

PROPERTIES OF \mathcal{H}_C -COMPACTNESS IN HEREDITARY m -SPACES

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ABSTRACT. Let (X, m, \mathcal{H}) be a hereditary m -space. A subset A of X is said to be \mathcal{H}_C -compact relative to X if for every cover \mathcal{U} of A by m -open sets of X , there exists a finite subset \mathcal{U}_0 of \mathcal{U} such that $A - \cup\{mCl(U_\alpha) : \alpha \in \mathcal{U}_0\} \in \mathcal{H}$. We obtain several properties of these sets. And also, we define and investigate nearly \mathcal{H}_C -compact relative to X .

1. Introduction

In 1967 Newcomb (1967) introduced the notion of compactness modulo an ideal. Ranchin (1972) and Hamlett and Janković (1990) further investigated this notion and obtained some more properties of compactness modulo an ideal. Császár (2007) introduced the notion of hereditary classes as a generalization of ideals. A minimal structure and a minimal space (X, m) were introduced and investigated by Popa and Noiri (2000).

In this paper, we define a subset A of a hereditary m -space (X, m, \mathcal{H}) to be \mathcal{H}_C -compact relative to X if for every cover \mathcal{U} of A by m -open sets of X , there exists a finite subset \mathcal{U}_0 of \mathcal{U} such that $A - \cup\{mCl(U_\alpha) : \alpha \in \mathcal{U}_0\} \in \mathcal{H}$. We obtain several properties of these sets. And also, we define and investigate nearly \mathcal{H}_C -compact relative to X . Some properties related to \mathcal{H}_C -compactness in hereditary m -spaces had been already introduced by Al-Omari and Noiri (2016, 2020, 2021).

2. Preliminaries

Definition 2.1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m -structure*) (Popa and Noiri 2000) on X if $\emptyset \in m$ and $X \in m$.

By (X, m) , we denote a nonempty set X with a minimal structure m on X and call it an *m -space*. Each member of m is said to be *m -open* and the complement of an m -open set is said to be *m -closed*. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in m\}$ is denoted by $m(x)$.

Definition 2.2. Let (X, m) be an m -space and A a subset of X . The *m -closure* of A and the *m -interior* of A are defined by Maki et al. (1999) as follows:

$$(1) \ mCl(A) = \cap\{F : A \subseteq F, X - F \in m\}$$

$$(2) \ mInt(A) = \cup\{U : U \subseteq A, U \in m\}$$

Definition 2.3. Let (X, m) be an m -space and A a subset of X . Then A is called

- (1) m -regular open set if $mCl(mCl(A)) = A$
- (2) m -regular closed set if $mCl(mInt(A)) = A$

Definition 2.4. A nonempty subfamily \mathcal{H} of $\mathcal{P}(X)$ is called a *hereditary class* on X (Császár 2007) if it satisfies the following properties: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class \mathcal{H} is called an *ideal* (Vaidyanathaswani 1944; Kuratowski 1966) if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$.

A minimal space (X, m) with a hereditary class \mathcal{H} on X is called a *hereditary minimal space* (briefly *hereditary m -space*) and is denoted by (X, m, \mathcal{H}) .

Definition 2.5. A minimal structure m of a set X is said to have property

- (1) \mathcal{B} (Maki et al. 1999) if the union of any collection of elements of m is an element of m .
- (2) \mathcal{F} (Al-Omari and Noiri 2019) if the finite intersection of elements of m is an element of m .

Lemma 2.6. Let (X, m, \mathcal{H}) be a hereditary m -space, $f : (X, m, \mathcal{H}) \rightarrow (Y, n)$ a function and $J_H = \{B \subset Y : f^{-1}(B) \in \mathcal{H}\}$. Then the following properties hold:

- (1) J_H is a hereditary class on Y ,
- (2) if f is injective, then $\mathcal{H} \subset f^{-1}(J_H)$,
- (3) if f is surjective, then $J_H \subset f(\mathcal{H})$,
- (4) if f is bijective, then $J_H = f(\mathcal{H})$.

