DOI: 10.1478/AAPP.1001A5

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 100, No. 1, A5 (2022)

EXISTENCE RESULTS FOR HIGHLY DISCONTINUOUS IMPLICIT ELLIPTIC EQUATIONS

PAOLO CUBIOTTI *

ABSTRACT. Let $n \in \mathbf{N}$, with $n \geq 3$, let $p \in]n/2, +\infty[$, and let $\Omega \subseteq \mathbf{R}^n$ be a bounded domain with smooth boundary. Let $Y \subseteq \mathbf{R}^n$, and let $\varphi : \Omega \times \mathbf{R}^h \to \mathbf{R}$ and $\psi : Y \to \mathbf{R}$ be two given functions, with ψ continuous. We study the existence of strong solutions $u = (u_1, \ldots, u_h) \in W^{2,p}(\Omega, \mathbf{R}^h) \cap W_0^{1,p}(\Omega, \mathbf{R}^h)$ of the implicit elliptic equation $\psi(-\Delta u) = \varphi(x, u)$, where $\Delta u = (\Delta u_1, \Delta u_2, \ldots, \Delta u_h)$. We prove existence results where φ is allowed to be highly discontinuous in both variables. In particular, a function $\varphi(x, z)$ satisfying our assumptions could be discontinuous, with respect to the second variable, even at all points $z \in \mathbf{R}^h$.

1. Introduction

Let $\Omega \subseteq \mathbf{R}^n$ (with $n \geq 3$) be a bounded domain, with smooth boundary $\partial\Omega$, let $Y \subseteq \mathbf{R}^h$, and let $\psi : Y \to \mathbf{R}$ and $\varphi : \Omega \times \mathbf{R}^h \to \mathbf{R}$ be two given functions. In this paper we deal with the implicit boundary-value problem

$$\begin{cases} -\Delta u \in Y & \text{in } \Omega, \\ \psi(-\Delta u) = \varphi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with $u = (u_1, u_2, \ldots, u_h)$, $\Delta u = (\Delta u_1, \Delta u_2, \ldots, \Delta u_h)$ and φ discontinuous in both variables. Up to our knowledge, there are not many results on this implicit problem. Actually, we can only cite Bielawski and Górniewicz (1986, 1989), Marano (1994, 1995, 1996), and Carl and Heikkilä (1998) (see also the survey by Marano 1997b). While the papers authored by Bielawski and Górniewicz (1986, 1989), Marano (1994), and Carl and Heikkilä (1998) deal with the case of a continuous nonlinearity φ, Marano (1995, 1996, 1997b) considers the case of a discontinuous φ, allowing the function φ itself to have a quite large (possibly uncountable) set of points of discontinuity.

Our aim in this paper is to prove some existence theorems for the solutions $u \in W^{2,p}(\Omega, \mathbf{R}^h) \cap W_0^{1,p}(\Omega, \mathbf{R}^h)$ of the problem (1) and of some of its special cases, where φ is allowed to have a set of points of discontinuity significantly larger than in the above quoted papers. Our research was originally motivated by the reading of the deep papers by Marano (1995, 1996). The basic idea of this paper is to combine the existence results for elliptic differential inclusions obtained by Marano (1995, 1996) with some results and techniques recently developed for the study of differential and integral problems associated with discontinuous functions (see, for instance, Cubiotti and Yao 2015, 2016, and references therein).

As a matter of fact, with respect to the original results obtained by Marano (1995, 1996), the kind of discontinuity allowed for φ is the main pecularity of our results. Therefore, in order to explain better the nature of our results, we now introduce some notations. For each $i \in \{1,\dots,h\}$, we denote be $P_{h,i} : \mathbf{R}^h \to \mathbf{R}$ the projection over the i-th axis. That is, for every $z = (z_1,\dots,z_h) \in \mathbf{R}^h$, we put $P_{h,i}(z) = z_i$. Moreover, we denote by $\mathscr{F}_h$ the family of all subsets $E \subseteq \mathbf{R}^h$ such that there exist sets $V_1,V_2,\dots,V_h \subseteq \mathbf{R}^h$, with $m_1(P_{h,i}(V_i)) = 0$ for all $i = 1\dots,h$, such that $E = \bigcup_{i=1}^h V_i$ (here and in the sequel, if $k \in \mathbf{N}$, m_k denotes the k-dimensional Lebesgue measure in $\mathbf{R}^k$). Of course, if $V \in \mathscr{F}_h$, then V is Lebesgue measurable and $m_h(V) = 0$. Now, for each $(x,z) = (x,z_1,\dots,z_h) \in \Omega \times \mathbf{R}^h$, let us put

$$\pi_0(x,z) = x, \qquad \pi_i(x,z) = P_{h,i}(z) = z_i \quad \text{if } i = 1,\dots,n,$$

and let

$$\mathscr{H}_{\Omega,h} := \big\{A \subseteq \Omega \times \mathbf{R}^h : \exists\, i \in \{0,\dots,h\} \text{ such that } \pi_i(A) \text{ has null Lebesgue measure}\big\}.$$

In the paper by Marano (1996), which is entirely devoted to the study of the implicit problem (1), the central assumption on $\varphi : \Omega \times \mathbf{R}^h \to \mathbf{R}$ is the following:

(a_1) φ is essentially bounded, and the set

$$D'_\varphi := \big\{(x,z) \in \Omega \times \mathbf{R}^h : \ \varphi \text{ is discontinuous at } (x,z)\ \big\}$$

belongs to $\mathscr{H}_{\Omega,h}$.

Thus, in the paper by Marano (1996) the set D'_φ of the discontinuities of the function φ can be significantly large, but it is forced to be a zero-measure set in $\mathbf{R}^{n+h}$, with a suitable geometry. Conversely, the central regularity assumption on φ that we require in our main result (as well as in its consequences) is the following:

(b_1) there exists a set $E \in \mathscr{F}_h$ such that for all $z \in \mathbf{R}^h \setminus E$ the function $\varphi(\cdot,z)$ is measurable, and for almost every $x \in \Omega$ the function $\varphi(x,\cdot)|_{\mathbf{R}^h \setminus E}$ is continuous.

It is routine matter to check that if the function φ satisfies assumption (a_1), then it satisfies our assumption (b_1), while the converse implication is not true in general. Moreover, it is easy to see that if a function $\varphi : \Omega \times \mathbf{R}^h \to \mathbf{R}$ satisfies assumption (b_1), then it may happen that for every $x \in \Omega$ the function $\varphi(x,\cdot)$ is discontinuous even at all points $z \in \mathbf{R}^h$. For instance, in the case $h = 1$, one can consider the function $\varphi : \Omega \times \mathbf{R} \to \mathbf{R}$ defined by

$$\varphi(x,z) = \begin{cases} 1 & \text{if } x \in \Omega \text{ and } z \in \mathbf{Q} \\ 2 & \text{if } x \in \Omega \text{ and } z \notin \mathbf{Q} \end{cases} \tag{2}$$

(where $\mathbf{Q}$ denotes the set of rational real numbers). In this case, for every $x \in \Omega$ the function $\varphi(x,\cdot)$ is discontinuous at all points $z \in \mathbf{R}$. Taking $E = \mathbf{Q}$, such a function φ satisfies (b_1), but not (a_1) since $D'_\varphi = \Omega \times \mathbf{R}$. Similar examples can be considered even for $h > 1$.

