ) is a space, then $\left(X, \omega_{\alpha(\mu)}\right)$ is a space.*
(ii) If (X,μ) is a μ-space, then $\left(X, \omega_{\alpha(\mu)}\right)$ is an $\omega_{\alpha(\mu)}$-space.
(iii) If A is a μ-α-open subset of a space (X,μ), then A is ω_μ-α-open, that is $\alpha(\mu) \subset \omega_{\alpha(\mu)}$.

(iv) Let A be a subset of a space (X,μ). Then A is ω_μ-α-open if and only if A is $\omega_{\omega_{\alpha(\mu)}}$-$\alpha$-open.
(v) Let A be a countable subset of a μ-space (X,μ). Then A is ω_μ-α-closed.

Proof. (i) Clearly, $\emptyset$ is ω_μ-α-open, that is $\emptyset \in \omega_{\alpha(\mu)}$. Now let U_α be ω_μ-α-open for each $\alpha \in \Lambda$, and let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha$. Then $x \in U_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Since U_{α_0} is ω_μ-α-open, there exists a μ-α-open set V_{α_0} containing x such that $V_{\alpha_0} \backslash U_{\alpha_0}$ is countable. Thus, $V_{\alpha_0} \backslash (\bigcup_{\alpha\in\Lambda} U_\alpha)$ is countable. Hence, $\bigcup_{\alpha\in\Lambda} U_\alpha$ is ω_μ-α-open, that is $\bigcup_{\alpha\in\Lambda} U_\alpha \in \omega_{\alpha(\mu)}$. Hence, $\left(X, \omega_{\alpha(\mu)}\right)$ is a space.

(ii) As (X,μ) is a μ-space, $X \in \mu$. Thus, X is μ-α-open. Let $x \in X$. Then $X \backslash X = \emptyset$ is countable. Thus, X is ω_μ-α-open, that is $X \in \omega_{\alpha(\mu)}$. Hence, $\left(X, \omega_{\alpha(\mu)}\right)$ is an $\omega_{\alpha(\mu)}$-space.

(iii) Let A be μ-α-open and $x \in A$. Then $A \backslash A = \emptyset$ is countable. Thus, A is ω_μ-α-open.

(iv) Suppose that A is ω_μ-α-open and let $x \in A$. Then there exists a μ-α-open set U containing x such that $U \backslash A$ is countable. By (iii), U is ω_μ-α-open. By Proposition **2.1**(v), U is $\omega_{\alpha(\mu)}$-α-open. Thus, A is $\omega_{\omega_{\alpha(\mu)}}$-$\alpha$-open. Conversely, suppose that A is $\omega_{\omega_{\alpha(\mu)}}$-$\alpha$-open and let $x \in A$. Then there exists an $\omega_{\alpha(\mu)}$-α-open set U containing x such that $U \backslash A$ is countable. By Proposition **2.1**(v), U is ω_μ-α-open. Thus, there exists a μ-α-open set V containing x such that $V \backslash U$ is countable. Now $V \backslash A = ((V \backslash U) \backslash A) \cup ((U \cap V) \backslash A)$. As $U \backslash A$ and $V \backslash U$ are both countable, $(V \backslash U) \backslash A$ and $(U \cap V) \backslash A$ are both countable, and thus, $V \backslash A$ is countable. Hence, A is ω_μ-α-open.

(v) Let $x \in X \backslash A$. As (X,μ) is a μ-space, $X \in \mu$, and thus, X is μ-α-open. Now $X \backslash (X \backslash A) = A$ is countable by assumption. Thus, $X \backslash A$ is ω_μ-α-open, that is A is ω_μ-α-closed. □

The next proposition discusses the property of the counterparts of Proposition **4.4**(i) and (ii) in classical topology.