Definition 2.7. A function $f : (X, m) \rightarrow (Y, n)$ is said to be

- (1) M -continuous (Popa and Noiri 2000) if for each $x \in X$ and each n -open set V containing $f(x)$, there exists an m -open set U containing x such that $f(U) \subset V$.
- (2) M -open if $f(U)$ is n -open in (Y, n) for every m -open set U of (X, m) .

Lemma 2.8. (Popa and Noiri 2000) Let m have property \mathcal{B} . Then a function $f : (X, m) \rightarrow (Y, n)$ is M -continuous if and only if $f^{-1}(V) \in m$ for every $V \in n$.

Definition 2.9. A topological space (X, τ) is said to be

- (1) C -compact (Vigilino 1969) if for each closed subset A of X and each τ -open cover $\{U_\alpha : \alpha \in \Delta\}$ of A , there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{Cl(U_\alpha) : \alpha \in \Delta_0\}$,
- (2) Nearly C -compact (Sharma and Namdeo 1977) if for each regular closed subset A of X and each τ -open cover $\{U_\alpha : \alpha \in \Delta\}$ of A , there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{Cl(U_\alpha) : \alpha \in \Delta_0\}$.

3. \mathcal{H}_C -compact spaces

Definition 3.1. Let (X, m, \mathcal{H}) be a hereditary m -space and A be a subset of X .

- (1) A is said to be \mathcal{H}_C -compact relative to X if for each cover \mathcal{U} of A by m -open sets of X , there exists a finite subfamily \mathcal{U}_0 of \mathcal{U} such that $A - \cup\{mCl(U_\alpha) : U_\alpha \in \mathcal{U}_0\} \in H$,

- (2) (X, m, \mathcal{H}) is said to be \mathcal{H}_C -compact if every m -closed set of X is \mathcal{H}_C -compact relative to X .

Theorem 3.2. Let $f : (X, m, \mathcal{H}) \rightarrow (Y, n, f(\mathcal{H}))$, where m has property \mathcal{B} , be a surjective M -continuous function. If X is \mathcal{H}_C -compact, then Y is $f(\mathcal{H})_C$ -compact.

Proof. Let $f : (X, m, \mathcal{H}) \rightarrow (Y, n, f(\mathcal{H}))$ be a surjective M -continuous function. Let A be any n -closed subset in Y and $\{U_\alpha : \alpha \in \Delta\}$ be any n -open cover of A in Y . Since f is M -continuous, $\{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is an m -open cover of $f^{-1}(A)$ in X . Since X is \mathcal{H}_C -compact and $f^{-1}(A)$ is m -closed in X , there is a finite subfamily Δ_0 of Δ and $\{f^{-1}(U_\alpha) : \alpha \in \Delta_0\}$ such that $f^{-1}(A) - \cup\{mCl(f^{-1}(U_\alpha)) : \alpha \in \Delta_0\} \in \mathcal{H}$. Therefore $f[f^{-1}(A) - \cup\{mCl(f^{-1}(U_\alpha)) : \alpha \in \Delta_0\}] \in f(\mathcal{H})$. Since $A - \cup\{f(mCl(f^{-1}(U_\alpha))) : \alpha \in \Delta_0\} \subseteq f[f^{-1}(A) - \cup\{mCl(f^{-1}(U_\alpha)) : \alpha \in \Delta_0\}]$, we have $A - \cup\{f(mCl(f^{-1}(U_\alpha))) : \alpha \in \Delta_0\} \in f(\mathcal{H})$. We note that $\{f^{-1}(U_\alpha) : \alpha \in \Delta_0\} \subseteq \{f^{-1}(mCl(U_\alpha)) : \alpha \in \Delta_0\}$ and $\{f^{-1}(mCl(U_\alpha)) : \alpha \in \Delta_0\}$ is m -closed in X . Hence $\{mCl(f^{-1}(U_\alpha)) : \alpha \in \Delta_0\} \subseteq \{f^{-1}(mCl(U_\alpha)) : \alpha \in \Delta_0\}$. Therefore $\{f(mCl(f^{-1}(U_\alpha))) : \alpha \in \Delta_0\} \subseteq \{mCl(U_\alpha) : \alpha \in \Delta_0\}$. Hence $A - \cup\{mCl(U_\alpha) : \alpha \in \Delta_0\} \in f(\mathcal{H})$. This shows that Y is $f(\mathcal{H})_C$ -compact.