Under assumption (b_1) and some other suitable assumptions (in particular, ψ is only assumed to be continuous and locally nonconstant), our main result guarantees the existence of a solution $u \in W^{2,p}(\Omega,\mathbf{R}^h) \cap W_0^{1,p}(\Omega,\mathbf{R}^h)$ of the problem (1), such that, in particular,

$$u(x) \in \mathbf{R}^h \setminus E \quad \text{for a.e.} \quad x \in \Omega.$$

At this point, it is worth noticing that in the statements of our results the function φ might even be defined only over the set $\Omega \times (\mathbf{R}^h \setminus E)$, since its behaviour over the set $\Omega \times E$ plays no role at all. In order to give a more precise idea of the nature of our results, we point out the following consequence of our main result (Theorem **3.1** below). In the statement of Theorem **1.1**, the constant B is defined as in Proposition **2.2** below.

Theorem 1.1. *Let $n \geq 3$, $p \in]n/2,+\infty[$, and let $\Omega \subseteq \mathbf{R}^n$ be a nonempty, open, bounded and connected set, with boundary $\partial\Omega$ of class $C^{1,1}$. Let $a > 0$, $\psi : [a,+\infty[\to \mathbf{R}$ a continuous function, and let $\varphi : \mathbf{R} \to \mathbf{R}$ be a given function. Finally, let $E \subseteq \mathbf{R}$, with $m_1(E) = 0$. Assume that:*

- (i$_1$) *the function $\varphi|_{\mathbf{R}\setminus E}$ is continuous;*
- (ii$_1$) $\mathrm{int}(\psi^{-1}(t)) = \emptyset$ *for all* $t \in \mathrm{int}(\psi([a,+\infty[))$*;*
- (iii$_1$) *there exists $\rho > a$ such that*

$$\varphi(]-B\rho\, m_n(\Omega)^{1/p}, B\rho\, m_n(\Omega)^{1/p}[\setminus E) \subseteq \psi([a,\rho]).$$

Then, there exists a positive function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that:

- (a) $-\Delta u(x) \in [a,\rho]$ *and* $\psi(-\Delta u(x)) = \varphi(u(x))$ *for a.e.* $x \in \Omega$*;*
- (b) $u(x) \in]0, B\rho\, m_n(\Omega)^{1/p}[\setminus E$ *for a.e.* $x \in \Omega$.

As we have already observed, a function $\varphi : \mathbf{R} \to \mathbf{R}$ satisfying assumption (i$_1$) of Theorem **1.1** can be discontinuous even at every point $z \in \mathbf{R}$. To see this, one can take $E = \mathbf{Q}$ and $\varphi : \mathbf{R} \to \mathbf{R}$ defined as in (2), that is

$$\varphi(z) = \begin{cases} 1 & \text{if } z \in \mathbf{Q} \\ 2 & \text{if } z \notin \mathbf{Q}. \end{cases} \tag{3}$$

At this point, a natural comparison can be made with the analogous nice result obtained - by means of a different argument - by Marano (1995) (see also Theorem 4.1 in the paper by Marano 1997b), that we now state explicitly for the reader's convenience.

Theorem 1.2. (Marano 1995, Theorem 4.2). *Let n, p, and Ω be as in Theorem **1.1**. Let $a > 0$, $\psi : [a,+\infty[\to \mathbf{R}$ a continuous function, and let $\varphi : \mathbf{R} \to \mathbf{R}$ be a given function. Assume that:*

- (i$_2$) *the set $D_\varphi = \{z \in \mathbf{R} :\ \varphi$ is discontinuous at $z\}$ has measure zero;*
- (ii$_2$) $\mathrm{int}(\psi^{-1}(t)) = \emptyset$ *for all* $t \in \mathrm{int}(\psi([a,+\infty[))$*;*
- (iii$_2$) *there exists $\rho > a$ such that*

$$\varphi([-B\rho\, m_n(\Omega)^{1/p}, B\rho\, m_n(\Omega)^{1/p}]) \subseteq \psi([a,\rho]).$$

Then, there exists a positive function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *such that* $\psi(-\Delta u(x)) = \varphi(u(x))$ *for a.e.* $x \in \Omega$.

It is clear how Theorem **1.1** improves Theorem **1.2**. That is, the regularity assumption (i_2) on φ in Theorem **1.2** is now replaced by the weaker assumption (i_1). This allows φ to have a significantly larger set of points of discontinuity. Just for instance, the function φ defined in (3) satisfies assumption (i_1), but it does not satisfy (i_2) since in this case we have $D_\varphi = \mathbf{R}$. As we pointed out above, Theorem **1.1** also guarantees that the solution u avoids almost everywhere che set E. Consequently, in the statement of Theorem **1.1**, the behaviour of the function φ over the set E plays no role at all.

Before ending this section, we briefly consider the explicit case where $h = 1$ and $\psi(y) = y$, that is the problem

$$\begin{cases} -\Delta u = \varphi(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

For this explicit problem and for its special cases, with φ discontinuous in u, more literature is available (see Stuart 1978; Stuart and Toland 1980; Cerami 1983a,b; Ambrosetti and Turner 1988; Ambrosetti and Badiale 1989; Lupo 1989; Carl and Heikkilä 1992a,b; Heikkilä and Lakshmikantham 1994; Bonanno and Marano 1996, 2000). However, it is interesting to observe that most literature deals with the case where the function $\varphi(x,\cdot)$ has a finite or at most countable set of discontinuity points (see Stuart 1978; Stuart and Toland 1980; Cerami 1983a,b; Ambrosetti and Turner 1988; Ambrosetti and Badiale 1989; Lupo 1989; Carl and Heikkilä 1992a,b; Heikkilä and Lakshmikantham 1994).

In the papers by Marano (1995), Bonanno and Marano (1996), Marano (1997a,b), and Bonanno and Marano (2000), conversely, some results are obtained for the case where φ may have a possibly uncountable set of points of discontinuity. As regards the continuity of the function $\varphi : \Omega \times \mathbf{R} \to \mathbf{R}$, we observe that the basic assumption on φ made by Marano (1995), Bonanno and Marano (1996), and Marano (1997b) is the following:

(a_2) there exists a set $\Omega_0 \subseteq \Omega$, with $m_n(\Omega_0) = 0$, such that the set

$$D_\varphi := \bigcup_{x \in \Omega \setminus \Omega_0} \{ z \in \mathbf{R} : \ \varphi(x,\cdot) \text{ is discontinuous at } z \}$$

has null Lebesgue measure, and for all $z \in \mathbf{R} \setminus D_\varphi$ the function $\varphi(\cdot, z)$ is measurable.

Conversely, as regards the explicit problem (4), our assumption (b_1) takes the following form:

(b_2) there exists a set $E \subseteq \mathbf{R}$, with $m_1(E) = 0$, such that for all $z \in \mathbf{R} \setminus E$ the function $\varphi(\cdot, z)$ is measurable, and for almost every $x \in \Omega$ the function $\varphi(x,\cdot)|_{\mathbf{R}\setminus E}$ is continuous.

Is is self-evident that if the function $\varphi : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies assumption (a_2), then it satifies, in particular, our assumption (b_2), while the converse is not true in general. To see this, it suffices to consider the function (2). Again, the same example shows that if a function φ satisfies (b_2), it may even happen that for all $x \in \Omega$ the function $\varphi(x,\cdot)$ is discontinuous *even at all points* $z \in \mathbf{R}$. However, we have to point out that most of the results obtained by

Marano (1995, 1996) are formally independent from ours, even if our ones allow φ to have a bigger set of discontinuity points.

