

ON THE RADIATION REACTION FORCE

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ABSTRACT. The radiation self-force of a point charge is obtained in a mathematically exact way in the framework of Distribution theory: the world line and the electromagnetic field of a point charge are treated as Distributions; the energy-momentum tensor of the point charge is too singular, but applying a known method (which I call pole taming), we can consider it a Distribution; the divergence of the pole tamed energy-momentum tensor is the force density Distribution from which we can read off the radiation reaction force in its usual form. It is pointed out to that this does not call forth that the spacetime motion of a point charge obeys the Lorentz–Abraham–Dirac equation.

Introduction

The radiation reaction and the equation of spacetime motion of a point charge under the action of an external force is an old problem of electromagnetism. The original work of Abraham (treated, *e.g.*, by Jackson 1998, Ch. 16.3) and then Dirac’s formulation (treated, *e.g.*, by Groot and Suttorp 1972, Ch. III.3) suffers from a mass renormalization according to which the finite mass of a particle is the difference of the positive infinite electric mass and the negative infinite mechanical mass. Other, quick deductions based on Larmor’s formula and some other ‘natural’ requirements (see Jackson 1998, Ch. 16.2) are not convincing either. Moreover, the Lorentz-Abraham-Dirac (LAD) equation admits physically untenable run-away solutions. The attempts to exclude such non-physical solutions by imposing extra conditions (*e.g.*, Mo and Papas 1971; Spohn 2000; Steane 2015; Matolcsi *et al.* 2017) do not carry much conviction.

There are a number of articles (*e.g.*, Baylis and Huschilt 1976; Rohrlich 1997, 2007; Gralla *et al.* 2009; Spohn 2011) considering various continuum charge distributions instead of a point charge and then taking or not the limit to a point. Though casting more and more new light on the problem and its possible solution, neither way seems satisfactory. A recent article (Bild *et al.* 2019) points out clearly to Dirac’s unjustified applications of Gauss–Stokes theorem, Taylor expansion and the limit procedure when an extended charge is shrunk to a point. In some papers (*e.g.*,

Taylor 1956, 1958; Rowe 1978; Gralla *et al.* 2009; Matolcsi and Ván 2020) special applications of Distribution Theory can be found when treating point charges.

In the present paper,¹ by a systematic use of Distribution Theory, it is shown that not a mathematically incorrect derivation of the radiation reaction force but a physical misapprehension is the source of why the LAD equation does not work well.

The formulation of spacetime free of coordinates and reference frames, expounded by Matolcsi (2020), is used which makes formulae shorter and more easily comprehensible; find enclosed a brief summary of the fundamental notions and some special notations.

Spacetime points and spacetime vectors (which are often confused in coordinates) are distinguished:

- \mathbf{M} , mathematically an affine space, is the set of spacetime points, its elements are denoted by normal letters, x, y etc.,
- \mathbf{M} , mathematically a vector space, is the set of spacetime vectors, its elements are denoted by boldface letters, \mathbf{x}, \mathbf{y} etc.,
- the set of time periods is \mathbb{T} ,
- the exact treatment requires the tensorial quotients of \mathbf{M} by \mathbb{T} , here we do not refer to them explicitly,
- the Lorentz product of the vectors \mathbf{x} and \mathbf{y} is denoted by $\mathbf{x} \cdot \mathbf{y}$ (which is $x_k y^k$ in coordinates); \mathbf{x} is timelike if $\mathbf{x} \cdot \mathbf{x} < 0$ (which means in coordinates the signature $(-1, 1, 1, 1)$ of the Lorentz form),
- an absolute velocity is a futurelike vector \mathbf{u} for which $\mathbf{u} \cdot \mathbf{u} = -1$,
- for an absolute velocity \mathbf{u} , $\mathbf{S}_{\mathbf{u}} := \{\mathbf{x} \in \mathbf{M} \mid \mathbf{u} \cdot \mathbf{x} = 0\}$, the set of \mathbf{u} -spacelike vectors, is a three dimensional Euclidean vector space,
- the action of linear and bilinear maps is denoted by a dot, too; e.g the action of a linear map \mathbf{L} on a vector \mathbf{x} is $\mathbf{L} \cdot \mathbf{x}$ (which is $L^i_k x^k$ in coordinates),
- the adjoint of a linear map \mathbf{L} is the linear map \mathbf{L}^* defined by $(\mathbf{L}^* \cdot \mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{L} \cdot \mathbf{y})$ for all vectors \mathbf{x}, \mathbf{y} ; shortly, $\mathbf{L}^* \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{L}$,
- \mathbf{L} is a Lorentz transformation if and only if $\mathbf{L}^* = \mathbf{L}^{-1}$,
- the tensor product $\mathbf{x} \otimes \mathbf{y}$ of the vectors \mathbf{x} and \mathbf{y} is a linear map whose action on a vector \mathbf{z} is $(\mathbf{x} \otimes \mathbf{y}) \cdot \mathbf{z} := \mathbf{x}(\mathbf{y} \cdot \mathbf{z})$ (in coordinates it is $x^i y^k z^i$); $(\mathbf{x} \otimes \mathbf{y})^* = \mathbf{y} \otimes \mathbf{x}$; $\mathbf{x} \wedge \mathbf{y} := \mathbf{x} \otimes \mathbf{y} - (\mathbf{x} \otimes \mathbf{y})^*$ and $\mathbf{x} \vee \mathbf{y} := \mathbf{x} \otimes \mathbf{y} + (\mathbf{x} \otimes \mathbf{y})^*$ are the antisymmetric and symmetric tensor product, respectively,
- the formulae of electromagnetism are written in the choice $c = 1$, $\hbar = 1$ and the electric charge is measured by real numbers.
- \mathbf{D} denotes the differentiation in spacetime (∂_k in coordinates). The antisymmetric derivative of a vector valued function \mathbf{f} is denoted by $\mathbf{D} \wedge \mathbf{f}$ ($\partial_k f_i - \partial_i f_k$ in coordinates). The spacetime divergence of \mathbf{f} is denoted by $\mathbf{D} \cdot \mathbf{f}$ ($\partial_k f^k$ in coordinates). Similarly, $\mathbf{D} \cdot \mathbf{T}$ is the spacetime divergence of a tensor valued function \mathbf{T} ($\partial_k T^{ik}$ in coordinates).
- \mathcal{D} denotes differentiation of other functions, *e.g.*, parameterizations.

The word ‘distribution’ will appear in two different senses: 1. having its everyday meaning (*e.g.*, distribution of charges); 2. being a mathematical notion (generalized

¹Preprint version: Matolcsi 2021.

function). To distinguish between them, I write distribution for the first one and Distribution for the second one.

The usual setting of Distribution Theory is based on \mathbb{R}^n (Gelfand and Shilov 1964; Demidov 2001; Horváth 2012; Dijk 2013). It is a quite simple generalization that spacetime and an observer space are taken instead of \mathbb{R}^4 and \mathbb{R}^3 . Another simple generalization is that vector and tensor Distributions are included, too. The present article can be understood without a thorough knowledge of Distributions; besides the elementary notions the only non trivial one is **pole taming**, described in the Appendices.

The guiding principle is that only quantities definable in Distribution Theory can make sense. Distributions are denoted by calligraphic letters. Locally integrable functions and the corresponding Distributions are distinguished in notation; *e.g.*, \mathbf{F} is an electromagnetic field function, \mathcal{F} is the corresponding electromagnetic field Distribution.

ELECTRODYNAMICS OF POINT CHARGES

1. The retarded proper time

The existence of a material point in spacetime is described by a **world line function** whose variable is the proper time; the range of a world line function is a **world line**, a one dimensional submanifold in spacetime.

Let r be a given continuously differentiable world line function. For a world point x , $\mathbf{s}_r(x)$ will denote the *retarded proper time corresponding to x i.e.*, the proper time point of r for which $x - r(\mathbf{s}_r(x))$ is future lightlike or zero; evidently, it is zero if and only if x is on the world line. In other words, $\mathbf{s}_r(x)$ is defined implicitly by the relations

$$(x - r(\mathbf{s}_r(x))) \cdot (x - r(\mathbf{s}_r(x))) = 0, \quad -\mathbf{u} \cdot (x - r(\mathbf{s}_r(x))) \geq 0$$

for an arbitrary absolute velocity \mathbf{u} . According to the implicit function theorem, \mathbf{s}_r is continuously differentiable outside the world line; differentiating the first equality above, we obtain that $2(x - r(\mathbf{s}_r(x))) \cdot (\mathbf{1} - \dot{r}(\mathbf{s}_r(x))) \otimes D\mathbf{s}_r[x] = 0$ i.e.

$$D\mathbf{s}_r[x] = \frac{x - r(\mathbf{s}_r(x))}{\dot{r}(\mathbf{s}_r(x)) \cdot (x - r(\mathbf{s}_r(x)))} \quad (x \notin \text{Ran}(r)).$$

Though \mathbf{s}_r is not differentiable on the world line, it is continuous there which can be seen as follows.

Let $\mathbf{s}_- < \mathbf{s}_0 < \mathbf{s}_+$ be arbitrary proper time points of the world line. If T^{\rightarrow} denotes the set of futurelike vectors then $(r(\mathbf{s}_-) + T^{\rightarrow}) \cap (r(\mathbf{s}_+) - T^{\rightarrow})$ is a bounded open set containing $r(\mathbf{s}_0)$. If x is in this set then $r(\mathbf{s}_r(x))$ is in it, too; consequently the function $x \mapsto x - r(\mathbf{s}_r(x))$ is bounded. The proper time passed between two points of the world line is less or equal than the inertial time, thus

$$\begin{aligned} |\mathbf{s}_r(x) - \mathbf{s}_0|^2 &\leq -(r(\mathbf{s}_r(x) - r(\mathbf{s}_0))) \cdot (r(\mathbf{s}_r(x) - r(\mathbf{s}_0))) = \\ &= -(r(\mathbf{s}_r(x) - x) + (x - r(\mathbf{s}_0))) \cdot ((r(\mathbf{s}_r(x) - x) + (x - r(\mathbf{s}_0)))) = \\ &= 2(x - r(\mathbf{s}_r(x))) \cdot (x - r(\mathbf{s}_0)) - (x - r(\mathbf{s}_0)) \cdot (x - r(\mathbf{s}_0)). \end{aligned}$$

Because of the mentioned boundedness, the right hand side tends to zero as x tends to $r(\mathbf{s}_0)$.

