DOI: 10.1478/AAPP.1011A2

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 101, No. 1, A2 (2023)

MATHEMATICS IN PHYSICS PROBLEM-SOLVING A KINEMATICS STUDY IN HIGH SCHOOL

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Abstract. One of the objectives of the high school is to support a domain of thought aimed at expanding the phenomenological field so that the experiences can give a greater basic efficacy to the scientific schemes to be adopted. This requires a conceptual change: teachers must guide students towards the awareness that the schemes of spontaneous knowledge and those of scientific knowledge have different contexts of use and this is not easy, also due to the limitations of linguistic tools. From a constructivist perspective, conceptual change mainly concerns the ability to recognize the proper context of a schema and the specific meaning of the words used in the different schemas. A methodological approach that allows us to go in this direction is that of problem-solving, often neglected in Italian secondary schools for various reasons. This contribution, which is placed in the aforementioned perspective, aims to underline the relationship between mathematical language and science. Specifically, through an example of didactic activity - based on problem-solving - it is shown how the study of kinematics can be abstracted into purely mathematical expressions.

«The strange thing about physics is that for the fundamental laws we still need mathematics . . . the more we investigate, the more laws we find, and the deeper we penetrate nature, the more the disease persists. Every one of our laws is a purely mathematical statement . . . »

Richard Feynman, *The Character of Physical Law*, The M.I.T. Press, 1967.

1. Introduction

Today, there is a broad consensus among scientists that Mathematics constitutes the language of modern Physics. Poincaré (1958, p. 76) expressed this view succinctly:

> *«. . . ordinary language is too poor, it is besides too vague, to express relations so delicate, so rich, and so precise. This therefore is one reason why the physicist can not do without Mathematics; it furnishes him the only language he can speak.»*

Lévy-Leblond (1992) argues that mathematics plays a much deeper role in physics and that the relation between Mathematics and Physics is not instrumental. He maintains that mathematics is internalized by physics and, like Bachelard (1965), he characterizes the relation of Mathematics to Physics as that of "constitutivity". The language of mathematics - rich and indispensable for representing the physical - is not only a linguistic tool but a scientific knowledge constitutive factor. The relationship between mathematical language and science is not optional: mathematics has taught modern science something profound that lies at the heart of the scientific method. The object of scientific investigation is always - in spite of its materiality - in a certain sense an abstract object. The foundation and mathematization of the modern concept of scientific object are based on approximations and idealizations, with the exclusion from the field of research of concrete subjectivity, implying the "*this*", the "*here*" and "*now*", and the essence of the object. For example, if we consider the classic mechanical experiment, of a body sliding on an inclined plane, some physical quantities of this object are observed, ready to be described mathematically. In high school, the relationship between Mathematics and Physics is an important element of teaching these disciplines. Since their beginnings in the ancient world, Physics and Mathematics have been deeply interrelated, and this mutual influence has played an essential role in both their developments; however, the image typically found in educational contexts is often quite different. As was well evidenced by Karam (2015, p. 487):

> «*In physics education, it is usual to find mathematics being seen as a mere tool to describe and calculate, whereas in mathematics education, physics is commonly viewed as a possible context for the application of mathematical concepts that were previously defined abstractly. This dichotomy creates significant learning problems for the students ... This problem demands a systematic research effort from experts in different fields, especially the ones who aim at informing educational practices by reflecting on historical, philosophical, and sociological aspects of scientific knowledge.*»

In particular, their deep epistemological affinities should be highlighted in both disciplinary contexts. In this frame of reference, Mathematics not only provides tools for Physics, but also guides intuition for physics; vice versa Physics, together with its laboratory activities, can provide easy access to mathematics topics that are not immediately understood (Caccamo and Magazù 2018). The transition from experience[1] to experiment[2] is functional to this transition from the real object to the object of scientific investigation. The object results from the clippings that every science operates on things by placing itself at a very specific point of view. Science delimits the field of its own investigation in reality, it cuts out the object of investigation, according to a particular point of view; studies abstract objects from reality, ready to be described mathematically. This means that there is a lot of abstraction in physics, and not only in modern and objectively very sophisticated physics, but also in the more basic version. In this abstraction, in addition to objects[3], we also find concepts[4] and situations[5], built starting from reality, but stripping the latter of all but the very few

[1]Observation of what happens in nature, in circumstances not created or desired by us.