Finally, we remark that our approach is based on set-valued analysis. In particular, it is based on a recent selection result (stated as Theorem 2.3 below) for lower semicontinuous multifunctions with possibly non-convex values, and on the precious previous work made by Marano (1995, 1996) on elliptic differential inclusions. This last work was based, in turn, on a deep existence theorem for operator inclusions, proved by Naselli Ricceri and Ricceri (1990, Theorem 1). Our main results and some of its applications will be stated and proved in Section 3, while in Section 2 we shall fix some notations and recall some preliminary results.

2. Preliminaries

Throughout this paper, n is a positive integer, with $n \geq 3$, and Ω is a nonempty, open, bounded and connected subset of $\mathbf{R}^n$, with boundary $\partial\Omega$ of class $C^{1,1}$. Moreover, p is any real number belonging to $]n/2, +\infty[$. Let $h \in \mathbf{N}$. We denote by $\mathscr{G}_h$ the family of all subsets $U \subseteq \mathbf{R}^h$ such that, for every $i = 1, \ldots, h$, the supremum and the infimum of the projection of $\overline{\text{conv}}\,(U)$ on the i-th axis are both positive or both negative, where $\overline{\text{conv}}\,(U)$ denotes the closed convex hull of the set U. Moreover, the projections $P_{h,i} : \mathbf{R}^h \to \mathbf{R}$ (for $i \in \{1, \ldots, h\}$) and the family $\mathscr{F}_h$ are defined as in Section 1. For every $z = (z_1, z_2, \ldots, z_h) \in \mathbf{R}^h$, we denote by $\|z\|_h$ the Euclidean norm of z in $\mathbf{R}^h$, and by $\|z\|_h^*$ the norm $\|z\|_h^* = \sum_{i=1}^h |z_i|$. The metrics induced in $\mathbf{R}^h$ by the norms $\|\cdot\|_h$ and $\|\cdot\|_h^*$ are denoted by d and d^*, respectively. Moreover, if $r > 0$, we denote by $\overline{B}(z, r)$ and $\overline{B}^*(z, r)$ the closed balls centered at z with radius r, with respect to the norms $\|\cdot\|_h$ and $\|\cdot\|_h^*$, respectively. Let V be a topological space and $(S, \mathscr{A})$ a measurable space. Following Himmelberg (1975), we say that a multifunction $F : S \to 2^V$ is *$\mathscr{A}$-measurable* (resp., *$\mathscr{A}$-weakly measurable*) in S if for any closed (resp., open) set $U \subseteq V$ one has

$$F^-(U) := \{x \in S : \ F(x) \cap U \neq \emptyset\} \in \mathscr{A}.$$

We recall that if V is a metric space, then the $\mathscr{A}$-measurability implies the $\mathscr{A}$-weak measurability. Moreover, if V is a σ-compact and separable metric space, and F has closed values, then the two notions of measurability are equivalent (Himmelberg 1975, Theorem 3.5). For the basic facts of the theory of multifunctions, we refer to Klein and Thompson (1984) and Denkowski *et al.* (2003). For what concerns measurable multifunctions, we also refer to Himmelberg (1975). Finally, we denote by $\mathscr{B}(V)$ the Borel family of the topological space V. As regards the definition of Souslin sets and their properties, we refer to Chapter 6 of Bogachev (2007).

In what follows, "measurable function" will mean "Lebesgue measurable function". Moreover, if $C \subseteq \mathbf{R}^n$ is a Lebesgue measurable set, we shall denote by $\mathscr{L}(C)$ the family of all Lebesgue measurable subsets of C. As usual, we denote by $L^p(\Omega, \mathbf{R}^h)$ the space of (equivalence classes of) functions $u : \Omega \to \mathbf{R}^h$, with $u = (u_1, u_2, \ldots, u_h)$, such that $u_i \in L^p(\Omega)$ for all $i = 1, \ldots, h$. The space $L^p(\Omega, \mathbf{R}^h)$ is endowed with the norm $\|u\|_{L^p(\Omega, \mathbf{R}^h)} = \sum_{i=1}^h \|u_i\|_{L^p(\Omega)}$, where $\|\cdot\|_{L^p(\Omega)}$ is the usual norm of $L^p(\Omega)$. If $k \in \mathbf{N}$, we denote by $W^{k,p}(\Omega)$ the space of all functions $v \in L^p(\Omega)$ whose weak derivatives $D^\alpha v$, up to the order k, belong to $L^p(\Omega)$ (see Adams and Fournier 2003). The space $W^{k,p}(\Omega)$ is endowed with the norm

$\|v\|_{W^{k,p}(\Omega)} = \sum_{0\leq|\alpha|\leq k} \|D^{\alpha}v\|_{L^p(\Omega)}$. We denote by $W_0^{1,p}(\Omega)$ the closure of C_0^{∞} in $W^{1,p}(\Omega)$. We now recall the following proposition.

Proposition 2.1. (Marano 1996, Proposition 2.1). *Let $u \in W^{2,p}(\Omega)$ and let $E \subseteq \mathbf{R}$ be a measurable set such that $m_1(E) = 0$.*

Then, $\Delta u(x) = 0$ for almost every $x \in u^{-1}(E)$.

The following fact, which is stated by Marano (1995), follows from Theorem 2 of Talenti (1976).

Proposition 2.2. *Let $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Then, one has*

$$\operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \leq B\,\|\Delta u(x)\|_{L^p(\Omega)},$$

where

$$B = [m_n(\Omega)]^{2/n-1/p}\,\frac{\Gamma(1+n/2)^{2/n}}{n(n-2)\pi}\times \\ \times\left[\frac{\Gamma(1+p/(p-1))\,\Gamma(n/(n-2)-p/(p-1))}{\Gamma(n/(n-2))}\right]^{1-1/p}$$

and Γ denotes the Gamma function.

As usual, we denote by $W^{k,p}(\Omega,\mathbf{R}^h)$ (resp., by $W_0^{1,p}(\Omega,\mathbf{R}^h)$) the space of (equivalence classes of) functions $u:\Omega\to\mathbf{R}^h$, with $u=(u_1,u_2,\ldots,u_h)$, such that $u_i \in W^{k,p}(\Omega)$ (resp., $u_i \in W_0^{1,p}(\Omega)$) for all $i=1,\ldots,h$. The space $W^{k,p}(\Omega,\mathbf{R}^h)$ is endowed with the norm $\|u\|_{W^{k,p}(\Omega,\mathbf{R}^h)} = \sum_{i=1}^{h} \|u_i\|_{W^{k,p}(\Omega)}$.

The following selection result, that we state explicitly for the sake of a better reading, will be a fundamental tool in the sequel (here, $\mathscr{T}_\mu$ denotes the completion of $\mathscr{B}(T)$ with respect to the measure μ).