2. The electromagnetic field produced by a point charge

Let r be a given twice differentiable world line function of a point charge e . Let us introduce the following functions:

$$\mathbf{R}_r(x) := x - r(\mathbf{s}_r(x)), \quad \mathbf{u}_r(x) := \dot{r}(\mathbf{s}_r(x)), \quad \mathbf{a}_r(x) := \ddot{r}(\mathbf{s}_r(x)), \quad (1)$$

$$\mathbf{L}_r := \frac{\mathbf{R}_r}{-\mathbf{u}_r \cdot \mathbf{R}_r} = -D\mathbf{s}_r, \quad \mathbf{d}_r := \mathbf{a}_r + (\mathbf{a}_r \cdot \mathbf{L}_r)\mathbf{u}_r. \quad (2)$$

Then it is known (see Jackson 1998) that the electromagnetic field produced by a point charge e with a **given world line function** r is the regular Distribution $\mathcal{F}[r]$ defined by the locally integrable function

$$\mathbf{F}[r] := \frac{e}{4\pi} \frac{\mathbf{u}_r \wedge \mathbf{L}_r}{(-\mathbf{u}_r \cdot \mathbf{R}_r)^2} + \frac{e}{4\pi} \frac{\mathbf{d}_r \wedge \mathbf{L}_r}{-\mathbf{u}_r \cdot \mathbf{R}_r}; \quad (3)$$

the first term is the *tied field* $\mathbf{F}[r]^{td}$ which is never zero, the second term is the *radiated field* $\mathbf{F}[r]^{rd}$ which is zero if and only if the charge is not accelerated.

It is emphasized by the notation $[r]$ that the formulae above make sense only if the world line function is **given** because *the retarded proper time is defined only in that case*.

3. ‘Energy-momentum tensor’ of a point charge

As usual, for an electromagnetic field function \mathbf{F} ,

$$\mathbf{T} := -\mathbf{F} \cdot \mathbf{F} - \frac{1}{4} \text{Tr}(\mathbf{F} \cdot \mathbf{F}) \quad (4)$$

is considered the energy-momentum tensor whose negative spacetime divergence $-D \cdot \mathbf{T}$ is the spacetime force density. The problem is that \mathbf{T} , in general, is not locally integrable, thus \mathbf{T} does not define a Distribution a priori, so it has no physical meaning. In particular, this occurs for a point charge; that is why the quotation mark appeared in the section title. According to (3), the ‘energy-momentum tensor’ is

$$\mathbf{T}[r] = \mathbf{T}[r]^{td} + \mathbf{T}[r]^{rtd} + \mathbf{T}[r]^{rd} \quad (5)$$

where

$$\begin{aligned} \mathbf{T}[r]^{td} &:= -\mathbf{F}[r]^{td} \cdot \mathbf{F}[r]^{td} - \frac{1}{4} \text{Tr} \mathbf{F}[r]^{td} \cdot \mathbf{F}[r]^{td}, \\ \mathbf{T}[r]^{rtd} &:= -(\mathbf{F}[r]^{td} \cdot \mathbf{F}[r]^{rd} + \mathbf{F}[r]^{rd} \cdot \mathbf{F}[r]^{td}) - \frac{1}{4} \text{Tr}(\mathbf{F}[r]^{td} \cdot \mathbf{F}[r]^{rd} + \mathbf{F}[r]^{rd} \cdot \mathbf{F}[r]^{td}), \\ \mathbf{T}[r]^{rd} &:= -\mathbf{F}[r]^{rd} \cdot \mathbf{F}[r]^{rd} - \frac{1}{4} \text{Tr} \mathbf{F}[r]^{rd} \cdot \mathbf{F}[r]^{rd}, \end{aligned}$$

for which we have the following equalities:

$$-\mathbf{F}^{td}[r] \cdot \mathbf{F}^{td}[r] = \frac{e^2}{16\pi^2} \frac{\mathbf{u}_r \otimes \mathbf{L}_r + \mathbf{L}_r \otimes \mathbf{u}_r - \mathbf{L}_r \otimes \mathbf{L}_r}{(\mathbf{u}_r \cdot \mathbf{R}_r)^4}, \quad (6)$$

$$-\text{Tr}(\mathbf{F}^{td}[r] \cdot \mathbf{F}^{td}[r]) = \frac{e^2}{16\pi^2} \frac{2}{(\mathbf{u}_r \cdot \mathbf{R}_r)^4}, \tag{7}$$

$$-(\mathbf{F}^{td}[r] \cdot \mathbf{F}^{rd}[r] + \mathbf{F}^{rd}[r] \cdot \mathbf{F}^{td}[r]) = \frac{e^2}{16\pi^2} \frac{\mathbf{d}_r \otimes \mathbf{L}_r + \mathbf{L}_r \otimes \mathbf{d}_r + 2(\mathbf{u}_r \cdot \mathbf{d}_r)\mathbf{L}_r \otimes \mathbf{L}_r}{(-\mathbf{u}_r \cdot \mathbf{R}_r)^3}, \tag{8}$$

$$\text{Tr}(\mathbf{F}^{td}[r] \cdot \mathbf{F}^{rd}[r]) = 0, \tag{9}$$

$$-\mathbf{F}^{rd}[r] \cdot \mathbf{F}^{rd}[r] = \frac{e^2}{16\pi^2} \frac{|\mathbf{d}_r|^2 \mathbf{L}_r \otimes \mathbf{L}_r}{(\mathbf{u}_r \cdot \mathbf{R}_r)^2}, \tag{10}$$

$$\text{Tr}(\mathbf{F}^{rd}[r] \cdot \mathbf{F}^{rd}[r]) = 0. \tag{11}$$

Evidently, $\mathbf{T}[r]$ is differentiable outside the world line; its properties near the world line will be examined in the forthcoming sections.

4. Radial expansion of the retarded proper time

The retarded proper time is not differentiable on the world line, so it has not a Taylor polynomial around the world line. Nevertheless, it can be expanded in ‘radial directions’ in powers of the ‘radial distance’ as follows. We take the parameterization (60) of spacetime around a given world line $\text{Ran}(r)$; in this parameterization, at every proper time point s of r , the spacelike vectors Lorentz orthogonal to $\dot{r}(s)$ are taken into account; then according to the notations introduced in (62), ρ and \mathbf{n} describe distances and directions, respectively, in that Euclidean space. The difference between the parameter proper time and the retarded proper time is

$$\Theta(\mathbf{s}, \mathbf{q}) := \mathbf{s} - \mathbf{s}_r(r(s) + \mathbf{L}(s)\mathbf{q}).$$

Θ is a continuous function, non-negative and $\Theta(\mathbf{s}, \mathbf{q}) = 0$ if and only if $\mathbf{q} = 0$. Thus,

$$\lim_{\mathbf{q} \rightarrow \mathbf{0}} \Theta(\mathbf{s}, \mathbf{q}) = 0 \tag{12}$$

holds as well.

In the sequel, r is supposed to be three times continuously differentiable, the variables are omitted for the sake of brevity and Ordo denotes a function whose limit at zero is zero. First, let us observe that according to (12)

$$\Theta = \text{Ordo}(\rho).$$

Then starting with $\mathbf{s}_r = \mathbf{s} - \Theta$, we have

$$r(\mathbf{s} - \Theta) = r(\mathbf{s}) - \dot{r}(\mathbf{s})\Theta + \frac{1}{2}\ddot{r}(\mathbf{s})\Theta^2 - \frac{1}{6}\ddot{\ddot{r}}(\mathbf{s})\Theta^3 + \Theta^3\text{Ordo}(\Theta). \tag{13}$$

Subtracting both sides from the parameterization (63), we get (see (1))

$$\mathbf{R}_r = \rho\mathbf{n} + \dot{r}\Theta - \frac{1}{2}\ddot{r}\Theta^2 + \frac{1}{6}\ddot{\ddot{r}}\Theta^3 + \Theta^3\text{Ordo}(\Theta). \tag{14}$$

The left hand side is a lightlike vector or zero. The Lorentz square of both sides, with the equalities

$$\dot{r} \cdot \dot{r} = -1, \quad \mathbf{n} \cdot \mathbf{n} = 1, \quad \dot{r} \cdot \mathbf{n} = 0, \quad \dot{r} \cdot \ddot{r} = 0, \quad \dot{r} \cdot \ddot{\ddot{r}} = -\ddot{r} \cdot \ddot{r},$$

results in

$$0 = \rho^2 - (1 + \rho \mathbf{n} \cdot \ddot{r})\Theta^2 + \frac{1}{3}\rho \mathbf{n} \cdot \ddot{\ddot{r}}\Theta^3 - \frac{1}{12}\ddot{r} \cdot \ddot{r}\Theta^4 + (\rho\Theta^3 + \Theta^4)\text{Ordo}(\Theta).$$

In another form,

$$1 = \frac{\Theta^2}{\rho^2} \left(1 + \rho \mathbf{n} \cdot \ddot{r} - \frac{1}{3}\rho \mathbf{n} \cdot \ddot{\ddot{r}}\Theta + \frac{1}{12}|\ddot{r}|^2\Theta^2 - (\rho\Theta + \Theta^2)\text{Ordo}(\Theta) \right), \quad (15)$$

which says that $\lim_{\rho \rightarrow 0} \frac{\Theta^2}{\rho^2} = 1$. Since Θ is non-negative, $\lim_{\rho \rightarrow 0} \frac{\Theta}{\rho} = 1$ holds as well. Then

$$\Theta = \rho(1 + A) \quad \text{where } A = \text{Ordo}(\rho) \quad (16)$$

and (15) can be rewritten in the form

$$1 = \frac{\Theta^2}{\rho^2} + \frac{\Theta^2}{\rho^2}(\rho \mathbf{n} \cdot \ddot{r} + \rho \text{Ordo}(\rho)).$$

Putting (16) in the first term on the right hand side, we get

$$\frac{2A + A^2}{\rho} = -\frac{\Theta^2}{\rho^2}(\mathbf{n} \cdot \ddot{r} + \text{Ordo}(\rho)).$$

Let ρ tend to zero. Since $A = \text{Ordo}(\rho)$, we get

$$\lim_{\rho \rightarrow 0} \frac{2A}{\rho} = -\mathbf{n} \cdot \ddot{r},$$

in other words, $A = \rho(-\frac{1}{2}\mathbf{n} \cdot \ddot{r} + \text{Ordo}(\rho))$; thus,

$$\Theta = \rho - \frac{\rho^2}{2}(\mathbf{n} \cdot \ddot{r} + B) \quad \text{where } B = \text{Ordo}(\rho). \quad (17)$$

Then (15) can be written in the form:

$$1 = \frac{\Theta^2}{\rho^2} (1 + \rho \mathbf{n} \cdot \ddot{r}) - \frac{\Theta^2}{\rho^2} \left(\frac{1}{3}\rho \mathbf{n} \cdot \ddot{\ddot{r}}\Theta - \frac{1}{12}|\ddot{r}|^2\Theta^2 + \rho^2 \text{Ordo}(\rho) \right). \quad (18)$$