[2]Circumstances are created and organized *ad hoc* by us to include certain factors and exclude others.

[3]*i.e.*: what does 'material point' mean?

[4]*i.e.*: what does 'force' mean? It is an abstract mathematical concept that represents a logical intermediary in physical laws to translate our idea of randomness. We observe the effects of forces, not the forces themselves.

[5]*i.e.*: absence of friction, falling into a vacuum, uniform rectilinear motion.

properties on which attention is focused. From the didactic point of view, this process is not easy because it requires of the teacher, in daily classroom practice, an adequate methodological approach as abstraction is often opposed to reality. What is abstract is not real or, better said, concrete. Therefore, it is necessary to provide a reading key rich in other points of view, otherwise the teacher risks reducing abstraction to a sterile, useless procedure, contrasted with a more profitable concreteness. On the contrary, 'manipulating' abstraction means recognizing mathematical regularities, recognizing them in reality and using them to decide and act in real life. Abstraction is not only useful but helps us to better interpret the complexity of reality that is before our eyes. To 'manipulate' the abstraction, the teacher must adopt a communicative choice that starts from the simplest realities, from the basic laws, from elementary principles, to gradually arrive at the more complex reality, maintaining the right balance between rigor and effectiveness. A communicative choice that must help the student to observe reality with a critical eye, that is an eye that ideally knows how to "subtract all but a few elements" from reality, to focus each time on a certain essential.

2. Methodological framework

Measurement is the gateway of reality into Physics. Once through this door, physical quantity is identified with a mathematical quantity. Until reality passes through that door, the game of science cannot begin. It is also noted that while the fundamental quantities (time, length, mass, *etc.*) can be measured, the derived quantities (average speed, density, *etc.*) are calculated starting from fundamental quantities. All these quantities cannot be defined without mathematics and, taking a further step, it is noted that certain derived quantities are defined with complex mathematical procedures[6]. Historical events remind us that modern science was born with Galileo Galilei[7], but matured with Isaac Newton[8] who gave it an even more modern form thanks to mathematics through the use of differential and integral calculus (and more). The fact that physical quantities are translated into mathematical quantities (Feynman 1967) highlights that mathematics plays a "constitutive" role in modern science. A role well described by Galilei in his scientific treatise *Dialogo Sopra i Massimi Sistemi del Mondo*[9] (Dialogue Concerning the Two Chief World Systems) in which - in the form of dialogue - the structure of the world and of reality is defined on the basis of experience and mathematical models (*sensible experiences and certain demonstrations*). In his argument, Galileo highlights a gradual methodology based on the processes of mathematical deduction and on experimental verification. In this analysis of the scientific method and the mathematization of Physics, genericity is made possible by the abstraction of the object we are talking about. The abstraction of mathematical objects makes it possible to reason with universal conclusions. The method of modern science, therefore, presents a profound analogy with the mathematical demonstrative method. It is through a common

[6] *i.e.*: the instantaneous velocity is a continuous function of time and gives the velocity at any point in time during a particle's motion; we can calculate the instantaneous velocity at a specific time by taking the derivative of the position function, which gives us the functional form of instantaneous velocity $v(t)$.

[7] Galileo Galilei (Pisa 1564 - Arcetri 1642).

[8] Isaac Newton (Woolsthorpe 1642 - Kensington 1727).

[9] Published in Florence in 1632.