Theorem 2.3. (Cubiotti and Yao 2015, Theorem 2.1). *Let T and $X_1,X_2,\ldots X_k$ be complete separable metric spaces, with $k\in\mathbf{N}$, and let $X := \prod_{j=1}^{k} X_j$ (endowed with the product topology). Let $\mu,\psi_1,\ldots,\psi_k$ be positive regular Borel measures over $T,X_1,X_2,\ldots X_k$, respectively, with μ finite and $\psi_1,\ldots,\psi_k$ σ-finite.*

Let S be a separable metric space, $W\subseteq X$ a Souslin set, and let $F: T\times W\to 2^S$ be a multifunction with nonempty complete values. Let $E\subseteq W$ be a given set. Finally, for all $i\in\{1,\ldots,k\}$, let $P_{*,i}: X\to X_i$ be the projection over X_i. Assume that:

(i) *the multifunction F is $\mathscr{T}_\mu\otimes\mathscr{B}(W)$-weakly measurable;*

(ii) *for a.e. $t\in T$, one has*

$$\left\{x=(x_1,\ldots,x_k)\in W : F(t,\cdot)\text{ is not lower semicontinuous at } x\right\}\subseteq E.$$

Then, there exist sets $Q_1,\ldots,Q_k$, with $Q_i\in\mathscr{B}(X_i)$ and $\psi_i(Q_i)=0$ for all $i=1,\ldots,k$, and a function $\phi: T\times W\to S$ such that:

(a) *$\phi(t,x)\in F(t,x)$ for all $(t,x)\in T\times W$;*

(b) *for all $x:=(x_1,x_2,\ldots,x_k)\in W\setminus\left[\left(\bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i)\right)\cup E\right]$, the function $\phi(\cdot,x)$ is $\mathscr{T}_\mu$-measurable over T;*

(c) *for a.e. $t \in T$, one has*

$$\Big\{x = (x_1, x_2, \dots, x_k) \in W : \phi(t, \cdot) \text{ is discontinuous at } x\Big\} \subseteq$$

$$\subseteq E \cup \Big[W \cap \Big(\bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i)\Big)\Big].$$

We also need the following proposition, which can be obtained by exactly the same proof of Proposition 2.6 of Cubiotti and Yao (2015).

Proposition 2.4. *Let $\Omega' \subseteq \mathbf{R}^n$ be a measurable set, $\varphi : \Omega' \times \mathbf{R}^h \to \mathbf{R}^k$ be a given function, $H^* \subseteq \mathbf{R}^h$ a Lebesgue measurable set, with $m_h(H^*) = 0$, and let D^* be a countable dense subset of $\mathbf{R}^h$, with $D^* \cap H^* = \emptyset$. Assume that:*

(i) *for all $x \in \Omega'$, the function $\varphi(x, \cdot)$ is bounded;*
(ii) *for all $z \in D^*$, the function $\varphi(\cdot, z)$ is measurable.*

Let $G : \Omega' \times \mathbf{R}^h \to 2^{\mathbf{R}^k}$ be the multifunction defined by setting, for each $(x,z) \in \Omega' \times \mathbf{R}^h$,

$$G(x,z) := \bigcap_{m \in \mathbf{N}} \overline{\text{conv}}\,\overline{\Big(\bigcup_{\substack{v \in D^* \\ \|v-z\|_h \le \frac{1}{m}}} \{\varphi(x,y)\}\Big)}.$$

Then, one has:

(a) *G has nonempty closed convex values;*
(b) *for all $z \in \mathbf{R}^h$, the multifunction $G(\cdot, z)$ is $\mathscr{L}(\Omega')$-measurable;*
(c) *for all $x \in \Omega'$, the multifunction $G(x, \cdot)$ has closed graph;*
(d) *if $x \in \Omega'$, and $\varphi(x, \cdot)|_{\mathbf{R}^h \setminus H^*}$ is continuous at $z \in \mathbf{R}^h \setminus H^*$, then one has*

$$G(x,z) = \{\varphi(x,z)\}.$$

3. Existence results

The following is our main result.

Theorem 3.1. *Let $Y \in \mathscr{G}_h$ be a closed, connected and locally connected subset of $\mathbf{R}^h$, and let $\psi : Y \to \mathbf{R}$ and $\varphi : \Omega \times \mathbf{R}^h \to \mathbf{R}$ be two given functions. Moreover, let $\xi \in L^p(\Omega)$ and let $E \in \mathscr{F}_h$. Assume that:*

(i) *the function ψ is continuous in Y, and $\text{int}_Y(\psi^{-1}(r)) = \emptyset$ for every $r \in \text{int}(\psi(Y))$.*
(ii) *for a.e. $x \in \Omega$, the function $\varphi(x, \cdot)|_{\mathbf{R}^h \setminus E}$ is continuous;*
(iii) *for all $z \in \mathbf{R}^h \setminus E$, the function $\varphi(\cdot, z)$ is measurable;*
(iv) *for a.e. $x \in \Omega$, one has $\varphi(x, \mathbf{R}^h \setminus E) \subseteq \psi(Y)$;*
(v) *for a.e. $x \in \Omega$ and for all $z \in \mathbf{R}^h \setminus E$, one has*

$$\sup\{\|y\|_h^* : y \in Y \text{ and } \psi(y) = \varphi(x,z)\} \le \xi(x).$$

Then, there exists $u \in W^{2,p}(\Omega, \mathbf{R}^h) \cap W_0^{1,p}(\Omega, \mathbf{R}^h)$ such that:

(a) *$-\Delta u(x) \in Y$ and $\psi(-\Delta u(x)) = \varphi(x, u(x))$ for a.e. $x \in \Omega$;*
(b) *$u(x) \in \mathbf{R}^h \setminus E$ for a.e. $x \in \Omega$;*
(c) *$\|\Delta u(x)\|_h^* \le \xi(x)$ for a.e. $x \in \Omega$.*

Proof. Without loss of generality we can assume that assumptions (ii), (iv) and (v) are satisfied for all $x \in \Omega$. Since $E \in \mathscr{F}_h$, there exist sets $E_1, E_2, \ldots, E_h \subseteq \mathbf{R}^h$ such that $m_1(P_{h,i}(E_i)) = 0$ for all $i = 1, \ldots, h$, and $E = \bigcup_{i=1}^{h} E_i$. For each $i = 1, \ldots, h$, there exists $B_i \in \mathscr{B}(\mathbf{R})$ such that $P_{h,i}(E_i) \subseteq B_i$ and $m_1(B_i) = 0$. Put

$$H := \bigcup_{i=1}^{h} P_{h,i}^{-1}(B_i),$$

and let

$$W := \mathbf{R}^h \setminus H = \prod_{i=1}^{h} (\mathbf{R} \setminus B_i).$$

We have that $W \in \mathscr{B}(\mathbf{R}^h)$, $E \subseteq H$ and $W \subseteq \mathbf{R}^h \setminus E$. By assumption (i) and Theorem 2.4 of Ricceri (1982), there exists a set $X \subseteq Y$ such that $\psi(X) = \psi(Y)$ and the function $\psi|_X : X \to \psi(Y)$ is open (it maps open subsets of X onto open subsets of $\psi(Y) = \psi(X)$). Hence, it follows easily that the multifunction $\Phi : \psi(Y) \to 2^X$ defined by setting, for each $t \in \psi(Y)$,

$$\Phi(t) := \psi^{-1}(t) \cap X,$$

is lower semicontinuous in $\psi(Y)$ with nonempty values.

Let $R : \Omega \times W \to 2^X$ be the multifunction defined by putting, for each $(x,z) \in \Omega \times W$,

$$R(x,z) := \Phi(\varphi(x,z)).$$

The multifunction R is well-defined by assumption (iv), with nonempty values. Moreover, by Theorem 7.3.11 of Klein and Thompson (1984) (taking into account assumption (ii) and the lower semicontinuity of Φ), we have that for each $x \in \Omega$ the multifunction $R(x,\cdot)$ is lower semicontinuous in W. Let us denote by $Q : \Omega \times W \to 2^Y$ (more precisely, $Q : \Omega \times W \to 2^{\overline{X}}$) the pointwise closure of Q. That is, for each $(x,z) \in \Omega \times W$ we put

$$Q(x,z) := \overline{R(x,z)} = \overline{(\Phi(\varphi(x,z)))} = \overline{(\psi^{-1}(\varphi(x,z)) \cap X)}.$$

By Proposition 7.3.3 of Klein and Thompson (1984), for each $x \in \Omega$ the multifunction $Q(x,\cdot)$ is lower semicontinuous in W, with nonempty closed (in $\mathbf{R}^h$) values.