Proposition 4.5. *Let (X,μ) be a topological space. Then $\left(X, \omega_{\alpha(\mu)}\right)$ is a topological space.*

Proof. By Proposition **4.4**(ii), it suffices to show that if A and B are ω_μ-α-open, then $A \cap B$ is ω_μ-α-open. Let $x \in A \cap B$. Then there exist μ-α-open sets U and V such that $x \in U \cap V$ and both $U \backslash A$ and $V \backslash B$ are countable. Since μ is a topology on X, $U \cap V$ is μ-α-open. Now,

$$(U \cap V) \backslash (A \cap B) \subset (U \backslash A) \cup (V \backslash B).$$

Since $U \backslash A$ and $V \backslash B$ are countable, $(U \cap V) \backslash (A \cap B)$ is countable. Hence, $A \cap B$ is ω_μ-α-open. □

The following easy example shows that the converse of Proposition **4.4**(iii) need not be correct, in general, even for topological spaces.

Example 4.6. *Let X be an uncountable set and $\mu = \{\emptyset, X, A\}$, where A is a non-empty countable subset of X. Then by Proposition **4.4**(v), $X \backslash A$ is ω_μ-α-open; however, $X \backslash A$ is not μ-α-open as it is not μ-preopen (observe that $X \backslash A \not\subset i_\mu\left(c_\mu(X \backslash A)\right) = i_\mu(X \backslash A) = \emptyset$).*

The following example shows that the converse of Proposition **4.4**(v) need not be correct, in general, even for topological spaces.

Example 4.7. *Let X be an uncountable set equipped with the discrete topology μ. Then clearly every subset of X is μ-α-open; thus by Proposition **4.4**(iii), every subset of X is ω_μ-α-open. Now let A be a subset of X such that $X\backslash A$ is uncountable. Then $X\backslash A$ is ω_μ-α-closed which is uncountable.*

5. More properties

The primary purpose of this section is to provide more properties and mapping properties of μ-α-Lindelöf sets.

Proposition 5.1. *Let A be an ω_μ-α-open subset of a μ-space (X,μ). If A is μ-α-Lindelöf, then $A = B\backslash C$ for some μ-α-open subset B of X and some countable subset C of X.*

Proof. Since A is ω_μ-α-open, it follows that for each $x \in A$ there exists a μ-α-open set U_x containing x such that $U_x\backslash A$ is countable, but A is μ-α-Lindelöf, so there exist $x_1, x_2, x_3, \ldots \in A$ such that $A \subset \bigcup_{i=1}^{\infty} U_{x_i}$. Thus, $A = (\bigcup_{i=1}^{\infty} U_{x_i}) \setminus (\bigcup_{i=1}^{\infty} (U_{x_i}\backslash A))$. Let $B = \bigcup_{i=1}^{\infty} U_{x_i}$ and $C = \bigcup_{i=1}^{\infty} (U_{x_i}\backslash A)$. Then B is μ-α-open and C is countable. □

Proposition 5.2. *Let A be a subset of a μ-space (X,μ). Then A is μ-α-Lindelöf if and only if A is $\omega_{\alpha(\mu)}$-Lindelöf.*

Proof. **Necessity.** Suppose that $\mathscr{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of A by $\omega_{\alpha(\mu)}$-open sets. Then for each $x \in A$, $x \in S_{\alpha(x)}$, for some $\alpha(x) \in \Lambda$. Thus, there exists a μ-α-open set U_x containing x such that $U_x\backslash S_{\alpha(x)}$ is countable. Now $\mathscr{A} = \{U_x : x \in A\}$ is a cover of A by μ-α-open sets, but A is μ-α-Lindelöf, so there exist $x_1, x_2, x_3, \ldots \in A$ such that $A \subset \bigcup_{i=1}^{\infty} (U_{x_i} \cap A) \subset \left(\bigcup_{i=1}^{\infty} \left(\left(U_{x_i}\backslash S_{\alpha(x_i)}\right) \cap A\right)\right) \cup \left(\bigcup_{i=1}^{\infty} S_{\alpha(x_i)}\right)$. Since $\bigcup_{i=1}^{\infty} \left(\left(U_{x_i}\backslash S_{\alpha(x_i)}\right) \cap A\right)$ is a countable subset of A, it is covered by a countable subcollection $\mathscr{B}$ of $\mathscr{S}$. Thus, $\mathscr{S}$ has $\mathscr{B} \cup \left\{S_{\alpha(x_i)} : i \in \mathbb{N}\right\}$ as a countable subcover. Hence, A is $\omega_{\alpha(\mu)}$-Lindelöf.