Putting (17) in the first term on the right hand side, we get

$$\begin{aligned} \left(1 - \rho \mathbf{n} \cdot \ddot{r} - \rho B + \frac{\rho^2}{4}((\mathbf{n} \cdot \ddot{r})^2 + 2\mathbf{n} \cdot \ddot{r}B + B^2) \right) (1 + \rho \mathbf{n} \cdot \ddot{r}) = \\ = 1 - \rho^2(\mathbf{n} \cdot \ddot{r})^2 - \rho B + \frac{\rho^2}{4}((\mathbf{n} \cdot \ddot{r})^2 + \text{Ordo}(\rho)). \end{aligned}$$

Then (18) divided by ρ^2 becomes

$$0 = -(\mathbf{n} \cdot \ddot{r})^2 - \frac{B}{\rho} + \frac{1}{4}((\mathbf{n} \cdot \ddot{r})^2 + \text{Ordo}(\rho)) - \frac{\Theta^2}{\rho^2} \left(\frac{1}{3}\mathbf{n} \cdot \ddot{\ddot{r}}\frac{\Theta}{\rho} - \frac{1}{12}|\ddot{r}|^2\frac{\Theta^2}{\rho^2} + \text{Ordo}(\rho) \right)$$

from which

$$\lim_{\rho \rightarrow 0} \frac{B}{\rho} = -\frac{3}{4}(\mathbf{n} \cdot \ddot{r})^2 - \frac{1}{3}\mathbf{n} \cdot \ddot{\ddot{r}} + \frac{1}{12}|\ddot{r}|^2.$$

Then

$$B = \rho \left(-\frac{3}{4}(\mathbf{n} \cdot \dot{r})^2 - \frac{1}{3}\mathbf{n} \cdot \ddot{r} + \frac{1}{12}|\ddot{r}|^2 + \text{Ordo}(\rho) \right).$$

Then we have the final result

$$\Theta = \rho \left(1 - \rho \frac{\mathbf{n} \cdot \ddot{r}}{2} + \frac{\rho^2}{2} \left(\frac{9(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2}{12} + \frac{\mathbf{n} \cdot \ddot{r} \ddot{r}}{3} \right) + \rho^2 \text{Ordo}(\rho) \right). \quad (19)$$

For the sake of simplicity, without the danger of confusion, we shall omit the terms $\text{Ordo}(\rho)$, writing approximative equalities:

$$\Theta \approx \rho \left(1 - \rho \frac{\mathbf{n} \cdot \ddot{r}}{2} + \frac{\rho^2}{2} \left(\frac{9(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2}{12} + \frac{\mathbf{n} \cdot \ddot{r} \ddot{r}}{3} \right) \right), \quad (20)$$

$$\Theta^2 \approx \rho^2(1 - \rho(\mathbf{n} \cdot \ddot{r})), \quad \Theta^3 \approx \rho^3. \quad (21)$$

It is emphasized again that these formulae do not come from a Taylor expansion because of the non differentiability of the retarded proper time on the world line.

5. Further radial expansions

Using the previous results, we give the radial expansion of the functions occurring in the formulae of the electromagnetic field (3). According to (20) and (21), the retarded velocity and acceleration introduced in (1) have radial expansion

$$\mathbf{u}_r \approx \dot{r} - \ddot{r}\Theta + \frac{1}{2}\ddot{r} \ddot{r} \Theta^2 \approx \dot{r} - \rho\ddot{r} + \frac{\rho^2}{2}((\mathbf{n} \cdot \ddot{r})\dot{r} + \ddot{r}), \quad (22)$$

$$\mathbf{a}_r \approx \ddot{r} - \ddot{r} \ddot{r} \Theta \approx \ddot{r} - \rho \ddot{r} \ddot{r}. \quad (23)$$

Further, (14) gives us

$$\begin{aligned} \mathbf{R}_r &\approx \rho\mathbf{n} + \dot{r}\Theta - \frac{1}{2}\ddot{r}\Theta^2 + \frac{1}{6}\ddot{r} \ddot{r} \Theta^3 \approx \\ &\approx \rho \left(\mathbf{n} + \dot{r} - \rho \frac{\ddot{r} + (\mathbf{n} \cdot \ddot{r})\dot{r}}{2} + \right. \\ &\quad \left. + \frac{\rho^2}{2} \left(\left(\frac{9(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2}{12} + \frac{\mathbf{n} \cdot \ddot{r} \ddot{r}}{3} \right) \dot{r} + (\mathbf{n} \cdot \dot{r})\ddot{r} + \frac{\ddot{r} \ddot{r}}{3} \right) \right), \\ (-\mathbf{u}_r \cdot \mathbf{R}_r) &\approx \rho \left(1 + \rho \frac{(\mathbf{n} \cdot \ddot{r})}{2} - \frac{\rho^2}{2} \left(\frac{(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2}{4} + \frac{2\mathbf{n} \cdot \ddot{r} \ddot{r}}{3} \right) \right), \\ \frac{1}{-\mathbf{u}_r \cdot \mathbf{R}_r} &\approx \frac{1}{\rho} \left(1 - \rho \frac{\mathbf{n} \cdot \ddot{r}}{2} + \frac{\rho^2}{2} \left(\frac{3(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2}{4} + \frac{2\mathbf{n} \cdot \ddot{r} \ddot{r}}{3} \right) \right), \\ \frac{1}{(\mathbf{u}_r \cdot \mathbf{R}_r)^2} &\approx \frac{1}{\rho^2} \left(1 - \rho\mathbf{n} \cdot \ddot{r} + \frac{\rho^2}{2} \left(\frac{4(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2}{2} + \frac{4\mathbf{n} \cdot \ddot{r} \ddot{r}}{3} \right) \right), \end{aligned}$$

$$\frac{1}{(-\mathbf{u}_r \cdot \mathbf{R}_r)^3} \approx \frac{1}{\rho^3} \left(1 - \rho \frac{3\mathbf{n} \cdot \ddot{r}}{2} + \frac{\rho^2}{2} \left(\frac{15(\mathbf{n} \cdot \ddot{r})^2 - 3|\ddot{r}|^2}{4} + 2\mathbf{n} \cdot \ddot{\ddot{r}} \right) \right),$$

$$\frac{1}{(\mathbf{u}_r \cdot \mathbf{R}_r)^4} \approx \frac{1}{\rho^4} \left(1 - \rho 2\mathbf{n} \cdot \ddot{r} + \frac{\rho^2}{2} \left(6(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2 + \frac{8}{3}\mathbf{n} \cdot \ddot{\ddot{r}} \right) \right).$$

Thus, for the quantities introduced in (2),

$$\begin{aligned} \mathbf{L}_r \approx & \mathbf{n} + \dot{r} - \rho \left(\frac{\ddot{r}}{2} + (\mathbf{n} \cdot \ddot{r}) \left(\dot{r} + \frac{\mathbf{n}}{2} \right) \right) + \\ & + \frac{\rho^2}{2} \left(\frac{6(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2}{3} + \mathbf{n} \cdot \ddot{\ddot{r}} \right) \dot{r} + \\ & + \frac{\rho^2}{2} \left(\frac{3(\mathbf{n} \cdot \ddot{r})^2 - |\ddot{r}|^2}{4} + \frac{2\mathbf{n} \cdot \ddot{\ddot{r}}}{3} \right) \mathbf{n} + \\ & + \frac{\rho^2}{2} \left(\frac{3(\mathbf{n} \cdot \ddot{r})\ddot{r}}{2} + \frac{\ddot{\ddot{r}}}{3} \right), \end{aligned} \quad (24)$$

$$\mathbf{d}_r \approx \ddot{r} + (\mathbf{n} \cdot \ddot{r})\dot{r} + \rho \left(\left(\frac{|\ddot{r}|^2 - (\mathbf{n} \cdot \ddot{r})^2}{2} - \mathbf{n} \cdot \ddot{\ddot{r}} \right) \dot{r} - (\mathbf{n} \cdot \ddot{r})\ddot{r} - \ddot{\ddot{r}} \right). \quad (25)$$

6. Radial expansion of the ‘energy-momentum tensor’

Let us take (6)-(11) and, for the sake of simplicity, let us rewrite (24) in the form

$$\mathbf{L}_r \approx \mathbf{n} + \dot{r} - \rho \mathbf{A} + \frac{\rho^2}{2} \mathbf{B}.$$

Then \vee denotes the symmetric tensor product -

$$\begin{aligned} (\mathbf{u}_r \vee \mathbf{L}_r) & \approx \mathbf{u} \vee (\mathbf{n} + \mathbf{u}) - \rho (\mathbf{u} \vee \mathbf{A} + \ddot{r} \vee (\mathbf{n} + \mathbf{u})) + \\ & + \frac{\rho^2}{2} \left(\dot{r} \vee \mathbf{B} + ((\mathbf{n} \cdot \ddot{r})\ddot{r} + \ddot{\ddot{r}}) \vee (\mathbf{n} + \dot{r}) + 2\ddot{r} \vee \mathbf{A} \right), \\ (\mathbf{L}_r \otimes \mathbf{L}_r) & \approx (\mathbf{n} + \dot{r}) \otimes (\mathbf{n} + \dot{r}) - \rho ((\mathbf{n} + \dot{r}) \vee \ddot{r}) + \\ & + \frac{\rho^2}{2} \left((\mathbf{n} + \dot{r}) \vee \mathbf{B} + \mathbf{A} \vee \mathbf{A} \right). \end{aligned}$$

Further, rewriting (25) in the form

$$\mathbf{d}_r \approx \ddot{r} + (\mathbf{n} \cdot \ddot{r})\dot{r} + \rho(\beta\dot{r} - (\mathbf{n} \cdot \ddot{r})\ddot{r} - \ddot{\ddot{r}})$$

we have

$$\begin{aligned} (\mathbf{d}_r \vee \mathbf{L}_r) & \approx (\ddot{r} + (\mathbf{n} \cdot \ddot{r})\dot{r}) \vee (\mathbf{n} + \dot{r}) + \\ & + \rho \left(-(\ddot{r} + (\mathbf{n} \cdot \ddot{r})\dot{r}) \vee \mathbf{A} + (\beta\dot{r} - (\mathbf{n} \cdot \ddot{r})\ddot{r} - \ddot{\ddot{r}}) \vee (\mathbf{n} + \dot{r}) \right), \\ (\mathbf{u}_r \cdot \mathbf{d}_r) & \approx -(\mathbf{n} \cdot \ddot{r} + \rho\beta), \end{aligned} \quad (26)$$

and

$$(\mathbf{u}_r \cdot \mathbf{d}_r)(\mathbf{L}_r \otimes \mathbf{L}_r) \approx -(\mathbf{n} \cdot \ddot{\mathbf{r}})((\mathbf{n} + \dot{\mathbf{r}}) \otimes (\mathbf{n} + \dot{\mathbf{r}})) + \rho((\mathbf{n} \cdot \ddot{\mathbf{r}})(\mathbf{n} + \dot{\mathbf{r}}) \vee \mathbf{A} - \beta(\mathbf{n} + \dot{\mathbf{r}}) \otimes (\mathbf{n} + \dot{\mathbf{r}}));$$

finally,

$$|\mathbf{d}_r|^2 \approx |\ddot{\mathbf{r}}|^2 - (\mathbf{n} \cdot \ddot{\mathbf{r}})^2.$$