By assumptions (ii) and (iii) and by the Lemma of Kucia (1991, p. 198), the function $\varphi|_{\Omega \times W}$ is $\mathscr{L}(\Omega) \otimes \mathscr{B}(W)$-measurable. Therefore, by the lower semicontinuity of Φ, the multifunction R is $\mathscr{L}(\Omega) \otimes \mathscr{B}(W)$-weakly measurable. By Proposition 2.6 of Himmelberg (1975), the multifunction Q is also $\mathscr{L}(\Omega) \otimes \mathscr{B}(W)$-weakly measurable (hence, $\mathscr{L}(\Omega) \otimes \mathscr{B}(W)$-measurable).

By assumption (v), we have that

$$Q(x,z) \subseteq \overline{B}^*(0_{\mathbf{R}^h}, \xi(x)) \cap Y \quad \text{for all} \quad (x,z) \in \Omega \times W. \tag{5}$$

Now, let $Q^* : \overline{\Omega} \times W \to 2^Y$ be defined by setting, for each $(x,z) \in \overline{\Omega} \times W$,

$$Q^*(x,z) = \begin{cases} Q(x,z) & \text{if } x \in \Omega \text{ and } z \in W, \\ Y & \text{if } x \in \partial\Omega \text{ and } z \in W. \end{cases}$$

Of course, Q^* is $\mathscr{L}(\overline{\Omega}) \otimes \mathscr{B}(W)$-weakly measurable. Taking into account that the set W is a Souslin set (Bogachev 2007, Corollary 6.6.7), we now apply Theorem **2.3**. Hence, there exist a set $\Omega_0^* \in \mathscr{L}(\overline{\Omega})$ and h sets $Z_1, \ldots, Z_h \in \mathscr{B}(\mathbf{R})$, with $m_n(\Omega_0^*) = 0$ and $m_1(Z_i) = 0$ for all $i = 1, \ldots, h$, and a function $\lambda^* : \overline{\Omega} \times W \to Y$ such that:

(i)$'$ $\lambda^*(x,z) \in Q^*(x,z)$ for all $(x,z) \in \overline{\Omega} \times W$;

(ii)$'$ for each $z \in W \setminus \big[\bigcup_{i=1}^h P_{h,i}^{-1}(Z_i)\big]$, the function $\lambda^*(\,\cdot\,,z)$ is measurable on $\overline{\Omega}$;

(iii)$'$ for each $x \in \overline{\Omega} \setminus \Omega_0^*$, one has

$$\{z \in W : \lambda^*(x,\cdot) \text{ is discontinuous at } z\} \subseteq W \cap \big[\bigcup_{i=1}^h P_{h,i}^{-1}(Z_i)\big].$$

Let $\Omega_0 := \Omega_0^* \cap \Omega$, and $\lambda := \lambda^*|_{\Omega \times W}$. By the above construction, we immediately get that the function $\lambda : \Omega \times W \to Y$ has the following properties:

(i)$''$ $\lambda(x,z) \in Q(x,z)$ for all $(x,z) \in \Omega \times W$;

(ii)$''$ for each $z \in W \setminus [\bigcup_{i=1}^h P_{h,i}^{-1}(Z_i)]$, the function $\lambda(\,\cdot\,,z)$ is measurable in Ω;

(iii)$''$ for each $x \in \Omega \setminus \Omega_0$, one has

$$\{z \in W : \lambda(x,\cdot) \text{ is discontinuous at } z\} \subseteq W \cap \big[\bigcup_{i=1}^h P_{h,i}^{-1}(Z_i)\big].$$

In particular, by (5), we have that

$$\|\lambda(x,z)\|_h^* \le \xi(x) \quad \text{for all} \quad (x,z) \in \Omega \times W. \tag{6}$$

We now extend λ to $\Omega \times \mathbf{R}^h$ by defining a function $\widetilde{\lambda} : \Omega \times \mathbf{R}^h \to \mathbf{R}^h$ by putting

$$\widetilde{\lambda}(x,z) = \begin{cases} \lambda(x,z) & \text{if } x \in \Omega \text{ and } z \in W \\ 0_{\mathbf{R}^h} & \text{if } x \in \Omega \text{ and } z \in \mathbf{R}^h \setminus W. \end{cases}$$

Of course, (6) implies that

$$\|\widetilde{\lambda}(x,z)\|_h^* \le \xi(x) \quad \text{for all} \quad (x,z) \in \Omega \times \mathbf{R}^h. \tag{7}$$

Since we have

$$W \setminus [\bigcup_{i=1}^h P_{h,i}^{-1}(Z_i)] = \mathbf{R}^h \setminus [\bigcup_{i=1}^h P_{h,i}^{-1}(B_i \cup Z_i)] = \prod_{i=1}^h [\mathbf{R} \setminus (B_i \cup Z_i)]$$

and

$$m_h(\bigcup_{i=1}^h P_{h,i}^{-1}(B_i \cup Z_i)) = 0,$$

there exists a countable set

$$D^* \subseteq W \setminus [\bigcup_{i=1}^h P_{h,i}^{-1}(Z_i)],$$

such that D^* is dense in $\mathbf{R}^h$. Let us define a multifunction $G : \Omega \times \mathbf{R}^h \to 2^{\mathbf{R}^h}$ by putting, for each $(x,z) \in \Omega \times \mathbf{R}^h$,

$$G(x,z) := \bigcap_{m \in \mathbf{N}} \overline{\text{conv}}\,\overline{\Big(\bigcup_{\substack{v \in D^* \\ \|v-z\|_h \le \frac{1}{m}}} \{\widetilde{\lambda}(x,v)\} \Big)} = \bigcap_{m \in \mathbf{N}} \overline{\text{conv}}\,\overline{\Big(\bigcup_{\substack{v \in D^* \\ \|v-z\|_h \le \frac{1}{m}}} \{\lambda(x,v)\} \Big)}.$$

Since the function λ takes its values in Y, by (6) we get

$$G(x,z) \subseteq \overline{\text{conv}}(Y) \cap \overline{B}^*(0_{\mathbf{R}^h}, \xi(x)) \quad \text{for every} \quad (x,z) \in \Omega \times \mathbf{R}^h. \tag{8}$$

Now, observe that, by (7), the function $\widetilde{\lambda}(x,\cdot)$ is bounded for each fixed $x \in \Omega$. Moreover, by (ii)$''$, for every $z \in D^*$ the function $\widetilde{\lambda}(\cdot,z)$ is measurable in Ω. By Proposition **2.4**, applied with $H^* = \bigcup_{i=1}^h P_{h,i}^{-1}(B_i \cup Z_i)$, we get:

(i)$'''$ G has nonempty closed convex values;
(ii)$'''$ for all $z \in \mathbf{R}^h$, the multifunction $G(\cdot,z)$ is $\mathscr{L}(\Omega)$-measurable;
(iii)$'''$ for all $x \in \Omega$, the multifunction $G(x,\cdot)$ has closed graph;
(iv)$'''$ if $x \in \Omega$, and the function $\widetilde{\lambda}(x,\cdot)|_{\mathbf{R}^h \setminus H^*} = \lambda(x,\cdot)|_{\mathbf{R}^h \setminus H^*}$ is continuous at $z \in \mathbf{R}^h \setminus H^*$, then one has $G(x,z) = \{\widetilde{\lambda}(x,z)\} = \{\lambda(x,z)\}$.