Sufficiency. Follows from Proposition **4.4**(iii). □

Proposition 5.3. *Let A be a μ-α-Lindelöf subset of a μ-space (X,μ) and B be an ω_μ-α-closed subset of X. Then $A \cap B$ is μ-α-Lindelöf. In particular, an ω_μ-α-closed subset A of a μ-α-Lindelöf space (X,μ) is μ-α-Lindelöf.*

Proof. Suppose that $\mathscr{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $A \cap B$ by μ-α-open sets. Then $\mathscr{A} = \{S_\alpha : \alpha \in \Lambda\} \cup \{X\backslash B\}$ is a cover of A by ω_μ-α-open sets, but A is μ-α-Lindelöf, so it follows from Proposition **5.2** that there exist $\alpha_1, \alpha_2, , \alpha_3, \ldots \in \Lambda$ such that $A \subset (\bigcup_{i=1}^{\infty} S_{\alpha_i}) \cup (X\backslash B)$. Thus, $A \cap B \subset \bigcup_{i=1}^{\infty} (S_{\alpha_i} \cap B) \subset \bigcup_{i=1}^{\infty} S_{\alpha_i}$. Hence, $A \cap B$ is μ-α-Lindelöf. □

Corollary 5.4. *Let A be a μ-α-Lindelöf subset of a μ-space (X,μ) and B be a μ-α-closed subset of X. Then $A \cap B$ is μ-α-Lindelöf. In particular, a μ-α-closed subset A of a μ-α-Lindelöf space (X,μ) is μ-α-Lindelöf.*

Proposition 5.5. *Let (X,μ) be a μ-space. If every proper μ-α-closed subset of X is μ-α-Lindelöf, then X is μ-α-Lindelöf.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of X by μ-α-open subsets of X. Choose $\alpha_0 \in \Lambda$ such that $U_{\alpha_0} \neq \emptyset$. Then $X\backslash U_{\alpha_0}$ is a proper μ-α-closed subset of X, thus by assumption, $X\backslash U_{\alpha_0}$ is μ-α-Lindelöf, so there exist $\alpha_1, \alpha_2, \alpha_3, \ldots \in \Lambda$ such that $X\backslash U_{\alpha_0} \subset \bigcup_{i=1}^{\infty} U_{\alpha_i}$, and thus, $X = \bigcup_{i=0}^{\infty} U_{\alpha_i}$. Hence, X is μ-α-Lindelöf. □

We will study now a sum property of μ-α-Lindelöf spaces, to proceed, we recall the following.

Definition 5.6. ***(Sarsak 2013)*** *Let (X_α, μ_α) be a GTS for each $\alpha \in \Lambda$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family of sets. The collection μ of subsets of $\bigcup X_\alpha$ is defined as follows:*

$$\mu = \left\{U \subset \bigcup X_\alpha : U \cap X_\alpha \in \mu_\alpha, \forall\ \alpha \in \Lambda\right\}.$$

Remark 5.7. ***(Sarsak 2013)*** *Let (X_α, μ_α) be a GTS for each $\alpha \in \Lambda$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family of sets, and let μ be as in Definition **5.6**. Then μ is a GT on $\bigcup X_\alpha$. The GTS $(\bigcup X_\alpha, \mu)$ will be called the generalized topological sum of $X_\alpha, \alpha \in \Lambda$, and will be denoted by $\oplus X_\alpha$.*

Remark 5.8. ***(Sarsak 2013)*** *Let (X_α, μ_α) be a μ_α-space for each $\alpha \in \Lambda$, and let $(\oplus X_\alpha, \mu)$ be the generalized topological sum of (X_α, μ_α), $\alpha \in \Lambda$. Then it is easy to see the following:*
(i) $(\oplus X_\alpha, \mu)$ is a μ-space;
(ii) X_α is both μ-open and μ-closed for each $\alpha \in \Lambda$.