process of approximation/abstraction that different physical phenomena are encompassed by analogous mathematizations. We note, further, that such convergences lead to the construction of new “multivalent” physical concepts, with operational validity in various domains[10]. It is quite clear that such physical concepts are indeed based on and constituted by a specific mathematical formulation. Abstraction is a real strength both for mathematics and for science and in the case of the latter, however, this determines an apparent paradox: to understand reality, one must sometimes move away from it and focus only on a few aspects. The ‘constitutive’ role of Mathematics in modern science represents an important objective of the teaching of Mathematics and Physics in high school. Communicating this aspect, documenting, and exemplifying the relationship between the two disciplines helps to understand the difference and to make the study of both more reasoned, less mechanical and, above all, less mnemonic. On the application level, the objective of the didactic action must be centred on mathematical language as a bearer of scientific knowledge, or rather to make students grasp the value of this aspect: a logical deduction, insofar as it leads us to a conclusion which we were not already aware of, actually increases our knowledge. This means giving space, in daily classroom practice, to ‘sensible experiences and mathematical demonstrations’ by structuring activities aimed at accompanying students in their knowledge and learning paths, stimulating their curiosity starting from the reality that surrounds them in a way to help them identify similarities and differences between Mathematics and Physics. A didactic action that shows the use of Mathematics for physical deductions implies a series of methodological choices in which the significant example has an importance to which it is worth sacrificing systematicity at all costs. In this sense, a methodology that lends itself is that of problem-solving or rather a didactic approach aimed at developing, on a rational and operational level, the ability to solve problems. Problem-solving tends to emphasize both the qualitative and quantitative aspects of problem-solving, and it is this dual emphasis which may help students not only to improve their problem-solving performance, but also to better understand concepts (Hewitt 2002). The scientific reflections which started in the first half of the last century (Duncker 1966; Köhler 1929; Polya 1945; Wertheimer 2020), up to the most recent European projects (Ryjchen and Salganik 2000) have identified the ability to solve problems as one of the more important transversal skills that the education system should promote (Asquini 2014), so much so as to place it at the centre of recent and important international surveys (OECD 2013). In addition, the operative nucleus of problem-solving is the discovery and mastering of problematic situations in general in order to develop the student’s heuristic potential and evaluation and objective judgment skills. Problem-solving transforms the problem into a well-defined project, to be managed according to the techniques of mathematical resolution. In fact, to solve the problem (problem-solving) it is necessary to become aware of the problem (reading the text - problem reading), then it is necessary to define the problem (problem-setting) and finally break down the problem into secondary problems (problem-analysis). The need to always operate at different levels requires abstractions on the phenomenon, abstractions on data and subdivision, and hierarchies of mathematical relationships. Adopting this methodology, in classroom practice, means creating a learning environment that responds to the canons of

[10]Consider, for instance, the notions of resonance, impedance, *etc*., as used in all (linear) oscillating phenomena, whether they are mechanical, acoustical, electrical, *etc*.

metacognitive teaching. In the real-life application of the method, the teacher must avoid mechanisms and repetitiveness; on the other hand, there is a need for flexibility of approach in order to gradually lead the student to gradually appropriate the method, to internalize it and use it critically several times.

3. Design of problem-solving process

In the case of a physics problem, the importance of the problem-solving methodology lies in providing key concepts and rational procedures that are valid far beyond physics itself. Given the procedural nature of the method, it is essential to always operate at three levels. They are as follows: abstraction on the phenomenon, abstraction on data, subdivision and hierarchies. Abstract is a verb that forces us to think, reflect, analyse what the phenomenon highlights. It means identifying a value by separating and isolating it, with a mental procedure, from other known values. Therefore, abstracting is a mental operation of thought that works to allow the physical-mathematical models to which the object refers to be identified in the body of science. Abstracting on the data means getting ideas on the quantitative aspects (properties) of the object, that is, it means having a clear idea of the group of physical quantities involved whose measurements are known. In addition, it gives an idea of the order of magnitude of what must be calculated. Subdividing a problem means dividing it into levels in order to be able to abstract its important aspects. To hierarchize the levels means to identify a precise hierarchy, an order, in the sense of levels that come first from levels that come afterwards. Furthermore, the abstraction must allow us to marginalize some details of the phenomenon in such a way as to be able to focus only on some aspects that are part of the problem. The subdivision of a complex problem into many small problems is not an easy operation as there is no single technique dedicated to doing this. However, subdivision is useful for obtaining simpler and reasonably independent problems. Polya (1945) established four general steps to follow as a problem-solving strategy (see Fig. 1)[11]. These steps encompass "the mental processes and unconscious questions experts explore as they themselves approach problem-solving" (Lederman 2009).