All these together give that the ‘energy-momentum tensor’ (5) has the radial expansion

$$\begin{aligned} \frac{16\pi^2}{e^2} \mathbf{T}[r] &= \frac{\dot{\mathbf{r}} \otimes \dot{\mathbf{r}} - \mathbf{n} \otimes \mathbf{n} + \frac{1}{2}}{\rho^4} - \\ &+ \frac{\frac{\mathbf{n} \otimes \ddot{\mathbf{r}} + \ddot{\mathbf{r}} \otimes \mathbf{n}}{2} - (\mathbf{n} \cdot \ddot{\mathbf{r}})(2\dot{\mathbf{r}} \otimes \dot{\mathbf{r}} - \mathbf{n} \otimes \mathbf{n} + \mathbf{1})}{\rho^3} - \\ &- \frac{1}{\rho^2} \left(\frac{\ddot{\mathbf{r}} \otimes \ddot{\mathbf{r}}}{4} + \frac{\dot{\mathbf{r}} \otimes \ddot{\mathbf{r}}' + \ddot{\mathbf{r}}' \otimes \dot{\mathbf{r}}}{2} \right) + \\ &- \frac{1}{\rho^2} \left(\frac{2(\mathbf{n} \otimes \ddot{\mathbf{r}}' + \ddot{\mathbf{r}}' \otimes \mathbf{n})}{3} + (\mathbf{n} \cdot \ddot{\mathbf{r}})(\mathbf{n} \otimes \ddot{\mathbf{r}} + \ddot{\mathbf{r}} \otimes \mathbf{n}) \right) + \\ &+ \frac{1}{\rho^2} \left(|\ddot{\mathbf{r}}|^2 + 3(\mathbf{n} \cdot \ddot{\mathbf{r}})^2 + \frac{4\mathbf{n} \cdot \ddot{\mathbf{r}}'}{3} \right) \dot{\mathbf{r}} \otimes \dot{\mathbf{r}} + \\ &+ \frac{1}{\rho^2} \left(\frac{3|\ddot{\mathbf{r}}|^2}{4} - (\mathbf{n} \cdot \ddot{\mathbf{r}})^2 \right) \mathbf{n} \otimes \mathbf{n} + \\ &+ \frac{1}{\rho^2} \left(\frac{2|\ddot{\mathbf{r}}|^2}{3} + \frac{\mathbf{n} \cdot \ddot{\mathbf{r}}'}{2} \right) (\dot{\mathbf{r}} \otimes \mathbf{n} + \mathbf{n} \otimes \dot{\mathbf{r}}) + \\ &+ \frac{1}{\rho^2} \left(\frac{|\ddot{\mathbf{r}}|^2}{4} + \frac{3(\mathbf{n} \cdot \ddot{\mathbf{r}})^2}{2} + \frac{2\mathbf{n} \cdot \ddot{\mathbf{r}}'}{3} \right) \mathbf{1} + \frac{\text{Ordo}(\rho)}{\rho^2}. \end{aligned} \tag{27}$$

7. Distribution of energy-momentum

The radial expansion (27) shows that the ‘energy-momentum tensor’ $\mathbf{T}[r]$, for every proper time value \mathbf{s} of the given world line function r , has a pole of fourth order and a pole of third order in the three dimensional Euclidean space Lorentz orthogonal to $\dot{\mathbf{r}}(\mathbf{s})$. We can tame its poles, obtaining $\text{tm}\mathbf{T}[r]$ (see Section C) as follows. For the sake of avoiding misunderstandings, we shall write simply \mathbf{T} instead of $\mathbf{T}[r]$ when integrating functions. For an x in the neighbourhood of the world line in question let $\hat{\mathbf{s}}(x)$ denote the proper time value for which

$$\dot{\mathbf{r}}(\hat{\mathbf{s}}(x)) \cdot (x - r(\hat{\mathbf{s}}(x))) = 0, \tag{28}$$

holds, *i.e.*, $\hat{\mathbf{s}}$ is the first component of the inverse of the parameterization (60).

Proposition 7.1. *If the support of the test function ϕ is disjoint from the range of r then*

$$(\text{tm}\mathbf{T}[r] | \phi) = \int_{\mathbf{M}} \mathbf{T}(x)\phi(x) dx;$$

if the support of the test function ϕ is in a world roll H_{I,R_I} then

$$\begin{aligned}
 (\text{tm}\mathbf{T}[r] | \phi) &= \\
 &= \int_{\mathbf{M}} \left(\mathbf{T}(x)\phi(x) - \frac{e^2}{16\pi^2} \frac{4\dot{r}(\hat{\mathbf{s}}(x)) \otimes \dot{r}(\hat{\mathbf{s}}(x)) + \mathbf{1}}{6|x - r(\hat{\mathbf{s}}(x))|^4} \phi(r(\hat{\mathbf{s}}(x))) \right) dx !! \quad (29)
 \end{aligned}$$

where the double exclamation mark says the integral must be taken in the radial parameterization of the world roll H_{I,R_I} and in a given order.

Proof The integral (29) in the radial parameterization reads

$$\begin{aligned}
 \int_{\mathbf{T}} \left(\int_0^\infty \left(\int_{S_1(0)} \left(\mathbf{T}(r(\mathbf{s}) + \rho\mathbf{n})\phi(r(\mathbf{s}) + \rho\mathbf{n}) - \right. \right. \right. \\
 \left. \left. \left. - \frac{e^2}{16\pi^2} \frac{4\dot{r}(\mathbf{s}) \otimes \dot{r}(\mathbf{s}) + \mathbf{1}}{6\rho^4} \phi(r(\mathbf{s})) \right) (1 + \rho(\mathbf{n} \cdot \ddot{r}(\mathbf{s}))) d\mathbf{n}_0 \right) \rho^2 d\rho \right) ds
 \end{aligned}$$

with $\mathbf{n} = \mathbf{L}(\mathbf{s})\mathbf{n}_0$.

Let us write the radial expansion of the ‘energy-momentum tensor’ in the form

$$\frac{A(\mathbf{n})}{\rho^4} + \frac{B(\mathbf{n})}{\rho^3} + \frac{C(\mathbf{n}) + \text{Ordo}(\rho)}{\rho^2}. \quad (30)$$

Then the function

$$\left(\frac{A(\mathbf{n})}{\rho^4} + \frac{B(\mathbf{n})}{\rho^3} + \frac{C(\mathbf{n}) + \text{Ordo}(\rho)}{\rho^2} \right) (1 + \rho\mathbf{n} \cdot \ddot{r}) = \quad (31)$$

$$= \frac{A(\mathbf{n})}{\rho^4} + \frac{A(\mathbf{n})(\mathbf{n} \cdot \ddot{r}) + B(\mathbf{n})}{\rho^3} + \frac{B(\mathbf{n})(\mathbf{n} \cdot \ddot{r}) + C(\mathbf{n}) + \text{Ordo}(\rho)}{\rho^2} \quad (32)$$

is to be pole tamed for each fixed \mathbf{s} . $A(\mathbf{n})$ is an even tensor product of \mathbf{n} (linear function of an even tensor multiple of \mathbf{n}_0), so the first term of (32) can be tamed: the function

$$\frac{A(\mathbf{n})(\phi(r(\mathbf{s}) + \rho\mathbf{n}) - \phi(r(\mathbf{s})))}{\rho^4}$$

is to be integrated in the radial variables (because the derivative of ϕ drops out from the Taylor polynomial). The part containing $\phi(r(\mathbf{s}) + \rho\mathbf{n})$ remains as it is,

$$\frac{A(\mathbf{n})}{\rho^4} \phi(r(\mathbf{s}) + \rho\mathbf{n}); \quad (33)$$

the part containing $\phi(r(\mathbf{s}))$ becomes

$$\frac{e^2}{16\pi^2} \left(\frac{4\dot{r}(\mathbf{s}) \otimes \dot{r}(\mathbf{s}) + \mathbf{1}}{6\rho^4} \right) \phi(r(\mathbf{s})) \quad (34)$$

which is obtained from the integral

$$\frac{1}{4\pi} \int_{S_1(0)} \left(\dot{r}(\mathbf{s}) \otimes \dot{r}(\mathbf{s}) - \mathbf{L}(\mathbf{s})\mathbf{n}_0 \otimes \mathbf{L}(\mathbf{s})\mathbf{n}_0 + \frac{\mathbf{1}}{2} \right) d\mathbf{n}_0$$

by the use of

$$\int_{S_1(0)} \mathbf{L}(\mathbf{s})\mathbf{n}_0 \otimes \mathbf{L}(\mathbf{s})\mathbf{n}_0 d\mathbf{n}_0 = \frac{4\pi}{3} (\mathbf{1} + \mathbf{L}(\mathbf{s})\dot{r}(0) \otimes \mathbf{L}(\mathbf{s})\dot{r}(0)) = \frac{4\pi}{3} (\mathbf{1} + \dot{r}(\mathbf{s}) \otimes \dot{r}(\mathbf{s})).$$

$A(\mathbf{n})\mathbf{n} \cdot \ddot{r} + B(\mathbf{n})$ is a linear function of an odd tensor product of \mathbf{n} , so the second term in (32) can be tamed: the function

$$\frac{A(\mathbf{n})\mathbf{n} \cdot \ddot{r} + B(\mathbf{n})}{\rho^3} \phi(r(\mathbf{s}) + \rho\mathbf{n}) \tag{35}$$

is to be integrated in the radial variables (because $\phi(r(\mathbf{s}))$ drops out from the Taylor polynomial). The third term in (32) defines a regular Distribution, so the function

$$\frac{B(\mathbf{n})\mathbf{n} \cdot \ddot{r} + C(\mathbf{n}) + \text{Ordo}(\rho)}{\rho^2} \phi(r(\mathbf{s}) + \rho\mathbf{n}) \tag{36}$$

is to be integrated.

Summarizing: the sum of (33), (35) and (36) gives the first term in (29), whereas (34) gives the second term because in the radial parameterization $\hat{\mathbf{s}}(x) = \mathbf{s}$ and $|x - r(\hat{\mathbf{s}}(x))| = \rho$. ■

$\text{tm}\mathbf{T}[r]$ will be called the *Distribution of energy-momentum* (which is not an energy-momentum distribution!) of the point charge with a given world line function r .