Corollary 3.3. If the product space $\prod X_\alpha$ of a nonempty family of minimal spaces (X_α, m_α) is \mathcal{H}_C -compact, then each (X_α, m_α) is $P_\alpha(\mathcal{H})_C$ -compact, where P_α is the projection map and \mathcal{H} is a hereditary class of $\prod X_\alpha$.

Definition 3.4. (Noiri and Popa 2018) Let (X, m, \mathcal{H}) be a hereditary m -space and A a subset of X .

- (1) The minimal local function $A_{mH}^*(\mathcal{H}, m)$ of A is defined as follows: $A_{mH}^*(\mathcal{H}, m) = \{x \in X : U \cap A \notin \mathcal{H} \text{ for every } U \in m(x)\}$, where $m(x) = \{U : x \in U \in m\}$. Hereafter, $A_{mH}^*(\mathcal{H}, m)$ is denoted by A_{mH}^* .
- (2) The minimal \star -closure $mCl_H^*(A)$ of A is defined as $mCl_H^*(A) = A \cup A_{mH}^*$. The m_H^* -structure is defined as follows: $m_H^* = \{U \subseteq X : mCl_H^*(X \setminus U) = X \setminus U\}$. Each member of m_H^* is said to be m_H^* -open and the complement of an m_H^* -open set is said to be m_H^* -closed.

Remark 3.5. (Noiri and Popa 2018) Let (X, m, \mathcal{H}) be a hereditary m -space and A a subset of X . If $\mathcal{H} = \{\emptyset\}$ (resp. $P(X)$), then $A_{mH}^* = mCl(A)$ (resp. $A_{mH}^* = \emptyset$).

Lemma 3.6. (Noiri and Popa 2018) For a hereditary m -space (X, m, \mathcal{H}) , the following properties hold:

- (1) m_H^* is an m -structure on X such that m_H^* has property \mathcal{B} and $m \subseteq m_H^*$.
- (2) $\beta(m, \mathcal{H}) = \{U \setminus H : U \in m, H \in \mathcal{H}\}$ is a basis for m_H^* such that $m \subseteq \beta(m, \mathcal{H})$.

A function $f : (X, m) \rightarrow (Y, n)$ is said to be $M(\theta)$ -continuous at a point $x \in X$ if for every n -open set V of Y containing $f(x)$, there exists an m -open set U of X containing x such that $f(mCl(U)) \subseteq nCl(V)$. A function $f : (X, m) \rightarrow (Y, n)$ is said to be $M(\theta)$ -continuous if f is $M(\theta)$ -continuous at every $x \in X$. $M(\theta)$ -continuous function are called θ - M -continuous by Noiri and Popa (2003).

An m -space (Y, n) is said to be n -Hausdorff if for each distinct points $x, y \in Y$, there exist $U, V \in n$ containing x, y , respectively, such that $U \cap V = \emptyset$.

Theorem 3.7. *Let $f : (X, m, \mathcal{H}) \rightarrow (Y, n, f(\mathcal{H}))$ be a surjective $M(\theta)$ -continuous function, where n has property \mathcal{B} . If X is \mathcal{H}_C -compact and Y is n -Hausdorff, then $f(A)$ is $n_{f(H)}^*$ -closed in Y for each m -closed set A of X .*