In particular, by (iv)$'''$ and (iii)$''$, taking into account that

$$\mathbf{R}^h \setminus H^* = W \setminus [\bigcup_{i=1}^h P_{h,i}^{-1}(Z_i)],$$

we get

$$G(x,z) = \{\widetilde{\lambda}(x,z)\} = \{\lambda(x,z)\} \quad \text{for all} \quad (x,z) \in \big(\Omega \setminus \Omega_0\big) \times \big[W \setminus (\bigcup_{i=1}^h P_{h,i}^{-1}(Z_i))\big]. \tag{9}$$

Now, fix any $r > \|\xi\|_{L^p(\Omega)}$, and let $\gamma_r : \Omega \to [0,+\infty]$ be defined by

$$\gamma_r(x) = \sup_{\|z\|_h^* \le Br} d^*(0_{\mathbf{R}^h}, G(x,z)) = \sup_{\|z\|_h^* \le Br} \inf_{v \in G(x,z)} \|v\|_h^*,$$

where B is defined according to Section 2. By (8) we get

$$\gamma_r(x) \le \xi(x) \quad \text{for all} \quad x \in \Omega,$$

hence $\gamma_r \in L^p(\Omega)$ and $\|\gamma_r\|_{L^p(\Omega)} \le \|\xi\|_{L^p(\Omega)} < r$ (as regards the measurability of the function γ_r, we refer to Naselli Ricceri and Ricceri (1990, p. 262). Therefore, all the assumptions of Theorem 2.1 of Marano (1996) are satisfied. Consequently, there exists $u \in W^{2,p}(\Omega,\mathbf{R}^h) \cap W_0^{1,p}(\Omega,\mathbf{R}^h)$ such that $-\Delta u(x) \in G(x,u(x))$ for a.e. $x \in \Omega$. Let $\Omega_1 \subseteq \Omega$, with $m_n(\Omega_1) = 0$, be such that

$$-\Delta u(x) \in G(x,u(x)) \quad \text{for all} \quad x \in \Omega \setminus \Omega_1. \tag{10}$$

In particular, by (8) we have

$$-\Delta u(x) \in \overline{\text{conv}}(Y) \cap \overline{B}^*(0,\xi(x)) \quad \text{for all} \quad x \in \Omega \setminus \Omega_1. \tag{11}$$

Fix $i \in \{1,\dots,h\}$. Since $Y \in \mathscr{G}_h$, by (11) the function $-\Delta u_i$ has constant sign in $\Omega \setminus \Omega_1$. Let us suppose that

$$-\Delta u_i(x) > 0 \quad \text{for all} \quad x \in \Omega \setminus \Omega_1 \tag{12}$$

(if, conversely, $-\Delta u_i(x) < 0$ for all $x \in \Omega \setminus \Omega_1$, then the argument is analogous). By Proposition 2.1. of Marano (1996), we have

$$\Delta u_i(x) = 0 \quad \text{for a.e.} \quad x \in u_i^{-1}(Z_i \cup B_i). \tag{13}$$

By (12) and (13) we easily get that

$$m_n(u_i^{-1}(Q_i \cup Z_i)) = 0.$$

Now, put

$$\Omega_2 := \Omega_1 \cup \Omega_0 \cup \Big[\bigcup_{i=1}^{h} u_i^{-1}(B_i \cup Z_i) \Big].$$

By the above construction we have that $m_n(\Omega_2) = 0$. Choose any $x \in \Omega \setminus \Omega_2$. By (10), we have that $-\Delta u(x) \in G(x, u(x))$. Moreover, we have $u_i(x) \notin B_i \cup Z_i$ for all $i = 1, \dots, h$, thus we get

$$u(x) \in \prod_{i=1}^{n} \big[\mathbf{R} \setminus (B_i \cup Z_i)\big] = W \setminus \Big[\bigcup_{i=1}^{h} P_{h,i}^{-1}(Z_i) \Big]. \tag{14}$$

In particular, we get that $u(x) \in \mathbf{R}^h \setminus E$. Since $x \notin \Omega_0$, by (9) and (14) we have that

$$G(x, u(x)) = \{\lambda(x, u(x))\},$$

hence

$$-\Delta u(x) = \lambda(x, u(x)) \in Q(x, u(x)) \subseteq Y. \tag{15}$$

By (15), and by the continuity of ψ and the closedness of Y, we get

$$-\Delta u(x) \in \overline{(\psi^{-1}(\varphi(x, u(x))) \cap X)} \subseteq \overline{\psi^{-1}(\varphi(x, u(x)))} = \psi^{-1}(\varphi(x, u(x))),$$

and thus

$$\psi(-\Delta u(x)) = \varphi(x, u(x)).$$

Finally, we observe that by (11) we have $\|\Delta u(x)\|_h^* \leq \xi(x)$. This completes the proof. □

Now we state explicitly some special cases and some corollaries of Theorem **3.1**. Firstly, we consider the case where $h = 1$ and φ does not depend on $x \in \Omega$.

Corollary 3.2. *Let $Y \subseteq \mathbf{R}$ be a closed interval, with* $\inf Y > 0$*, $\psi : Y \to \mathbf{R}$ and $\varphi : \mathbf{R} \to \mathbf{R}$ two given functions, and let $E \subseteq \mathbf{R}$, with $m_1(E) = 0$. Assume that:*

(i) *ψ is continuous and* $\operatorname{int}(\psi^{-1}(t)) = \emptyset$ *for all $t \in$* $\operatorname{int}(\psi(Y))$*;*
(ii) *$\varphi|_{\mathbf{R}\setminus E}$ is continuous;*
(iii) *one has $\varphi(\mathbf{R} \setminus E) \subseteq \psi(Y)$, and* $\sup \psi^{-1}(\varphi(\mathbf{R} \setminus E)) < +\infty$.

Then, there exists a positive $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

(a) *$-\Delta u(x) \in Y$ and $\psi(-\Delta u(x)) = \varphi(u(x))$ for a.e. $x \in \Omega$;*
(b) *$u(x) \in \mathbf{R} \setminus E$ for a.e. $x \in \Omega$;*
(c) *$|\Delta u(x)| \leq \sup \psi^{-1}(\varphi(\mathbf{R} \setminus E))$ for a.e. $x \in \Omega$.*

Proof. It follows at once from Theorem **3.1**. In particular, the fact that u is positive in Ω follows immediately from Proposition 2.2 of Marano (1996), since $-\Delta u(x) \geq \inf Y > 0$ for almost every $x \in \Omega$. □

Corollary **3.2** can be usefully compared with Theorem 3.2 of Marano (1996), where the same problem is considered under the stronger condition that the set $D_\varphi = \{z \in \mathbf{R} :$ *φ is discontinuous at z*$\}$ has measure zero. Anyway, these two results are formally independent, and also their proofs are very different. We also remark that a result of the same kind has been also established in Theorem 3.4 of Marano (1994), assuming the continuity of φ. We can now give the proof of Theorem **1.1**.