8. The radiation reaction force

The Distribution of energy-momentum of a point charge is a mathematical object. Has it some physical meaning? The negative spacetime divergence of the energy-momentum tensor of a continuous charge-current density is the force density; that is why we can hope here a similar physical meaning. Recall that $\lambda_{\text{Ran}(r)}$ is the Lebesgue measure of the world line $\text{Ran}(r)$ (see equality (53)).

Proposition 8.1.

$$-\text{D} \cdot \text{tm}\mathbf{T}[r] = \frac{1}{4\pi} \frac{2e^2}{3} (\ddot{r} + (\dot{r} \cdot \ddot{r})\dot{r}) \lambda_{\text{Ran}(r)} = \tag{37}$$

$$= \frac{1}{4\pi} \frac{2e^2}{3} (\dot{r} \wedge \ddot{r}) \cdot \dot{r} \lambda_{\text{Ran}(r)}. \tag{38}$$

Proof The Distribution $\text{tm}\mathbf{T}[r]$ restricted to an open subset disjoint from the world line equals the regular Distribution corresponding to the restriction of $\mathbf{T}[r]$. Since $\text{D} \cdot \mathbf{T}[r] = 0$ except the world line, we have to take test functions ϕ whose

support is in a world roll H_{I,R_I} around the world line. For them

$$\begin{aligned} (-D \cdot \text{tm} \mathbf{T}[r] \mid \phi) &= (\text{tm} \mathbf{T}[r] \mid D\phi) = \\ &= \int_M \left(\mathbf{T}(x) \cdot D\phi[x] - \frac{e^2}{16\pi^2} \frac{4\dot{r}(\hat{\mathbf{s}}(x)) \otimes \dot{r}(\hat{\mathbf{s}}(x)) + \mathbf{1}}{6|x - r(\hat{\mathbf{s}}(x))|^4} \cdot D\phi[r(\hat{\mathbf{s}}(x))] \right) dx !! = \\ &= \lim_{R \rightarrow 0} \int_{M \setminus H_{I,R}} \dots !! \end{aligned}$$

Because of $D \cdot \mathbf{T} = 0$ in $M \setminus H_{I,R}$, the first term in the integrand equals $D \cdot (\mathbf{T}\phi)$. For examining the second term let us consider ρ and $\hat{\mathbf{n}}$ as functions of the spacetime points x instead of the radial parameters (\mathbf{s}, \mathbf{q}) , *i.e.*, let us introduce

$$\hat{\rho}(x) := |x - r(\hat{\mathbf{s}}(x))|, \quad \hat{\mathbf{n}}(x) := \frac{x - r(\hat{\mathbf{s}}(x))}{|x - r(\hat{\mathbf{s}}(x))|}; \tag{39}$$

moreover, let

$$\hat{\mathbf{u}}(x) := \dot{r}(\hat{\mathbf{s}}(x)), \quad \hat{\mathbf{a}}(x) := \ddot{r}(\hat{\mathbf{s}}(x)).$$

Then

$$(\hat{\rho}\hat{\mathbf{n}})(x) = x - r(\hat{\mathbf{s}}(x))$$

and

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1, \quad \hat{\mathbf{u}} \cdot \hat{\mathbf{n}} = 0. \tag{40}$$

By differentiation we get

$$\mathbf{1} - \hat{\mathbf{u}} \otimes D\hat{\mathbf{s}} = D(\hat{\rho}\hat{\mathbf{n}}) = \hat{\mathbf{n}} \otimes D\hat{\rho} + \hat{\rho}D\hat{\mathbf{n}}, \tag{41}$$

from which, by (40) we deduce

$$D\hat{\rho} = \hat{\mathbf{n}}. \tag{42}$$

Further,

$$D\hat{\mathbf{u}} = \hat{\mathbf{a}} \otimes D\hat{\mathbf{s}},$$

and

$$0 = D(\hat{\mathbf{u}} \cdot \hat{\rho}\hat{\mathbf{n}}) = \hat{\rho}\hat{\mathbf{n}} \cdot (\hat{\mathbf{a}} \otimes D\hat{\mathbf{s}}) + \hat{\mathbf{u}} \cdot (\mathbf{1} - \hat{\mathbf{u}} \otimes D\hat{\mathbf{s}}),$$

therefore

$$D\hat{\mathbf{s}} = -\frac{\hat{\mathbf{u}}}{1 + \hat{\rho}\hat{\mathbf{n}} \cdot \hat{\mathbf{a}}} \tag{43}$$

and as a consequence,

$$D \cdot (\hat{\rho}\hat{\mathbf{n}}) = 4 - \frac{1}{1 + \hat{\rho}\hat{\mathbf{n}} \cdot \hat{\mathbf{a}}}, \tag{44}$$

Further, let

$$\mathbf{Z}(\mathbf{s}) := \frac{(4\dot{r}(\mathbf{s}) \otimes \dot{r}(\mathbf{s}) + \mathbf{1}) \cdot D\phi[r(\mathbf{s})]}{6}, \quad \hat{\mathbf{Z}}(x) := \mathbf{Z}(\hat{\mathbf{s}}(x)).$$

Then we find that $D\hat{\mathbf{Z}} = \hat{\mathbf{Z}}(\hat{\mathbf{s}}) \otimes D\hat{\mathbf{s}}$ from which by (43) $(D\hat{\mathbf{Z}}) \cdot \hat{\mathbf{n}} = 0$ follows. Then, taking into account (44) and (42),

$$\begin{aligned} D \cdot \frac{\hat{\mathbf{Z}} \otimes \hat{\rho} \hat{\mathbf{n}}}{\hat{\rho}^4} &= \hat{\mathbf{Z}} \left(\left(4 - \frac{1}{1 + \hat{\rho} \hat{\mathbf{n}} \cdot \hat{\mathbf{a}}} \right) \frac{1}{\hat{\rho}^4} - \frac{\hat{\rho} \hat{\mathbf{n}} \cdot 4 \hat{\mathbf{n}}}{\hat{\rho}^5} \right) = \\ &= - \frac{\hat{\mathbf{Z}}}{(1 + \hat{\rho} \hat{\mathbf{n}} \cdot \hat{\mathbf{a}}) \hat{\rho}^4} = - \frac{\hat{\mathbf{Z}}}{\hat{\rho}^4} + \frac{\hat{\mathbf{Z}}(\hat{\mathbf{n}} \cdot \hat{\mathbf{a}})}{(1 + \hat{\rho} \hat{\mathbf{n}} \cdot \hat{\mathbf{a}}) \hat{\rho}^3}, \end{aligned}$$

in other words,

$$- \frac{\hat{\mathbf{Z}}}{\hat{\rho}^4} = D \cdot \frac{\hat{\mathbf{Z}} \otimes \hat{\mathbf{n}}}{\hat{\rho}^3} - \frac{\hat{\mathbf{Z}}(\hat{\mathbf{n}} \cdot \hat{\mathbf{a}})}{(1 + \hat{\rho} \hat{\mathbf{n}} \cdot \hat{\mathbf{a}}) \hat{\rho}^3}.$$

Thus, we arrive at

$$\begin{aligned} (-D \cdot \text{tm} \mathbf{T}[r] | \phi) &= \\ &= \lim_{R \rightarrow 0} \int_{M \setminus H_{I,R}} \left(D \cdot \left(\mathbf{T}\phi + \frac{e^2}{16\pi^2} \frac{\hat{\mathbf{Z}} \otimes \hat{\mathbf{n}}}{\hat{\rho}^3} \right) - \frac{e^2}{16\pi^2} \frac{\hat{\mathbf{Z}}(\hat{\mathbf{n}} \cdot \hat{\mathbf{a}})}{(1 + \hat{\rho} \hat{\mathbf{n}} \cdot \hat{\mathbf{a}}) \hat{\rho}^3} \right) !!. \end{aligned}$$

The integral of the second term in the parameterization around the world line becomes a multiple of

$$\int_I \int_R^{R_I} \rho^2 \int_{S_1(\mathbf{0})} \frac{\mathbf{Z}(\mathbf{s})(\mathbf{L}(\mathbf{s}) \cdot \mathbf{n}_0) \cdot \ddot{r}(\mathbf{s})}{\rho^3} d\mathbf{n}_0 d\rho ds$$

which is zero because of equality (54). By Gauss' theorem, the integral of the first term can be transformed to an integral on the corresponding world tube in such a way that the tube is directed 'inwards' *i.e.*, the normal vector is $-\hat{\mathbf{n}}$:

$$\begin{aligned} (-D \cdot \text{tm} \mathbf{T}[r] | \phi) &= \lim_{R \rightarrow 0} \int_{M \setminus H_{I,R}} D \cdot \left(\mathbf{T}\phi + \frac{e^2}{16\pi^2} \frac{\hat{\mathbf{Z}} \otimes \hat{\mathbf{n}}}{\hat{\rho}^3} \right) !! = \\ &= - \lim_{R \rightarrow 0} \int_{P_{I,R}} \left(\mathbf{T}\phi + \frac{e^2}{16\pi^2} \frac{\hat{\mathbf{Z}} \otimes \hat{\mathbf{n}}}{\hat{\rho}^3} \right) \cdot \hat{\mathbf{n}} d\lambda_{P_{I,R}} !! = \quad (45) \\ &= - \lim_{R \rightarrow 0} \int_I R^2 \int_{S_1(0)} \dots(\mathbf{n}_0, \mathbf{s}) d\mathbf{n}_0 ds ! \end{aligned}$$

where the integrand is

$$\left((\mathbf{T}\phi)(r(\mathbf{s}) + R\mathbf{n}(\mathbf{s})) + \frac{e^2}{16\pi^2} \frac{\mathbf{Z}(\mathbf{s}) \otimes \mathbf{n}(\mathbf{s})}{R^3} \right) \cdot \mathbf{n}(\mathbf{s})(1 + R\mathbf{n}(\mathbf{s}) \cdot \ddot{r}(\mathbf{s})) \quad (46)$$

with $\mathbf{n}(\mathbf{s}) := \mathbf{L}(\mathbf{s})\mathbf{n}_0$. Then

$$\frac{\mathbf{Z} \otimes \mathbf{n}}{R^3} \cdot \mathbf{n}(1 + R\mathbf{n} \cdot \ddot{r}) = \frac{(4\dot{r} \otimes \dot{r} + \mathbf{1}) \cdot D\phi[r]}{6} \left(\frac{1}{R^3} + \frac{\mathbf{n} \cdot \ddot{r}}{R^2} \right),$$

and on the base of the radial expansion (27)

$$\begin{aligned}
 (\mathbf{T}(r + R\mathbf{n}) \cdot \mathbf{n}(1 + R\mathbf{n} \cdot \dot{r})) &= \\
 &= \frac{e^2}{16\pi^2} \left(-\frac{\mathbf{n}}{2R^4} + \frac{\dot{r}}{2R^3} + \right. \\
 &+ \frac{|\dot{r}|^2 \mathbf{n} + \frac{2}{3} |\dot{r}|^2 \dot{r} - \frac{3}{4} (\mathbf{n} \cdot \dot{r}) \dot{r} - \frac{2}{3} \ddot{r}}{R^2} + \\
 &\left. + \frac{\text{Ordo}(R)}{R^2} \right).
 \end{aligned}$$