Proof. Let A be any m -closed set in X and $a \notin f(A)$. For each $x \in A$, there exists an n -open set V_y containing $y = f(x)$ such that $a \notin nCl(V_y)$. Now because f is $M(\theta)$ -continuous, there exists an m -open set U_x containing x such that $f(mCl(U_x)) \subseteq nCl(V_y)$. The family $\{U_x : x \in A\}$ is an m -open cover of A . Therefore, there exists a finite subfamily $\{U_{x_i} : i = 1, 2, \dots, n\}$ such that $A - \cup_{i=1}^n mCl(U_{x_i}) \in \mathcal{H}$. Then $f[A - \cup_{i=1}^n mCl(U_{x_i})] \in f(\mathcal{H})$ and $f(A) - f(\cup_{i=1}^n mCl(U_{x_i})) \in f(\mathcal{H})$. Hence $f(A) - (\cup_{i=1}^n nCl(V_{y_i})) \in f(\mathcal{H})$. Now $a \notin nCl(V_{y_i})$ for any i implies that $a \in Y - \cup_{i=1}^n nCl(V_{y_i})$ which is n -open in Y and $(Y - \cup_{i=1}^n nCl(V_{y_i})) \cap f(A) = f(A) - \cup_{i=1}^n nCl(V_{y_i}) \in f(\mathcal{H})$. Hence $a \notin (f(A))_{nf(H)}^*$. Thus $(f(A))_{nf(H)}^* \subseteq f(A)$ and so $f(A)$ is $n_{nf(H)}^*$ -closed.

Theorem 3.8. *Let (X, m, \mathcal{H}) be a hereditary m -space, m have Property \mathcal{F} , and (X, m) be m -Hausdorff. If Y is \mathcal{H}_C -compact relative to X , then Y is m_H^* -closed.*

Proof. Let $x \in X - Y$. Since X is m -Hausdorff, for each $y \in Y$, there exist m -open sets U_y, V_y of x and y , respectively, such that $U_y \cap V_y = \emptyset$. The family $\{V_y : y \in Y\}$ is an m -open cover of Y . Since Y is \mathcal{H}_C -compact, there exists a finite subcollection $\{V_{y_i} : i = 1, 2, \dots, n\}$ such that $Y - \cup_{i=1}^n mCl(V_{y_i}) \in \mathcal{H}$. Let $H = Y - \cup_{i=1}^n mCl(V_{y_i})$, Define $U = \cap_{i=1}^n U_{y_i}$. Then U is an m -open set containing x and $U \cap (\cup_{i=1}^n mCl(V_{y_i})) = \emptyset$. Therefore $U \cap Y \subseteq H \in \mathcal{H}$ and hence $x \notin Y_{mH}^*$. Thus $x \notin Y$ implies that $x \notin Y_{mH}^*$. Hence Y is m_H^* -closed.

A family \mathcal{F} of subsets of X is said to have the finite-intersection property modulo a hereditary class \mathcal{H} on X or just \mathcal{H} -FIP if the intersection of no finite subfamily of \mathcal{F} is a member of \mathcal{H} .

Theorem 3.9. *For a hereditary m -space (X, m, \mathcal{H}) , the following properties are equivalent:*

- (1) (X, m, \mathcal{H}) is \mathcal{H}_C -compact;
- (2) for each m -closed subset A of X and each family \mathcal{F} of m -closed subsets of X such that $\cap\{F \cap A : F \in \mathcal{F}\} = \emptyset$, there exists a finite subfamily $\{F_1, F_2, \dots, F_n\}$ such that $\cap_{i=1}^n \{mInt(F_i) \cap A\} \in \mathcal{H}$;
- (3) for each m -closed subset A of X and each family \mathcal{F} of m -closed subsets of X such that $\{mInt(F) \cap A : F \in \mathcal{F}\}$ has \mathcal{H} -FIP, one has $\cap\{F \cap A : F \in \mathcal{F}\} \neq \emptyset$.