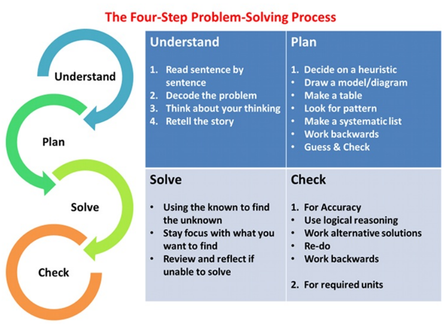


FIGURE 1. Pólya's phases of problem-solving.

[11]Image source credit: https://www.cleanpng.com/png-problem-solving-worksheet-mathematical-problem-soc-2993375.

Nevertheless, even though the four steps in Fig. 1 seem very simple, their generality makes it hard for students to follow them. For this reason, we have chosen to detail the four steps as much as possible in order to make them more accessible. The justification for utilizing a more detailed problem-solving strategy can be found in the words of (Schoenfeld 1980):

> *«First, the strategies are more complex than their simple descriptions would seem to indicate. If we want students to use them, we must describe them in detail and teach them with the same seriousness that we would teach any other mathematics.»*

The example shown below highlights the importance of a structured, systemic methodology to solve physics problems.

3.1 A Kinematics Introduction Study

Kinematics is the branch of classical mechanics that describes the motion of points, objects, and systems of groups of objects, without reference to the causes of motion (*i.e.*, forces). The study of kinematics is often referred to as the "geometry of motion". Determining the types of physical problems that are most effective in promoting student use of problem-solving strategies is the first step the teacher must take.

Problem 1: *A car starts from a standstill and, moving along a straight road, accelerates uniformly until it reaches a velocity of* 5.0 *m/s in* 10 *s. Determine the acceleration and the space travelled in this time interval.*

STEP 1 – UNDERSTAND

After a careful reading and understanding of the text, a list of the information provided and the unknown elements is prepared (Table 1).

TABLE 1. Information provided by the problem.

INFORMATION	
Known	- The car starts from a standstill. - It moves on a straight road. - In 10 s it reaches the velocity of 5.0 m/s.
Unknown	- The acceleration of the car. - The space travelled in 10 s.

Subsequently, the information is explained in the respective physical quantities (Table 2).

TABLE 2. List of physical quantities.

PHYSICAL QUANTITIES	
Known	Zero initial velocity Motion in one dimension Time interval
Unknown	Acceleration Displacement

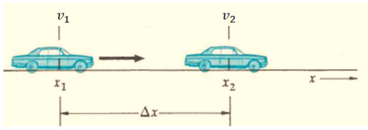


FIGURE 2. Car that moves along the x-axis.

In this way, a diagram is drawn with the relative notation system so as to visualize the physical quantities (Fig. 2). The drawing in Fig. 2 will help students focus attention on motion in one dimension, *i.e.*, motion along a straight line. This leads us to reason on the fact that for such a limited motion there are only two possible oriented directions (one positive and the other negative). To describe motion, the concepts of *displacement*, *velocity*, and *acceleration* are needed. The teacher stimulates the discussion in order to help the students to generalize: the motion of a particle in three dimensions is characterized by vector quantities, which are defined by an oriented direction (*orientation*), and a numerical value (*module*).

A motion that has a straight trajectory is called a straight line, but in general, a motion's space-time graph can be any curve. The fact that it takes place in a single direction makes it possible to use a single axis to represent positions and displacements. The considerations made on the movement can be summarized by the students by means of a table (Table 3).

TABLE 3. Space and time in one-dimensional motion.

Physical Quantity	GEOMETRICAL REPRESENTATION			SI Units
	Local reference systems	Points	Variations	
SPACE	x $x-$axis	x Position	Δx Displacement	m meter
TIME	t $t-$axis	x Instant in time	Δt Elapsed time	s second

STEP 2 – PLAN

Students must reason on the fact that to evaluate the motion of a body it is important to know its position and the corresponding instant in time. The concept of velocity makes it possible to compare distances travelled and time intervals spent. However, the velocity reveals nothing about the direction of movement. To describe both the velocity with which an object moves and the direction of its movement, it is necessary to introduce the vector concept of velocity. Two other concepts already encountered will be utilized: displacement and time. Building new concepts from more basic ones is a common theme in physics. Indeed, the great strength of Physics as a science is that it builds a coherent understanding of nature through the development of interrelated concepts. The teacher will specify that the velocity we are talking about is an average velocity, that is, a quantity that refers to motion as a whole. More specifically, the average velocity depends only on the starting and ending points of the motion and not on what happens between them. Students will thus be able to deduce that the average velocity is a derived quantity and can assume a positive or negative value. Also, in this case, a synoptic table is useful (Table 4).