This, together with the expansion

$$\phi(r + R\mathbf{n}) = \phi(r) + R\mathbf{n} \cdot D\phi[r] + R^2 \text{Ordo}(R),$$

give that the integral (45) of the Ordo terms in (46), because of the three times differentiability of r , is less than a multiple of R , so their limit is zero when R tends to zero; moreover, the integrals of the terms linear and trilinear in \mathbf{n} in (46) are zero. The integral of the terms independent of \mathbf{n} results in a multiplication by 4π , the integral of the terms bilinear in \mathbf{n} results in a multiplication by $\frac{4\pi}{3}(\mathbf{1} + \dot{r} \otimes \dot{r})$. Thus, $D\phi[r]$ will be multiplied by

$$-\frac{4\pi}{3} \frac{\mathbf{1} + \dot{r} \otimes \dot{r}}{2R} + 4\pi \frac{4\dot{r} \otimes \dot{r} + \mathbf{1}}{6R} = \frac{4\pi}{2R} \dot{r} \otimes \dot{r}$$

and $\phi(r)$ will be multiplied by $\frac{4\pi}{2R} \dot{r} + \frac{2}{3} (|\dot{r}|^2 \dot{r} - \ddot{r})$. Since

$$\frac{4\pi}{2R} \left(\dot{r}(\mathbf{s})(\dot{r}(\mathbf{s}) \cdot D\phi[r(\mathbf{s})]) + \frac{4\pi}{2R} \ddot{r}(\mathbf{s}) \right) = \frac{4\pi}{2R} \frac{d}{ds} (\dot{r}(\mathbf{s})\phi(r(\mathbf{s}))) \tag{47}$$

and its integral is zero because the support of ϕ is in H_{I,R_I} , it remains only

$$-\frac{e^2}{16\pi^2} \int_I 4\pi \frac{2}{3} (|\ddot{r}(\mathbf{s})|^2 \dot{r}(\mathbf{s}) - \ddot{r}(\mathbf{s})) \phi(r(\mathbf{s})) ds$$

which gives the desired result by $|\ddot{r}|^2 = -\dot{r} \cdot \ddot{r}$. ■

9. Discussion

9.1. On the self-force. The usual formula (38) of the radiation reaction force is obtained in a mathematically exact way, without unjustified application of Gauss–Stokes theorem, of Taylor expansion and without a doubtful limit to zero. It is an important fact that the self-force is obtained for a **given** world line function r . Consequently, its physical meaning is the following:

If the world line function is r then the self-force is $\frac{1}{4\pi} \frac{2e^2}{3} (\dot{r} \wedge \ddot{r}) \cdot \dot{r}$.

Accordingly, if the force \mathbf{f} is necessary to get a given r without radiation *i.e.*, $m\ddot{r} = \mathbf{f}(r, \dot{r})$ would valid without radiation, then besides \mathbf{f} the force opposite to the self-force must be applied for getting the desired world line function r .

An actual example is when an elementary particle is revolved in a cyclotron by a homogeneous static magnetic field. In spacetime formulation: there is given a constant electromagnetic field \mathbf{F} and the world line without radiation would satisfy $m\ddot{r} = e\mathbf{F}\dot{r}$. Then

$$\ddot{r} = \frac{e^2}{m^2}\mathbf{F}\cdot\mathbf{F}\cdot\dot{r}, \quad \dot{r}\cdot\ddot{r} = \frac{e^2}{m^2}(\dot{r}\cdot\mathbf{F})\cdot(\mathbf{F}\cdot\dot{r}) = -\frac{e^2}{m^2}(\mathbf{F}\cdot\dot{r})\cdot(\mathbf{F}\cdot\dot{r}),$$

thus the self-force is

$$\frac{1}{4\pi}\frac{2e^4}{3m^2}(\mathbf{F}\cdot\mathbf{F}\cdot\dot{r} - |\mathbf{F}\cdot\dot{r}|^2\dot{r}); \quad (48)$$

to keep the particle on a prescribed orbit, the opposite of the above force must be applied besides the magnetic field. For better seeing, let us take the standard inertial frame \mathbf{u} in which the cyclotron is at rest. Then the \mathbf{u} -spacelike component of \mathbf{F} , for which $\mathbf{F}\cdot\mathbf{u} = 0$, is the magnetic field $\underline{\mathbf{B}}$, a three dimensional antisymmetric tensor. Using the relative velocity $\mathbf{v} := \frac{\dot{r}}{-\mathbf{u}\cdot\dot{r}} - \mathbf{u}$ with respect to the inertial frame, we have

$$\mathbf{F}\cdot\dot{r} = \mathbf{F}\cdot\left(\frac{\dot{r}}{-\mathbf{u}\cdot\dot{r}} - \mathbf{u}\right)(-\mathbf{u}\cdot\dot{r}) = \frac{\underline{\mathbf{B}}\cdot\mathbf{v}}{\sqrt{1-|\mathbf{v}|^2}}$$

and the self-force relative to the inertial frame is

$$\frac{1}{4\pi}\frac{2e^4}{3m^2}\frac{(\mathbf{1} + \mathbf{u}\otimes\mathbf{u})}{-\mathbf{u}\cdot\dot{r}}(\mathbf{F}\cdot\mathbf{F}\cdot\dot{r} - |\mathbf{F}\cdot\dot{r}|^2\dot{r}) = \frac{1}{4\pi}\frac{2e^4}{3m^2}\left(\underline{\mathbf{B}}\cdot\underline{\mathbf{B}}\cdot\mathbf{v} - \frac{|\underline{\mathbf{B}}\cdot\mathbf{v}|^2}{1-|\mathbf{v}|^2}\mathbf{v}\right).$$

Introducing the customary axial vector \mathbf{B} corresponding to $\underline{\mathbf{B}}$ we have $|\mathbf{B}| = |\underline{\mathbf{B}}|$, $\underline{\mathbf{B}}\cdot\mathbf{v} = \mathbf{v}\times\mathbf{B}$, $\underline{\mathbf{B}}\cdot\underline{\mathbf{B}}\cdot\mathbf{v} = (\mathbf{v}\times\mathbf{B})\times\mathbf{B}$. If the relative velocity is orthogonal to \mathbf{B} then $\underline{\mathbf{B}}\cdot\underline{\mathbf{B}}\cdot\mathbf{v} = -|\mathbf{B}|^2\mathbf{v}$, $|\underline{\mathbf{B}}\cdot\mathbf{v}|^2 = |\mathbf{B}|^2|\mathbf{v}|^2$ and the self-force relative to the inertial frame

$$-\frac{1}{4\pi}\frac{2e^4}{3m^2}\frac{|\mathbf{B}|^2\mathbf{v}}{1-|\mathbf{v}|^2} \quad (49)$$

brakes the motion; the same opposite force must be applied additionally to avoid braking and to ensure that the particle moves on the prescribed orbit.

9.2. On the LAD equation. Besides the unjustified application of Gauss–Stokes theorem, of Taylor expansion and a doubtful limit to zero there is another, a fundamental error in usual derivations of the LAD equation which is clearly seen now.

*The self-force $\frac{1}{4\pi}\frac{2e^2}{3}(\dot{r}\wedge\ddot{r})\cdot\dot{r}$ is valid for a **given** world line function r ; therefore it is unjustified to put it in a Newtonian-like equation to obtain the LAD equation which would serve to determine r . Thus, the LAD equation is a **misconception** and its pathological properties are not surprising.*

The well known fundamental problem (Bergmann 1962) is that both electrodynamics and mechanics in their known forms are **theories of action**:

- the Maxwell equations define the electromagnetic field \mathcal{F} produced by a **given** world line function r of a particle,

- the Newtonian equation defines the world line function r of a particle in a **given** force (*e.g.*, the Lorentz force in an extraneous electromagnetic field \mathbf{F}),

and at present we have not a well working **theory of interaction** which **would define** both the electromagnetic field \mathcal{F} and the world line function r together. To see the problem more closely, let us consider a point charge under the action of a given force \mathbf{f} . Then according to the general conception (see, *e.g.*, Gralla *et al.* 2009) - formulated in our language - the Newtonian equation is replaced with the balance equation

$$-D \cdot (\mathcal{T}_m[r] + \mathcal{T}_e[\mathcal{F}]) + \mathbf{f}(r, \dot{r})\lambda_{\text{Ran}(r)} = 0 \quad (50)$$

where $\mathcal{T}_m[r]$ is the tensor Distribution of mechanical energy-momentum constructed somehow of r and $\mathcal{T}_e[\mathcal{F}]$ is the tensor Distribution of electromagnetic energy-momentum constructed somehow of \mathcal{F} . It seems evident that $\mathcal{T}_m[r] = m\dot{r} \otimes \dot{r}\lambda_{\text{Ran}(r)}$ and then the balance equation

$$m\ddot{r}\lambda_{\text{Ran}r} = \mathbf{f}(r, \dot{r})\lambda_{\text{Ran}(r)} - D \cdot \mathcal{T}_e[\mathcal{F}] \quad (51)$$

together with the Maxwell equations

$$D \cdot \mathcal{F} = \dot{r}\lambda_{\text{Ran}(r)}, \quad D \wedge \mathcal{F} = 0 \quad (52)$$

would describe that r and \mathcal{F} determine each other mutually. It is not sure that there is a conveniently defined $\mathcal{T}_e[\mathcal{F}]$ with which (51) and (52) form a consistent system of equations. It is sure, however, that $\mathcal{T}_e[\mathcal{F}]$ cannot be replaced with $\text{tm}\mathbf{T}[r]$ which is valid for a **given** r ; the LAD ‘equation’ is obtained by committing this replacement.

9.3. Equation, equality. At last, let the LAD ‘equation’ be looked at from another point of view. Namely, we have to distinguish clearly between an equation and an equality, both denoted by the same symbol $=$. An equation is a **definition**: it defines a set (the set of its solutions). An equality is a **statement**: it states that two sets are the same.