Proof. (1) \Rightarrow (2): Let (X, m) be \mathcal{H}_C -compact, A be an m -closed subset of X , and \mathcal{F} a family of m -closed subsets with $\cap\{F \cap A : F \in \mathcal{F}\} = \emptyset$. Then $\{X - F : F \in \mathcal{F}\}$ is an m -open cover of A and hence admits a finite subfamily $\{X - F_i : i = 1, 2, \dots, n\}$ such that $A - \cup_{i=1}^n mCl(X - F_i) \in \mathcal{H}$. So $\cap_{i=1}^n \{mInt(F_i) \cap A\} \subseteq A - \cup_{i=1}^n mCl(X - F_i) \in \mathcal{H}$. Hence $\cap_{i=1}^n \{mInt(F_i) \cap A\} \in \mathcal{H}$.

(2) \Rightarrow (3): This is easy to be established.

(3) \Rightarrow (1): Let A be an m -closed subset and \mathcal{U} be an m -open cover of A with the property that for no finite subfamily $\{U_1, U_2, \dots, U_n\}$ of \mathcal{U} one has $A - \cup_{i=1}^n mCl(U_i) \in \mathcal{H}$. Then $\{X - U : U \in \mathcal{U}\}$ is a family of m -closed sets. Since

$$\cap_{i=1}^n \{X - mCl(U_i)\} \cap A = \cap_{i=1}^n \{A - mCl(U_i)\} = A - \cup_{i=1}^n mCl(U_i),$$

the family $\{mInt(X - U) \cap A : U \in \mathcal{U}\}$ has \mathcal{H} -FIP. By the hypothesis $\cap\{(X - U) \cap A : U \in \mathcal{U}\} \neq \emptyset$. But then $A - \cup\{U : U \in \mathcal{U}\} \neq \emptyset$ and \mathcal{U} is not cover of A . This is a contradiction.

Theorem 3.10. For a hereditary m -space (X, m, \mathcal{H}) , where m has property \mathcal{B} , the following properties are equivalent:

- (1) (X, m, \mathcal{H}) is \mathcal{H}_C -compact;
- (2) for each m -closed subset A of X , each m -open cover \mathcal{U} of $X - A$, and each m -open neighborhood V of A , there exists a finite subfamily $\{U_1, U_2, \dots, U_n\}$ of \mathcal{U} such that $X - (V \cup (\cup_{i=1}^n mCl(U_i))) \in \mathcal{H}$.

Proof. (1) \Rightarrow (2): Let A be an m -closed set, V an m -open neighborhood of A , and \mathcal{U} an m -open cover of $X - A$. Since $X - V \subseteq X - A$, \mathcal{U} is also an m -open cover of the closed set $X - V$. Let $\{U_1, U_2, \dots, U_n\}$ be a finite subcollection of \mathcal{U} such that $(X - V) - \cup_{i=1}^n mCl(U_i) \in \mathcal{H}$. However, $X - (V \cup (\cup_{i=1}^n mCl(U_i))) \in \mathcal{H}$.

(2) \Rightarrow (1): Let A be an m -closed subset of X and \mathcal{U} an m -open covering of A . If H denotes the union of members of \mathcal{U} , then $F = X - H$ is an m -closed set and $X - A$ is an m -open neighborhood of F . Also \mathcal{U} is an m -open cover of $X - F$. By hypothesis, there is a finite subcollection $\{U_1, U_2, \dots, U_n\}$ of \mathcal{U} such that $X - ((X - A) \cup (\cup_{i=1}^n mCl(U_i))) \in \mathcal{H}$. However, $A - \cup_{i=1}^n mCl(U_i) \subseteq X - ((X - A) \cup (\cup_{i=1}^n mCl(U_i))) \in \mathcal{H}$. Hence (X, m, \mathcal{H}) is \mathcal{H}_C -compact.

4. Nearly \mathcal{H}_C -compact sets

We now introduce the concept of nearly \mathcal{H}_C -compact sets.

Definition 4.1. A hereditary m -space (X, m, \mathcal{H}) is said to be nearly \mathcal{H}_C -compact if for each m -regular closed set A of X and an m -open cover $\{U_\alpha : \alpha \in \Delta\}$ of A , there exists a finite subset Δ_0 of Δ such that $A - \cup\{mCl(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$.