Proposition 5.9. *Let (X_α, μ_α) be a μ_α-space for each $\alpha \in \Lambda$, and let $(\oplus X_\alpha, \mu)$ be the generalized topological sum of $(X_\alpha, \mu_\alpha), \alpha \in \Lambda$. Then $\oplus X_\alpha$ is μ-α-Lindelöf if and only if X_α is μ-α-Lindelöf for each $\alpha \in \Lambda$ and Λ is countable.*

Proof. **Necessity.** Observe first by Remark **5.8**(i) that $(\oplus X_\alpha, \mu)$ is a μ-space. By Remark **5.8**(ii), X_α is μ-α-closed for each $\alpha \in \Lambda$. Since $\oplus X_\alpha$ is μ-α-Lindelöf, it follows by Corollary **5.4** that X_α is μ-α-Lindelöf for each $\alpha \in \Lambda$. Also by Remark **5.8**(ii), X_α is μ-α-open for each $\alpha \in \Lambda$, thus, $\mathscr{A} = \{X_\alpha : \alpha \in \Lambda\}$ is a cover of $\oplus X_\alpha$ by μ-α-open sets, but $\oplus X_\alpha$ is μ-α-Lindelöf, so, Λ is countable.

Sufficiency. Follows from Remark **3.3**. ☐

At the end of this section, we study several mapping properties of μ-α-Lindelöf sets.

Recall that a function $f : (X, \mu) \to (Y, \kappa)$ is called (μ, κ)-continuous (Sarsak 2011) if the inverse image of each κ-open set is μ-open, and called (μ, κ)-closed (Sarsak 2011) if the image of every μ-closed set is κ-closed.

Proposition 5.10. *Let $f : (X, \mu) \to (Y, \kappa)$ be a $(\omega_{\alpha(\mu)}, \alpha(\kappa))$-continuous function, where (X, μ) is a μ-space and (Y, κ) is a κ-space. If A is μ-α-Lindelöf, then $f(A)$ is κ-α-Lindelöf.*

Proof. Suppose that $\mathscr{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $f(A)$ by κ-α-open sets. Then $\mathscr{A} = \left\{f^{-1}(S_\alpha) : \alpha \in \Lambda\right\}$ is a cover of A, but f is $(\omega_{\alpha(\mu)}, \alpha(\kappa))$-continuous, so $f^{-1}(S_\alpha)$ is ω_μ-α-open for each $\alpha \in \Lambda$. Since A is μ-α-Lindelöf, it follows from Proposition **5.2** that there exist $\alpha_1, \alpha_2, \alpha_3, ... \in \Lambda$ such that $A \subset \bigcup_{i=1}^{\infty} f^{-1}(S_{\alpha_i})$. Thus $f(A) \subset \bigcup_{i=1}^{\infty} f\left(f^{-1}(S_{\alpha_i})\right) \subset \bigcup_{i=1}^{\infty} S_{\alpha_i}$. Hence, $f(A)$ is κ-α-Lindelöf. ☐

Corollary 5.11. *Let $f : (X, \mu) \to (Y, \kappa)$ be an $(\alpha(\mu), \alpha(\kappa))$-continuous function, where (X, μ) is a μ-space and (Y, κ) is a κ-space. If A is μ-α-Lindelöf, then $f(A)$ is κ-α-Lindelöf.*

Proposition 5.12. *Let $f : (X, \mu) \to (Y, \kappa)$ be an $(\alpha(\mu), \omega_{\alpha(\kappa)})$-closed function, where (X, μ) is a μ-space and (Y, κ) is a κ-space. If for each $y \in Y$, $f^{-1}(y)$ is μ-α-Lindelöf, then $f^{-1}(A)$ is μ-α-Lindelöf whenever A is κ-α-Lindelöf.*