TABLE 4. Average velocity is a derived quantity.

Derived Quantity	Physical Fundamental Quantities	Formula	SI Units
Average velocity	Length	$v = \frac{\Delta x}{\Delta t}$	m/s (meters per second)
	Time		

At this point, it is necessary to pass from the physical phenomenon to the mathematical model; this means identifying physical quantities in terms of the formal mathematics language (Table 5). This can be done by updating Table 2.

TABLE 5. Identification of the formal mathematics language.

	PHYSICAL QUANTITIES	MATHEMATICAL SYMBOLS
KNOWN DATA	Zero initial velocity It moves in a straight line In 10 s it reaches the velocity of 5.0 m/s	$v_1 = 0$ Motion along *x-direction* For Δt=10 s v_2= 5.0 m/s
UNKNOWN DATA	Acceleration Displacement	a =? Δx =?

What equations are used to determine the acceleration a and the displacement Δx? Starting from the physical quantity of velocity, defined as the variation of the position over time, the operation of variation over time is applied again and a new quantity is obtained: acceleration. Acceleration is therefore a variation of a change in position, and this complication makes it difficult to use in everyday life. Since t is always positive, the sign of the average acceleration depends on the velocity v, which can also be negative or zero. If $a = 0$, the v is constant and the motion is uniform. If $a < 0$, it means that the velocity variation is negative: this happens when the object slows down while proceeding in the same direction as the space axis, or when the velocity increases in the opposite direction (Fig. 3).

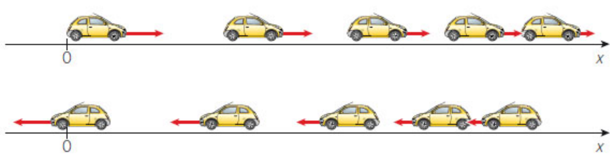


FIGURE 3. Variations of the velocity of the car.

Just as the average velocity was defined as the change in position in a unit of time (one second), acceleration is defined as the change in velocity in a unit of time. This allows students to reuse all the arguments they made previously. If two different physical quantities are represented by equal equations, then they behave in the same way, because the equations are solved in the same way. The teacher points out that the analogy based on the similarity of the mathematical equations (Fig. 4) is a widely used tool in physics and allows the elaboration of different theories of unknown phenomena starting from the knowledge of different phenomena.

FIGURE 4. The analogy between velocity and acceleration.

Once the theory has been developed, the comparison between the mathematical representation and the reality it wants to describe is always and, in any case, necessary. Once again, the drawing up of a synoptic table (see Table 6) will help the students to take stock of the situation.

TABLE 6. Comparison between velocity and acceleration.

	VELOCITY	ACCELERATION
Formula	$v = \frac{\Delta x}{\Delta t}$	$a = \frac{\Delta v}{\Delta t}$
Explanation	CHANGE OF: $\frac{\textit{Position}}{\textit{Elapsed time}}$	CHANGE OF: $\frac{\textit{Velocity}}{\textit{Elapsed time}}$
SI Units	$\frac{\mathrm{m}}{\mathrm{s}}$	$\frac{\mathrm{m}}{\mathrm{s}^2}$

At this point, students are aware of the fact that equations are needed which contain both the given quantities (the initial velocity v_1, the final velocity v_2, and the time t) and the unknowns (displacement Δx and acceleration a). Ultimately, we need the following equations:

$$x = vt \tag{1}$$

Equation (1) that provides x contains the average velocity v, consequently, the latter must be determined. For the uniformly accelerated motion, the average velocity is given by:

$$v = \frac{1}{2}(v_1 + v_2) \tag{2}$$

STEP 3 – SOLVE

The data for the motion in the x-direction are listed in Table 7.

TABLE 7. Data for the motion in the x-direction.

x-Direction Data				
x	v_1	v_2	Δt	a
?	0	5.0 m/s	10 s	?