Let $S(e, m, \mathbf{f})$ be the set of processes of a point charge e with mass m under the action of a force \mathbf{f} . Because of the non-physical run-away solutions, the LAD ‘equation’ does not define $S(e, m, \mathbf{f})$, thus it is not an equation from a physical point of view. The various attempts to find satisfactory conditions to exclude non-physical solutions suggest in this context that, in fact, the LAD ‘equation’ would be an equality, stating that $S(e, m, \mathbf{f})$ is a subset of all the solutions. Unfortunately, this is not true, either, as it is shown in the excellent paper by Eliezer (1943): \mathbf{f} being the Coulomb force of a point charge, no physical motion in radial direction is a solution of the LAD ‘equation’.

9.4. An illustrative example. The following example can illustrate the problem of describing the interaction. Let two particles with the same charge and mass rest in the space of a standard inertial frame. They repulse each other by the Coulomb forces which are compensated by some constraint to attain the rest. If the constraint is left off at an instant then the particles start to move. Figure 1 shows what

happens (time passes from the left to the right). While resting, the particles send

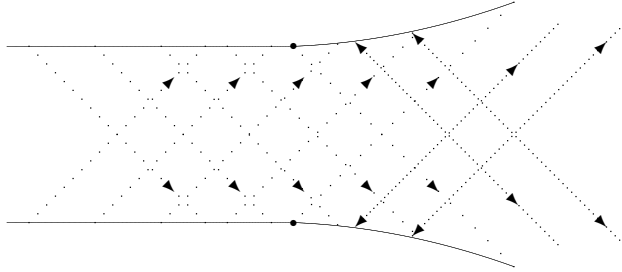


FIGURE 1. Interaction of two point charges

Coulomb-actions to each other in the positive light cone, represented by sparsely dotted lines. At the instant when the constraint is left off, the acceleration of each particles is determined by the Coulomb force of the other. Then for some time the acceleration of a particle is determined by the Coulomb force of the other particle and the radiation self-force, and then it sends some action, different from the Coulomb force to the other particle; this is represented by the densely dotted lines. We have no formula for this acceleration and action. Later the situation becomes more complicated: it is not known which action and self-force give rise to an acceleration.

9.5. Final remarks. It is emphasized that the retarded field is meaningful only for a **given** world line, therefore it cannot be used for **determining** the world line. The various attempts to obtain a physically acceptable equation with continuum charge distributions instead of a point charge suffer from the same fault of using the retarded field (and sometimes the advanced one, too).

Appendix A. Measures

Elementary notions of (vector) measures and integration by them can be found in the book by Dinculeanu (1967).

The Lebesgue measure of an affine space is the measure given by the usual integration (customary for \mathbb{R}^n). The Lebesgue measure λ_H of a submanifold H in spacetime is given by the formula

$$\int_H f d\lambda_H := \int_{\text{Dom}(p)} f(p(\xi)) \sqrt{|\det(\mathcal{D}p[\xi]^* \mathcal{D}p[\xi])|} d\xi$$

where p is an arbitrary parameterization of the submanifold, *i.e.*, a continuously differentiable injective map defined in an affine space, having H as its range and $\mathcal{D}p[\xi]$, the derivative of p at ξ is injective for all ξ . In particular, for a world line function r , the Lebesgue measure of the corresponding world line is $\lambda_{\text{Ran}(r)}$ which gives the integration along the world line. In particular, it is considered a Distribution in spacetime,

$$(\lambda_{\text{Ran}(r)} | \phi) = \int_{\mathbb{T}} \phi(r(\mathbf{s})) \, ds \quad (53)$$

for arbitrary test functions ϕ .

Appendix B. Three dimensional integration

Let \mathbf{u}_0 be an absolute velocity in spacetime and \mathbf{S}_0 the three dimensional Euclidean vector space Lorentz orthogonal to \mathbf{u}_0 ; $S_1(\mathbf{0})$ is the unit sphere around zero, its elements are denoted by \mathbf{n}_0 . In the usual parameterization given by the angles ϑ, φ relative to an orthonormal basis, the Lebesgue measure of $S_1(\mathbf{0})$ is obtained by the symbolic formula $d\mathbf{n}_0 \sim \sin \vartheta \, d\vartheta \, d\varphi$ with which we easily find the equalities

$$\int_{S_1(\mathbf{0})} \mathbf{n}_0 \, d\mathbf{n}_0 = 0, \quad \int_{S_1(\mathbf{0})} \mathbf{n}_0 \otimes \mathbf{n}_0 \otimes \mathbf{n}_0 \, d\mathbf{n}_0 = 0, \quad (54)$$

$$\int_{S_1(\mathbf{0})} \mathbf{n}_0 \otimes \mathbf{n}_0 \, d\mathbf{n}_0 = \frac{4\pi}{3} (\mathbf{1} + \mathbf{u}_0 \otimes \mathbf{u}_0) \quad (55)$$

where $\mathbf{1}$ is the identity map of spacetime vectors; and so on, the integral of an even tensor products is a non-zero tensor, the integral of an odd tensor product is zero. The radial coordinates of $\mathbf{q} \in \mathbf{S}_0$ are

$$\rho(\mathbf{q}) := |\mathbf{q}| \geq 0, \quad \mathbf{n}_0(\mathbf{q}) := \frac{\mathbf{q}}{|\mathbf{q}|} \in S_1(\mathbf{0}).$$

Then the radial parameterization of \mathbf{S}_0 is

$$(\rho, \mathbf{n}_0) \mapsto \rho \mathbf{n}_0.$$

We applied the usual ambiguity that ρ and \mathbf{n}_0 denote functions in some respect and variables in an other respect. The other well known integral formula

$$\int_{\mathbf{S}_0} f(\mathbf{q}) \, d\mathbf{q} = \int_0^\infty \rho^2 \int_{S_1(\mathbf{0})} f(\rho \mathbf{n}_0) \, d\mathbf{n}_0 \, d\rho \quad (56)$$

will be often used, too.

Appendix C. Pole taming

There is a well defined method to attach a Distribution to a locally non-integrable function, having a pole singularity in a Euclidean space. Originally the method was called regularization (Gelfand and Shilov 1964). Since the Distribution obtained in this way is not regular, I call the method pole taming². Only a special case is necessary for us.

For the function $\rho(\mathbf{q}) := |\mathbf{q}|$ ($\mathbf{q} \in \mathbf{S}_0$), $\frac{1}{\rho^{2+m}}$ is not locally integrable if $m > 0$. For m being an even positive integer, a Distribution can be attached to it by pole

²Áron Szabó proposed this name.

taming defined as follows. For a test function ψ let $\mathbf{q} \mapsto T_\psi^{(m-1)}(\mathbf{q})$ denote the Taylor polynomial of order $m - 1$ at zero. In a neighbourhood of zero, $\psi - T_\psi^{(m-1)}$ is a continuous function of order ρ^m , therefore $\frac{\psi - T_\psi^{(m-1)}}{\rho^{2+m}}$ is integrable there. Outside the support of ψ , however, the term containing the $(m - 1)$ -th derivative is not integrable; m is even; in the radial parameterization (B), $\mathcal{D}^{(m-1)}\psi(\mathbf{0})(\rho\mathbf{n}_0, \dots, \rho\mathbf{n}_0)$ is a linear function of an odd tensor product of \mathbf{n}_0 , thus, integrating in the order given by (56), this term will drop out and an integrable function remains.

We repeat for avoiding misunderstanding: it is known that, for an integrable function, the order of integration by \mathbf{n}_0 and ρ can be interchanged in (56). If a function is not integrable, it can occur that in one of the orders the integral exists (but in the other order not). Here we take the advantage that the integral exists in the given order. In this way the pole taming of $\frac{1}{\rho^{2+m}}$ results in the Distribution defined by

$$\left(\text{tm} \frac{1}{\rho^{2+m}} \mid \psi \right) := \int_0^\infty \left(\rho^2 \int_{S_1(\mathbf{0})} \frac{\psi(\rho\mathbf{n}_0) - T_\psi^{(m-1)}(\rho\mathbf{n}_0)}{\rho^{2+m}} d\mathbf{n}_0 \right) d\rho ! \quad (57)$$

where the exclamation mark calls attention to that the order of integration cannot be interchanged. For $m = 0$ we mean the pole taming formally by taking the corresponding regular Distribution. Later, for the sake of brevity, we also use the formula

$$\left(\text{tm} \frac{1}{\rho^{2+m}} \mid \psi \right) := \int_{S_0} \frac{\psi(\mathbf{q}) - T_\psi^{(m-1)}(\mathbf{q})}{|\mathbf{q}|^{2+m}} d\mathbf{q} !! \quad (58)$$

where the double exclamation mark calls attention to that the integral must be taken in radial parameterization and in the given order.

We have to extend the notion of pole taming as follows. For a k -th tensor power $\mathbf{n}_0^{\otimes k} := \mathbf{n}_0 \otimes \mathbf{n}_0 \otimes \dots \otimes \mathbf{n}_0$ we put

$$\left(\text{tm} \frac{\mathbf{n}_0^{\otimes k}}{\rho^{2+m}} \mid \psi \right) := \int_0^\infty \rho^2 \left(\int_{S_1(\mathbf{0})} \frac{\mathbf{n}_0^{\otimes k}(\psi(\rho\mathbf{n}_0) - T_\psi^{(m-1)}(\rho\mathbf{n}_0))}{\rho^{2+m}} d\mathbf{n}_0 \right) d\rho ! \quad (59)$$

which makes sense if $m - 1 + k$ is odd; therefore,

- if m is even then k must be even, too;
- if m is odd then k must be odd, too.

Appendix D. World rolls and world tubes around a world line

Let r be a twice continuously differentiable world line function. The Lorentz boost (see Matolcsi 2020)

$$\mathbf{L}(\mathbf{s}) := \mathbf{1} + \frac{(\dot{r}(0) + \dot{r}(\mathbf{s})) \otimes (\dot{r}(0) + \dot{r}(\mathbf{s}))}{1 - \dot{r}(0) \cdot \dot{r}(\mathbf{s})} - 2\dot{r}(\mathbf{s}) \otimes \dot{r}(0)$$

maps $\dot{r}(0)$ into $\dot{r}(\mathbf{s})$. Further, putting $\mathbf{u}_0 := \dot{r}(0)$, we use the earlier notations; then $\{\mathbf{L}(\mathbf{s}) \cdot \mathbf{q} \mid \mathbf{q} \in \mathbf{S}_0\}$ is the subspace consisting of vectors Lorentz orthogonal to $\dot{r}(\mathbf{s})$. Then

$$p: \mathbb{T} \times \mathbf{S}_0 \rightarrow \mathbb{M}, \quad (\mathbf{s}, \mathbf{q}) \mapsto r(\mathbf{s}) + \mathbf{L}(\mathbf{s}) \cdot \mathbf{q} \quad (60)$$

is a parameterization of a neighbourhood of the world line. This is a reformulation of a usual setting: (\mathbf{s}, \mathbf{q}) correspond to the so called Fermi coordinates. To be precise, the following details are necessary. p is evidently continuously differentiable and its derivative

$$\mathcal{D}p[\mathbf{s}, \mathbf{q}] = \left(\dot{r}(\mathbf{s}) + \dot{\mathbf{L}}(\mathbf{s}) \cdot \mathbf{q} \quad \mathbf{L}(\mathbf{s})|_{\mathbf{S}_0} \right) \quad (61)$$

$(\mathbf{L}(\mathbf{s})|_{\mathbf{S}_0}$ is the restriction of $\mathbf{L}(\mathbf{s})$ to \mathbf{S}_0) is injective for all \mathbf{s} and for $\mathbf{q} = \mathbf{0}$, thus, according to the inverse function theorem and the fact that $p(\mathbf{s}, \mathbf{0}) = r(\mathbf{s})$, there is an open subset in $\mathbb{T} \times \mathbf{S}_0$ which contains $\mathbb{T} \times \{\mathbf{0}\}$ where both p and all the values $\mathcal{D}p[\cdot]$ are injective.