Proof of Theorem 1.1. Choose any $z^* \in]-B\rho\, m_n(\Omega)^{1/p}, B\rho\, m_n(\Omega)^{1/p}[\setminus E$. Let $Y := [a,\rho]$, $\hat{\psi} := \psi|_Y$, and let $\hat{\varphi} : \mathbf{R} \to \mathbf{R}$ be defined by

$$\hat{\varphi}(z) = \begin{cases} \varphi(z) & \text{if } z \in [-B\rho\, m_n(\Omega)^{1/p}, B\rho\, m_n(\Omega)^{1/p}], \\ \varphi(z^*) & \text{otherwise.} \end{cases}$$

If we put

$$E^* := E \cup \{ -B\rho\, m_n(\Omega)^{1/p}, B\rho\, m_n(\Omega)^{1/p} \},$$

we have that $\hat{\varphi}|_{\mathbf{R}\setminus E^*}$ is continuous and $\hat{\varphi}(\mathbf{R}\setminus E^*) \subseteq \hat{\psi}([a,\rho])$. Moreover, $\operatorname{int}(\hat{\psi}^{-1}(t)) = \emptyset$ for all $t \in \operatorname{int}(\hat{\psi}(Y))$. Finally, we have $\sup \hat{\psi}^{-1}(\hat{\varphi}(\mathbf{R}\setminus E^*)) \le \rho$. By Corollary **3.2**, there exists a positive $u \in W^{2,p}(\Omega) \cap W^{1,p}(\Omega)$ such that for a.e. $x \in \Omega$ one has $-\Delta u(x) \in [a,\rho]$, $\hat{\psi}(-\Delta u(x)) = \hat{\varphi}(u(x))$ and $u(x) \in \mathbf{R}\setminus E^*$. By Proposition **2.2**, we have that

$$\operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \le B\|\Delta u\|_{L^p(\Omega)} \le B\rho\, m_n(\Omega)^{1/p}.$$

Hence, in particular, we get

$$u(x) \in [-B\rho\, m_n(\Omega)^{1/p}, B\rho\, m_n(\Omega)^{1/p}] \setminus E^* =]-B\rho\, m_n(\Omega)^{1/p}, B\rho\, m_n(\Omega)^{1/p}[\setminus E$$

for almost every $x \in \Omega$. By the definition of $\hat{\varphi}$ we get

$$\psi(-\Delta u(x)) = \hat{\psi}(-\Delta u(x)) = \hat{\varphi}(u(x)) = \varphi(u(x)) \quad \text{for a.e.} \quad x \in \Omega,$$

and this ends the proof. □

We now consider the explicit problem (4). In this case, Theorem **3.1** immediately gives the following result.

Theorem 3.3. *Let $\varphi : \Omega \times \mathbf{R} \to \mathbf{R}$ be a given function. Let $\alpha > 0$, $E \subseteq \mathbf{R}$, with $m_1(E) = 0$, and $\xi \in L^p(\Omega)$ be such that:*

(i) *for a.e. $x \in \Omega$, the function $\varphi(x,\cdot)|_{\mathbf{R}\setminus E}$ is continuous;*
(ii) *for all $z \in \mathbf{R}\setminus E$, the function $\varphi(\cdot, z)$ is measurable;*
(iii) *for a.e. $x \in \Omega$, one has* $\inf \varphi(x, \mathbf{R}\setminus E) \ge \alpha$ *and* $\sup \varphi(x, \mathbf{R}\setminus E) \le \xi(x)$.

Then, there exists a positive $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

(a) $-\Delta u(x) \in [\alpha, \xi(x)]$ *and* $u(x) \in \mathbf{R}\setminus E$ *for a.e. $x \in \Omega$;*
(b) $-\Delta u(x) = \varphi(x, u(x))$ *for a.e. $x \in \Omega$.*

Proof. Choose $Y := [\alpha, +\infty[$ and $\psi(y) = y$. Then, the conclusion follows by at once by Theorem **3.1**. As regards the positivity of u, it follows by conclusion (a) and Proposition 2.2 of Marano (1996). □

A result of the same kind of Theorem **3.3** has been established in Theorem 3.3 of Marano (1996), under the stronger regularity assumption (a_1) on φ. However, the two results are independent. By Theorem **3.3** we immediately get the following corollary, which concerns the problem $-\Delta u = g(x,u) + \lambda(x)$.

Corollary 3.4. *Let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be a given function. Let $\lambda, \eta \in L^p(\Omega)$, $\alpha > 0$ and $E \subseteq \mathbf{R}$, with $m_1(E) = 0$, be such that:*

(i) *for a.e.* $x \in \Omega$, *the function* $g(x,\cdot)|_{\mathbf{R}\setminus E}$ *is continuous;*
(ii) *for all* $z \in \mathbf{R}\setminus E$, *the function* $g(\cdot,z)$ *is measurable;*
(iii) *for a.e.* $x \in \Omega$, *one has*

$$\alpha - \lambda(x) \leq \inf g(x, \mathbf{R}\setminus E), \qquad \sup g(x,\mathbf{R}\setminus E) \leq \eta(x).$$

Then, there exists a positive $u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ *such that* $u(x) \in \mathbf{R}\setminus E$, $-\Delta u(x) \in [\alpha, \eta(x)+\lambda(x)]$, *and* $-\Delta u(x) = g(x,u(x)) + \lambda(x)$ *for almost every* $x \in \Omega$.

Proof. It follows at once by Theorem **3.3**, by choosing $\varphi(x,z) = g(x,z)+\lambda(x)$ and $\xi = \eta+\lambda$. □

When the function g does not depend on $x \in \Omega$, Corollary **3.4** immediately gives the following existence result for the problem $-\Delta u = f(u) + \lambda(x)$, where $f : \mathbf{R} \to \mathbf{R}$ can be discontinuous even at all points $z \in \mathbf{R}$.

Corollary 3.5. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a given function. Let* $\lambda \in L^p(\Omega)$, $\alpha > 0$ *and* $E \subseteq \mathbf{R}$, *with* $m_1(E) = 0$, *be such that:*

(i) *the function* $f|_{\mathbf{R}\setminus E}$ *is continuous and bounded;*
(ii) *for a.e.* $x \in \Omega$, *one has*

$$\lambda(x) \geq \alpha - \inf f(\mathbf{R}\setminus E).$$

Then, there exists a positive $u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ *such that* $u(x) \in \mathbf{R}\setminus E$ *and* $-\Delta u(x) = f(u(x)) + \lambda(x)$ *for a.e.* $x \in \Omega$.

Remark 3.6. In order to give counterexamples to some possible improvements of our results, let us consider the function $\varphi : \mathbf{R} \to \mathbf{R}$ be defined by

$$\varphi(z) = \begin{cases} 1 & \text{if } z \in \mathbf{Q}, \\ 0 & \text{if } z \in \mathbf{R}\setminus\mathbf{Q}. \end{cases}$$

In Remark 3.2 of Marano (1995) it is showed that, for such a function φ, the problem $-\Delta u = \varphi(u)$ has no solutions $u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. It is self-evident that, in this case, the restriction $\varphi|_{\mathbf{R}\setminus\mathbf{Q}}$ is continuous, since it is constant. Hence, if we choose $h = 1$, $Y := [0,1]$, $E = \mathbf{Q}$, $\psi(y) = y$, $\xi(x) \equiv 0$, all the assumptions of our Theorem **3.1** (except for $Y \in \mathscr{G}_1$) are satisfied. Therefore, Remark 3.2 of Marano (1995) shows that the assumption $Y \in \mathscr{G}_h$ cannot be dropped from the statement of Theorem **3.1**. At the same time, it shows that the assumptions $\inf Y > 0$ and $\inf \varphi(x,\mathbf{R}\setminus E) \geq \alpha > 0$ cannot be dropped from Corollary **3.2** and Theorem **3.3**, respectively. Moreover, it shows that neither the assumption $\alpha - \lambda(x) \leq \inf g(x,\mathbf{R}\setminus E)$ in Corollary **3.4**, nor the assumption $\lambda(x) \geq \alpha - \inf f(\mathbf{R}\setminus E)$ in Corollary **3.5**, can be dropped from the statements.