Proof. Suppose that $\mathscr{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $f^{-1}(A)$ by μ-α-open sets. Then it follows by assumption that for each $y \in A$ there exists a countable subcollection $\mathscr{S}^y$ of $\mathscr{S}$ such that $f^{-1}(y) \subset \bigcup \mathscr{S}^y$. Let $V_y = \bigcup \mathscr{S}^y$. Then V_y is μ-α-open. Let $H_y = Y \backslash f(X \backslash V_y)$. Then H_y is ω_κ-α-open as f is $(\alpha(\mu), \omega_{\alpha(\kappa)})$-closed, also $y \in H_y$ for each $y \in A$ as $f^{-1}(y) \subset V_y$. Thus, $\mathscr{H} = \{H_y : y \in A\}$ is a cover of A by ω_κ-α-open sets, but A is κ-α-Lindelöf, so it follows from Proposition **5.2** that there exist $y_1, y_2, y_3, ... \in A$ such that $A \subset \bigcup_{i=1}^\infty H_{y_i}$. Thus, $f^{-1}(A) \subset \bigcup_{i=1}^\infty f^{-1}(H_{y_i}) \subset \bigcup_{i=1}^\infty V_{y_i}$. Since $\mathscr{S}^{y_i}$ is a countable subcollection of $\mathscr{S}$ for each $i \in \mathbb{N}$, it follows that $\bigcup_{i=1}^\infty \mathscr{S}^{y_i}$ is a countable subcollection of $\mathscr{S}$. Hence, $f^{-1}(A)$ is μ-α-Lindelöf. ☐

Corollary 5.13. *Let $f : (X, \mu) \to (Y, \kappa)$ be an $(\alpha(\mu), \alpha(\kappa))$-closed function, where (X, μ) is a μ-space and (Y, κ) is a κ-space. If for each $y \in Y$, $f^{-1}(y)$ is μ-α-Lindelöf, then $f^{-1}(A)$ is μ-α-Lindelöf whenever A is κ-α-Lindelöf.*

The next issue is to study a product property of μ-α-Lindelöf spaces; to proceed, we recall the following.

Proposition 5.14. ***(Sarsak 2013)*** *Let (X, μ) and (Y, κ) be GTSs, and let $\mathscr{U} = \{U \times V : U \in \mu, V \in \kappa\}$. Then $\lambda = \{\bigcup \mathscr{V} : \mathscr{V} \subset \mathscr{U}\}$ is a GT on $X \times Y$.*

Remark 5.15. ***(Sarsak 2013)*** *Let (X, μ) and (Y, κ) be GTSs. Then λ from Proposition **5.14** is called the generalized product topology on $X \times Y$, and denoted by $\mu \times \kappa$. Let $A \subset X$, $B \subset Y$, and $K \subset X \times Y$. Then it is easy to see the following:*
(i) $c_\lambda(A \times B) = c_\mu(A) \times c_\kappa(B)$;
(ii) $i_\lambda(A \times B) = i_\mu(A) \times i_\kappa(B)$.

Remark 5.16. ***(Sarsak 2013)*** *Let (X, μ) be a μ-space, (Y, κ) be a κ-space, and λ be the generalized product topology on $X \times Y$. Then it is clear that $(X \times Y, \lambda)$ is a λ-space.*

Lemma 5.17. *Let (X, μ) be a μ-space, (Y, κ) be a κ-space, and λ be the generalized product topology on $X \times Y$. Then the projection $P_X : (X \times Y, \lambda) \to (X, \mu)$ (resp. $P_Y : (X \times Y, \lambda) \to (Y, \kappa)$) is $(\alpha(\lambda), \alpha(\mu))$-continuous (resp. $(\alpha(\lambda), \alpha(\kappa))$-continuous).*

Proof. We will show that the projection $P_X : (X \times Y, \lambda) \to (X, \mu)$ is $(\alpha(\lambda), \alpha(\mu))$-continuous, the other case is similar. Let A be a μ-α-open subset of X. Then $(P_X)^{-1}(A) = A \times Y$. We want to show that $A \times Y$ is λ-α-open. Now by Remark **5.15**, $i_\lambda(c_\lambda(i_\lambda(A \times Y))) = i_\mu(c_\mu(i_\mu(A))) \times Y \supset A \times Y$. Thus, $A \times Y$ is λ-α-open. ☐

Corollary 5.18. *Let (X, μ) be a μ-space and (Y, κ) be a κ-space, and λ be the generalized product topology on $X \times Y$. If $X \times Y$ is λ-α-Lindelöf, then (X, μ) is μ-α-Lindelöf and (Y, κ) is κ-α-Lindelöf.*

Proof. Observe first by Remark **5.16** that since (X, μ) is a μ-space and (Y, κ) is a κ-space, $(X \times Y, \lambda)$ is a λ-space. The result follows from Corollary **5.11** and Lemma **5.17**. ☐

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