Remark 4.2. The following results can be verified easily.

- (1) Every \mathcal{H}_C -compact is nearly \mathcal{H}_C -compact.
- (2) Every m -regular closed subspace of a nearly \mathcal{H}_C -compact space is nearly \mathcal{H}_C -compact.
- (3) Let (X, m, \mathcal{H}) be nearly \mathcal{H}_C -compact. If \mathcal{I} is a hereditary classes on X such that $\mathcal{H} \subseteq \mathcal{I}$, then (X, m, \mathcal{I}) is nearly \mathcal{I}_C -compact.
- (4) Let (X, m, \mathcal{H}) be an m -Housdorff space. If $Y \subseteq X$ is nearly \mathcal{H}_C -compact, then Y is m_H^* -closed.

We recall the following definitions:

Definition 4.3. A subset A of an m -space (X, m) is said to be N_m -closed if for every cover of A by m -regular open sets of X , there exists a finite subcover of A .

Definition 4.4. An m -space (X, m) is said to be almost m -regular if for every m -regular closed set A and a point $x \notin A$, there exist m -open sets U and V such that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$, or equivalently, if for each point $x \in X$ and each m -regular open set V containing x , there exists an m -regular open set U such that $x \in U \subseteq mCl(U) \subseteq V$.

We now introduce the following definition:

Definition 4.5. Let (X, m, \mathcal{H}) be a hereditary m -space. A subset A of X is said to be \mathcal{H}_N -closed if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m -regular open sets of X , there exists a finite subset Δ_0 of Δ such that $A - \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$.

Theorem 4.6. *Let (X, m, \mathcal{H}) be almost m -regular and nearly \mathcal{H}_C -compact. Then every m -regular subset of X is \mathcal{H}_N -closed.*

Proof. Let A be an m -regular closed subset of the almost m -regular nearly \mathcal{H}_C -compact space X . Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of A by m -regular open sets of X . For each $x \in A$, there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since X is almost m -regular, there exists an m -regular open set $V_{\alpha(x)}$ such that $x \in V_{\alpha(x)} \subseteq mCl(V_{\alpha(x)}) \subseteq U_{\alpha(x)}$. Therefore the collection $\{V_{\alpha(x)} : x \in A\}$ is an m -regular open cover of A . Since X is nearly \mathcal{H}_C -compact, there exists a finite subset Δ_0 of Δ such that $A - \cup\{mCl(V_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$. Since $mCl(V_\alpha) \subseteq U_\alpha$ for each $\alpha \in \Delta_0$, we have $A - \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This implies that A is \mathcal{H}_N -closed.

The A function $f : (X, m) \rightarrow (Y, n)$ is said to be almost m -open (resp. almost m -closed) if the image of every m -regular open (resp. m -regular closed) set of X is an n -open (resp. n -closed) set of Y .

Theorem 4.7. *Let (X, m, \mathcal{H}) be a hereditary m -space and (Y, n) be an n -space, where n has property \mathcal{F} . If $f : (X, m, \mathcal{H}) \rightarrow (Y, n)$ is an $M(\theta)$ -continuous function from a nearly \mathcal{H}_C -compact space into an n -Housdorff space, then $f : (X, m, \mathcal{H}) \rightarrow (Y, n_{f(\mathcal{H})}^*)$ is almost m -closed.*