Let I be a bounded and closed time interval. Then $\{r(\mathbf{s}) \mid \mathbf{s} \in I\}$ is a compact set and each of its points has a neighbourhood in the range of the parameterization; thus, $\{r(\mathbf{s}) \mid \mathbf{s} \in I\}$ can be covered by finite many such neighbourhoods. As a consequence, there is an $R_I > 0$ in such a way that the part $\{r(\mathbf{s}) \mid \mathbf{s} \in I\}$ of the world line is contained in

$$H_{I,R} := \{r(\mathbf{s}) + \mathbf{L}(\mathbf{s}) \cdot \mathbf{q} \mid \mathbf{s} \in I, |\mathbf{q}| < R\}$$

for all $0 < R \leq R_I$. $H_{I,R}$ is called a **world roll** of length I and radius R . With the notations

$$\rho(\mathbf{q}) := |\mathbf{L}(\mathbf{s}) \cdot \mathbf{q}| = |\mathbf{q}|, \quad \mathbf{n}(\mathbf{s}, \mathbf{q}) := \frac{\mathbf{L}(\mathbf{s}) \cdot \mathbf{q}}{\rho(\mathbf{q})}, \quad \mathbf{n}_0(\mathbf{q}) := \frac{\mathbf{q}}{\rho(\mathbf{q})}, \quad (62)$$

the parameterization of $H_{I,R}$ can be rewritten in the form

$$r(\mathbf{s}) + \rho(\mathbf{q})\mathbf{n}(\mathbf{s}, \mathbf{q}) = r(\mathbf{s}) + \mathbf{L}(\mathbf{s}) \cdot \mathbf{n}_0(\mathbf{q}). \quad (63)$$

The cylinder around $H_{I,R}$,

$$P_{I,R} := \{r(\mathbf{s}) + \mathbf{L}(\mathbf{s}) \cdot \mathbf{q} \mid \mathbf{s} \in I, |\mathbf{q}| = R\}$$

is called the corresponding **world tube**. The tangent space of $P_{I,R}$ at the point $r(\mathbf{s}) + \mathbf{L}(\mathbf{s}) \cdot \mathbf{q}$ is

$$\{(\dot{r}(\mathbf{s}) + \dot{\mathbf{L}}(\mathbf{s}) \cdot \mathbf{q})\mathbf{t} + \mathbf{L}(\mathbf{s}) \cdot \mathbf{h} \mid \mathbf{t} \in \mathbb{T}, \mathbf{h} \in \mathbf{S}_0, \mathbf{h} \cdot \mathbf{q} = 0\}.$$

For $\mathbf{q} \in \mathbf{S}_0$ we have $(\mathbf{L}(\mathbf{s}) \cdot \mathbf{q}) \cdot \dot{r}(\mathbf{s}) = \mathbf{0}$, $(\mathbf{L}(\mathbf{s}) \cdot \mathbf{q}) \cdot (\mathbf{L}(\mathbf{s}) \cdot \mathbf{h}) = \mathbf{q} \cdot \mathbf{h} = 0$ and $(\dot{\mathbf{L}}(\mathbf{s}) \cdot \mathbf{q}) \cdot (\mathbf{L}(\mathbf{s}) \cdot \mathbf{q}) = \frac{1}{2} \frac{\partial}{\partial \mathbf{s}} |\mathbf{L}(\mathbf{s}) \cdot \mathbf{q}|^2 = 0$; thus,

$$\left((\dot{r}(\mathbf{s}) + \dot{\mathbf{L}}(\mathbf{s}) \cdot \mathbf{q})\mathbf{t} + \mathbf{L}(\mathbf{s}) \cdot \mathbf{h} \right) \cdot (\mathbf{L}(\mathbf{s}) \cdot \mathbf{q}) = 0$$

and we find that the outward normal vector of $P_{I,R}$ at the point $r(\mathbf{s}) + \rho(\mathbf{q})\mathbf{n}(\mathbf{s}, \mathbf{q})$ is

$$\mathbf{n}(\mathbf{s}, \mathbf{q}). \tag{64}$$

Appendix E. Integration in world rolls

Integrals in a world roll will be computed in the parameterization (60) for which we have

$$\sqrt{|\det \mathcal{D}p[\mathbf{s}, \mathbf{q}]^* \mathcal{D}p[\mathbf{s}, \mathbf{q}]|} = 1 + \rho(\mathbf{q})\mathbf{n}(\mathbf{s}, \mathbf{q}) \cdot \dot{r}(\mathbf{s}). \tag{65}$$

To show it, for the sake of brevity, we introduce the notation

$$\mathbf{z}(\mathbf{s}, \mathbf{q}) := \dot{r}(\mathbf{s}) + \dot{\mathbf{L}}(\mathbf{s}) \cdot \mathbf{q}.$$

Then from (61) we get

$$\mathcal{D}p[\mathbf{s}, \mathbf{q}]^* \mathcal{D}p[\mathbf{s}, \mathbf{q}] = \begin{pmatrix} \mathbf{z}(\mathbf{s}, \mathbf{q}) \\ (\mathbf{L}(\mathbf{s})|_{\mathbf{S}_0})^* \end{pmatrix} \cdot \begin{pmatrix} \mathbf{z}(\mathbf{s}, \mathbf{q}) & \mathbf{L}(\mathbf{s})|_{\mathbf{S}_0} \end{pmatrix} = \tag{66}$$

$$= \begin{pmatrix} \mathbf{z}(\mathbf{s}, \mathbf{q}) \cdot \mathbf{z}(\mathbf{s}, \mathbf{q}) & \mathbf{z}(\mathbf{s}, \mathbf{q}) \cdot \mathbf{L}(\mathbf{s})|_{\mathbf{S}_0} \\ (\mathbf{L}(\mathbf{s})|_{\mathbf{S}_0})^* \cdot \mathbf{z}(\mathbf{s}, \mathbf{q}) & \mathbf{1}_{\mathbf{S}_0} \end{pmatrix}. \tag{67}$$

Omitting the variables for the sake of perspicuity, we can write $\mathbf{L}|_{\mathbf{S}_0} = \mathbf{L} \cdot (\mathbf{1} + \mathbf{u}_0 \otimes \mathbf{u}_0)$, thus $(\mathbf{L}|_{\mathbf{S}_0})^* = (\mathbf{1} + \mathbf{u}_0 \otimes \mathbf{u}_0) \cdot \mathbf{L}^*$; accordingly, $(\mathbf{1} + \mathbf{u}_0 \otimes \mathbf{u}_0) \cdot \mathbf{L}^* \cdot \mathbf{z} = \mathbf{L}^* \cdot \mathbf{z} + \mathbf{u}_0(\mathbf{u}_0 \cdot \mathbf{L}^*) \cdot \mathbf{z}$ which equals $\mathbf{z} \cdot \mathbf{L} + \mathbf{u}_0(\dot{r} \cdot \mathbf{z}) = \mathbf{z} \cdot (\mathbf{L} \cdot (\mathbf{1} + \mathbf{u}_0 \otimes \mathbf{u}_0))$. The block matrix (67) is symmetric, its determinant is $\mathbf{z} \cdot \mathbf{z} - (\mathbf{z} \cdot \mathbf{L}|_{\mathbf{S}_0}) \cdot ((\mathbf{L}|_{\mathbf{S}_0})^* \cdot \mathbf{z})$. The second term here equals $(\mathbf{z} \cdot \mathbf{L}) \cdot (\mathbf{z} \cdot \mathbf{L}) + 2(\mathbf{z} \cdot \mathbf{L}) \cdot \mathbf{u}_0(\dot{r} \cdot \mathbf{u}_0) - (\dot{r} \cdot \mathbf{u}_0)^2 = \mathbf{z} \cdot \mathbf{z} + (\dot{r} \cdot \mathbf{u}_0)^2$, thus we have

$$\det(\mathcal{D}p^* \mathcal{D}p) = -(\dot{r} \cdot \mathbf{z})^2 = -(1 + \rho \mathbf{n} \cdot \dot{r})^2;$$

the last equality comes from

$$\dot{r}(\mathbf{s}) \cdot \mathbf{z}(\mathbf{s}, \mathbf{q}) = -1 + \dot{r}(\mathbf{s}) \cdot \dot{\mathbf{L}}(\mathbf{s}) \cdot \mathbf{q} = -1 + \frac{\partial}{\partial \mathbf{s}} (\dot{r}(\mathbf{s}) \cdot \mathbf{L}(\mathbf{s}) \cdot \mathbf{q}) - \ddot{r}(\mathbf{s}) \cdot \mathbf{L}(\mathbf{s}) \cdot \mathbf{q}$$

and the quantity in the parenthesis in the middle term on the right hand side is zero.

Finally, if $\rho(\mathbf{q})$ is ‘sufficiently small’ then $1 + \rho(\mathbf{q})\mathbf{n}(\mathbf{s}, \mathbf{q}) \cdot \dot{r}(\mathbf{s})$ is positive, thus it is positive for all the possible (\mathbf{s}, \mathbf{q}) because $\mathcal{D}p[\mathbf{s}, \mathbf{q}]$ is injective, the determinant cannot be zero. Further, because of $\mathbf{n}(\mathbf{s}, \mathbf{q}) = \mathbf{L}(\mathbf{s})\mathbf{n}_0(\mathbf{q})$, using the radial parameterization (B) of \mathbf{S}_0 we have

$$\int_{H_{I,R}} \dots(x) dx = \int_0^R \rho^2 \int_{S_1(0)} \dots(r(\mathbf{s}) + \rho \mathbf{L}(\mathbf{s})\mathbf{n}_0) (1 + \rho \dot{r}(\mathbf{s}) \cdot \mathbf{L}(\mathbf{s})) d\mathbf{n}_0 d\rho. \tag{68}$$

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