By substituting the numerical values and carrying out the appropriate algebraic passages, we arrive at the solution of the problem.

The acceleration module is:

$$a = \frac{5.0\,\mathrm{m/s} - 0.0\,\mathrm{m/s}}{10\,\mathrm{s}} = 0.50\,\mathrm{m/s}^2$$

The module of the average velocity is:

$$a = \frac{1}{2}(0.0\,\mathrm{m/s} - 5.0\,\mathrm{m/s}) = 2.50\,\mathrm{m/s}^2$$

The displacement by the car in $10\,\mathrm{s}$ is:

$$x = (2.5\,\mathrm{m/s})(10\,\mathrm{s}) = 25\,\mathrm{m}$$

STEP 4 – CHECK

Students must check the solution found, making sure it is correct. This also means making sure of the units of measurement used and, if necessary, rounding the final result to the same number of significant figures that appear in the problem data. Having made all the problem-solving steps more detailed makes the students aware of the following key aspects:

- kinematics is the branch of mechanics concerned with objects in motion, but not with the forces involved;
- to describe motion, kinematics studies the trajectories of points, lines, and other geometric objects;
- the study of kinematics can be abstracted into purely mathematical expressions;
- kinematic equations can be used to calculate various aspects of motion such as velocity, acceleration, displacement, and time.

Table 8 reports the key points and the key terms.

In the logic of the methodological approach chosen, a second problem-solving activity is subsequently proposed in which students, based on the experience gained previously, will be more autonomous on a rational and operational level.

Problem 2: *A car traveling at a velocity of* 5.00 *m/s is brought to a halt in the space of* 20.0 *m. Determine its acceleration and the time it takes to stop, assuming that the motion occurs along the x-axis and that the acceleration is constant.*

TABLE 8. Key points and key terms.

Key Points	Choosing a local reference system is particularly important when describing an object's displacement. Displacement is the change in position of an object relative to its local reference system.
Key Terms	Displacement: A vector quantity that denotes distance with a directional component. Local reference system: A coordinate system or set of axes within which to measure the position, orientation, and other properties of objects in it.

The following reasoning strategy gives an overview of how the equations of kinematics are applied to describe motion in one dimension, such as that in Problem 1.

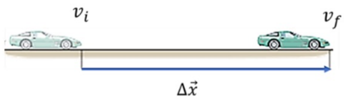
(1) Make a drawing to represent the situation being studied.
(2) Decide which directions are to be called positive (+) and negative (–) relative to a conveniently chosen coordinate origin.
(3) Remember that the time variable t has the same value for the part of the motion along the x-axis.
(4) In an organized way, write down the values (with appropriate + and – signs) that are given for any of the kinematic variables associated with the x-direction. Be on the alert for "implied data", such as the phrase "starts from rest," which means that the values of the initial velocity components are zero: $v_i = 0$ m/s. The data summary boxes used in the examples are a good way of keeping track of this information. In addition, identify the variables to determine.
(5) Before attempting to solve a problem, verify that the given information contains values for at least three of the kinematic variables. Do this for the x-direction of the motion. Once the known variables are identified along with the desired unknown variable, the appropriate relations can be located.
(6) Try to visualize the different physical situations to which the answers correspond.

Questions & Answers

Q: What information is given? How should we interpret the statement of the problem?
A1: The car travels at a velocity of 5.00 *m/s, this means that* $v_i = 5.00$ *m/s.*
A2: 'It is stopped' means that the final velocity: $v_f = 0.0$ *m/s. Then in the space of* 20.0 *m means that the change of velocity (with constant acceleration occurs in a stretch of* 20.0 *m).*
Q: What are the physical quantities that need to be determined?
A: The acceleration a and the time t that the car takes to stop.

Students should be aware by now that there is no one formula for everything! In fact, they studied the relationships between the quantities used to describe motion. Some of these relations contain the time t. Now, the students are able to sketch the physical situation (Fig. 5).