Proof. Let A be any m -regular closed subset of X . Consider an element $y \notin f(A)$. Since Y is n -Housdorff, for each $x \in A$, we can choose an n -open set O_x containing $f(x)$ such that $y \notin nCl(O_x)$. Since f is $M(\theta)$ -continuous, there exists an m -open set U_x containing x such that $f(mCl(U_x)) \subseteq nCl(O_x)$. Now $\{U_x : x \in A\}$ is an m -open cover of A . Since X is nearly \mathcal{H}_C -compact, there exists a finite subset A_0 of A such that $A - \cup\{mCl(U_x) : x \in A_0\} \in \mathcal{H}$. Hence $f[A - \cup\{mCl(U_x) : x \in A_0\}] \in f(\mathcal{H})$, consequently, $f(A) - \cup f[\{mCl(U_x) : x \in A_0\}] \in f(\mathcal{H})$ implies that $f(A) - \cup\{nCl(O_x) : x \in A_0\} \in f(\mathcal{H})$ and $f(A) - \cup\{nCl(O_x) : x \in A_0\} = f(H)$ for some $f(H) \in f(\mathcal{H})$. Therefore, $f(A) - f(H) \subseteq \cup\{nCl(O_x) : x \in A_0\}$ and $Y - \cup\{nCl(O_x) : x \in A_0\}$ is an n -open set containing y which is disjoint from $f(A) - f(H)$. Thus $[Y - \cup\{nCl(O_x) : x \in A_0\}] - f(H)$ is an $n_{f(\mathcal{H})}^*$ -open set in Y containing y with respect to the minimal structure $n_{f(\mathcal{H})}^*$ on Y , which is disjoint from $f(A)$. This proves that $f(A)$ is $n_{f(\mathcal{H})}^*$ -closed in $(Y, n_{f(\mathcal{H})}^*)$. Thus f is almost m -closed.

Since every M -continuous function is $M(\theta)$ -continuous, it follows that

Corollary 4.8. *Let (X, m, \mathcal{H}) be a hereditary m -space and (Y, n) be an n -space. If $f : (X, m, \mathcal{H}) \rightarrow (Y, n)$ is an M -continuous function from a nearly \mathcal{H}_C -compact space into an n -Housdorff space, then f is almost m -closed as a function $f : (X, m, \mathcal{H}) \rightarrow (Y, n_{f(\mathcal{H})}^*)$.*

Corollary 4.9. *Let (X, m, \mathcal{H}) be a hereditary m -space and (Y, n) be an n -space. If $f : (X, m, \mathcal{H}) \rightarrow (Y, n)$ is an M -continuous function from an \mathcal{H}_C -compact space into an n -Housdorff space, then f is almost m -closed as a function $f : (X, m, \mathcal{H}) \rightarrow (Y, n_{f(\mathcal{H})}^*)$.*

A function $f : (X, m) \rightarrow (Y, n)$ is said to be completely M -continuous if the inverse image of every n -open subset of Y is an m -regular open subset of X . We obtain a theorem on the completely M -continuous image of a nearly \mathcal{H}_C -compact space.

Theorem 4.10. *Let (X, m, \mathcal{H}) be a hereditary m -space and (Y, n) be an n -space. Then the completely M -continuous image of a nearly \mathcal{H}_C -compact space is $f(\mathcal{H})_C$ -compact.*

Proof. Let $f : (X, m, \mathcal{H}) \rightarrow (Y, n)$ be a completely M -continuous surjection from a nearly \mathcal{H}_C -compact space X onto Y . Let A be an n -closed subset of Y and $\{O_\alpha : \alpha \in \Delta\}$ be an n -open cover of A . Since f is completely M -continuous, $f^{-1}(A)$ is an m -regular closed set of X and $\{f^{-1}(O_\alpha) : \alpha \in \Delta\}$ is an m -regular open covering of $f^{-1}(A)$. Since X is nearly \mathcal{H}_C -compact, there exists a finite subset Δ_0 of Δ such that $f^{-1}(A) - \cup\{mCl(f^{-1}(O_\alpha)) : \alpha \in \Delta_0\} \in \mathcal{H}$ and hence $f[f^{-1}(A) - \cup\{mCl(f^{-1}(O_\alpha)) : \alpha \in \Delta_0\}] \in f(\mathcal{H})$.

Consequently, $A - \cup f\{mCl(f^{-1}(O_\alpha)) : \alpha \in \Delta_0\} \in f(\mathcal{H})$ and hence $A - \cup\{nCl(O_\alpha) : \alpha \in \Delta_0\} \in f(\mathcal{H})$. This implies that Y is $f(\mathcal{H})_C$ -compact.

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