Q: How can you find the value of t? We don't have a formula that provides this. Which equation should be used to calculate a? Is there any other equation that contains the value of t?
A: By manipulating the first-degree equation $\Delta\vec{x} = vt$ *we obtain* $t = x/v$.



x- Direction Data				
$\Delta\vec{x}$	v_i	v_f	t	a
20.0 m	5.0 m/s	0.0 m/s	?	?

FIGURE 5. Drawing and organizing the physical situation.

Q: How can we derive the velocity v from the data?
A: Using the following equation:

$$v = \frac{1}{2}(v_i - v_f)$$

Q: Which equation must we use to find a?
A: The following equation:

$$a = (v_f - v_i)/t$$

The students solve the problem by highlighting that the negative sign indicates that the direction of a is opposite to that of the velocity v, this describes the fact that the car slows down.

$$v = \frac{1}{2}(v_i - v_f) \rightarrow v = 2.50\,\mathrm{m/s}$$

$$t = (\Delta\vec{x})/v \rightarrow t = \frac{(20.0\,\mathrm{m})}{2.50\,\mathrm{m/s}} = 8.00\,\mathrm{s}$$

$$a = (v_f - v_i)/t \rightarrow a = \frac{(0 - 5.00)\,\mathrm{m/s}}{8.00\,\mathrm{s}} = -0.625\,\mathrm{m/s}^2.$$

At the end of this second activity, it is useful to have the students draw up structural maps to hierarchically retrace the concepts encountered throughout the course. According to Cañas *et al.* (2003), maps play a role of schematic synthesis of what students know, they can therefore be used as advanced organizers to visualize their foreknowledge on a topic or as post organizers to visually summarize what they have learned or as a tool to support the studies. Concept mapping is a creative activity, in which the student must strive cognitively to clarify and rearrange meanings, identifying important concepts, relationships and the structure of knowledge. In this specific case, the concept mapping favours further reflection on the construction of the meaning of the first elementary concepts that characterize the study of kinematics and at the same time helps students to monitor and manage their own learning metacognitively.

4. Conclusions

Physicists and mathematicians are trained differently - and its normal - since they have different ways of doing things, and different ways of saying or writing things. According to Lévy-Leblond (1992), what gives science its vitality, whether within a discipline or at the interface between different disciplines, is not a convergence towards becoming unified and tending towards a single formulation or towards identical statements; on the contrary, it is the diversity. The fact that scientific language is inherently difficult does not exempt educators from the commitment not to add further difficulties and to remove all unnecessary ones (Ferrari 2020). The problem-solving strategy is presented here to provide a methodological approach aimed at helping students to improve their mathematical reasoning abilities, which are essential for reinforcing students' knowledge of conceptual physics. Implementing educational situations based on problem-solving allows to reduce the distance between Mathematics and Physics and, therefore, facilitates the transfer of concepts from one discipline to another. The intent of the authors is to provide teachers with ways to highlight the role of knowledge in problem-solving while maintaining the implementation of the methodological approach flexible and easily adaptable to different curricular needs. From the perspective of the problem-resolution dialectic, the problem-solving methodology obliges us with an expressive rigor to clearly and unambiguously explain the hypotheses, to identify a solution process and its subsequent formalization (Serpe and Frassia 2018). Furthermore, it induces us to appropriate concepts with a critical spirit and to use the methodologies in a non-repetitive way, positively influencing the training of young people because it creates self-esteem and awareness of one's abilities (Serpe 2020). In teaching practice, problem-solving represents a valid and effective pedagogical tool not only because it makes the active dynamics of teaching constructive, varied, and articulated, but above all because it allows the mathematical formalism to be exploited in a balanced way. In fact, if on the one hand, the mathematical formula constitutes an effective possibility of synthesis, and its analysis can be a starting point for further investigating a given aspect of physical reality, on the other hand, there is the risk of generating in the student the false belief of having fully learned the subject in question. It is not enough to remember a law if you do not know how to apply it, just as it is not enough to remember formulas if you are unable to calculate solutions. The methodological depth of problem-solving provides students with the opportunity to increase their logical-deductive faculties, to stimulate their rational abilities of analysis and synthesis and, ultimately, to provide them with the tools of a rational nature, capable of making them interpret the complex physical phenomenology (Serpe and Frassia 2021).

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Communicated 28 May 2021; manuscript received 28 June 2022; published online 24 January 2023