However, Corollary **3.5** shows that for such a function φ, and for every $\lambda \in L^p(\Omega)$, with $\operatorname{essinf}_{x\in\Omega}\lambda(x) > 0$, the problem $-\Delta u = \varphi(u) + \lambda(x)$ has a positive solution $u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. We point out that, even for such a simple case, the results of Marano (1995), as well as those of Marano (1996), cannot be applied, since in this case the set $D_\varphi = \{z \in \mathbf{R} : \varphi \text{ is discontinuous at } z\} = \mathbf{R}$ has not null Lebesgue measure.

Remark 3.7. It is natural to ask if the set $E \in \mathscr{F}_h$ in the statement of Theorem **3.1** can depend on $x \in \Omega$. That is, one can ask if the following assertion holds.

Assertion A. *Let $Y \in \mathscr{G}_h$ be a closed, connected and locally connected subset of $\mathbf{R}^h$, $\psi : Y \to \mathbf{R}$ and $\varphi : \Omega \times \mathbf{R}^h \to \mathbf{R}$ two given functions, and $\xi \in L^p(\Omega)$.*
Assume that:

(i) *the function ψ is continuous in Y, and $\operatorname{int}_Y(\psi^{-1}(r)) = \emptyset$ for every $r \in \operatorname{int}(\psi(Y))$;*
(ii) *for each $x \in \Omega$, there exists a set $E_x \in \mathscr{F}_h$ such that the function $\varphi(x,\cdot)|_{\mathbf{R}^h \setminus E_x}$ is continuous;*
(iii) *for all $z \in \mathbf{R}^h$, the function $\varphi(\cdot, z)$ is measurable;*
(iv) *for every $x \in \Omega$, one has $\varphi(x, \mathbf{R}^h) \subseteq \psi(Y)$;*
(v) *for every $x \in \Omega$ and for every $z \in \mathbf{R}^h$, one has*

$$\sup \{ \|y\|_h^* : \ y \in Y \text{ and } \ \psi(y) = \varphi(x,z) \} \ \le \ \xi(x).$$

Then, there exists $u \in W^{2,p}(\Omega, \mathbf{R}^h) \cap W_0^{1,p}(\Omega, \mathbf{R}^h)$ such that $-\Delta u(x) \in Y$ and $\psi(-\Delta u(x)) = \varphi(x, u(x))$ for almost every $x \in \Omega$.

The Example 4.3 of Marano (1996) shows that for $p > n$ Assertion A is false, even if $h = 1$ and $\psi(y) = y$. Therefore, the set $E \in \mathscr{F}_h$ in the statement of Theorem **3.1** cannot depend on $x \in \Omega$.

Remark 3.8. Another peculiarity of Theorem **3.1**, as well as of its corollaries, is represented by the following simple observation. Assume that all the assumptions of Theorem **3.1** are satisfied by suitable Ω, Y, ψ, φ, ξ and E. If we choose any further set $K \in \mathscr{F}_h$, the assumptions of Theorem **3.1** are still satisfied by replacing the set E with the set $E \cup K$ (since we still have $E \cup K \in \mathscr{F}_h$). Consequently, by conclusions (a) and (b), we get the existence of a solution $u \in W^{2,p}(\Omega, \mathbf{R}^h) \cap W_0^{1,p}(\Omega, \mathbf{R}^h)$ of the problem $-\Delta u \in Y$ and $\psi(-\Delta u) = \varphi(x, u)$ which, in addition, satisfies $u(x) \notin E \cup K$ for almost every $x \in \Omega$. In particular, the vector $u(x)$ avoids the set K for almost every $x \in \Omega$.

Taking this in mind, we now give two simple concrete examples of applications of Theorem **3.1**.

Corollary 3.9. *Let $g \in L^\infty(\Omega)$, with $\alpha := \operatorname{essinf}_{x \in \Omega} g(x) > 0$, and let $\gamma > 0$, $\mu \ge 0$, $b \ge 1$, $\lambda \in [0,1]$. Let $\varphi : \Omega \times \mathbf{R} \to \mathbf{R}$ be defined by*

$$\varphi(x,z) = \begin{cases} g(x) + \mu(b + \cos z)^\gamma & \text{if } x \in \Omega \text{ and } z \le 0 \\ g(x) + \mu(b + \cos z)^\gamma + 1 & \text{if } x \in \Omega \text{ and } z > 0. \end{cases}$$

Then, there exists a function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that:

(a) *$-\Delta u(x) = \varphi(x, u(x)) + \lambda \sin(-\Delta u(x))$ for almost every $x \in \Omega$;*
(b) *$u(x) \in \mathbf{R} \setminus \mathbf{Q}$ for almost every $x \in \Omega$.*

Proof. Of course, there exists a set $\Omega_0 \subseteq \Omega$, with $m_n(\Omega_0) = 0$, such that

$$\alpha \le \varphi(x,z) \le \|g\|_{L^\infty(\Omega)} + \mu(b+1)^\gamma + 1 \quad \text{for all} \quad x \in \Omega \setminus \Omega_0, \ z \in \mathbf{R}.$$

Moreover, since $\lim_{y\to 0^+}(y-\lambda\sin y)=0$, there exists $y_0>0$ such that $y_0-\lambda\sin y_0<\alpha$. On the other hand, since $\lim_{y\to+\infty}(y-\lambda\sin y)=+\infty$, there exists $y_1>y_0$ such that

$$y_1-\lambda\sin y_1>\|g\|_{L^\infty(\Omega)}+\mu(b+1)^\gamma+1.$$

Now, we want to apply Theorem **3.1** with $h=1$, $Y=[y_0,y_1]$, $\xi(x)\equiv y_1$, $E=\mathbf{Q}$, $\psi(y)=y-\lambda\sin y$, and φ defined as above. Of course, $Y\in\mathscr{G}_1$, and assumption (i) is satisfied since ψ' never vanishes identically on a interval. Moreover, by the above construction we have that

$$\varphi(x,\mathbf{R}\setminus\mathbf{Q})\subseteq[\alpha,\|g\|_{L^\infty(\Omega)}+\mu(b+1)^\gamma+1]\subseteq[\psi(y_0),\psi(y_1)]\subseteq\psi(Y)\quad\text{for all}\quad x\in\Omega\setminus\Omega_0,$$

hence assumption (iv) is satisfied. Finally, assumptions (ii), (iii) and (v) are obviously satisfied. Consequently, by Theorem **3.1** there exists a function $u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ such that $-\Delta u(x)-\lambda\sin(-\Delta u(x))=\varphi(x,u(x))$ and $u(x)\in\mathbf{R}\setminus\mathbf{Q}$ for almost every $x\in\Omega$, that is our conclusion. □

Corollary 3.10. *Let $g\in L^\infty(\Omega)$, with $\alpha:=\operatorname{essinf}_{x\in\Omega}g(x)>0$, and let $\gamma>0$, $\mu\geq 0$, $b\geq 1$, $\lambda\in[0,\alpha[$. Let $c\in\mathbf{R}\setminus\mathbf{Q}$, and let $\varphi:\Omega\times\mathbf{R}\to\mathbf{R}$ be defined by*

$$\varphi(x,z)=\begin{cases} g(x)+\mu(b+\cos z)^\gamma & \text{if } x\in\Omega,\ z<0 \text{ and } z+c\in\mathbf{R}\setminus\mathbf{Q}\\ g(x)+\mu(b+\cos z)^\gamma+1 & \text{if } x\in\Omega \text{ and } z+c\in\mathbf{Q}\\ g(x)+\mu(b+\cos z)^\gamma+2 & \text{if } x\in\Omega,\ z\geq 0 \text{ and } z+c\in\mathbf{R}\setminus\mathbf{Q}.\end{cases}$$

Then, there exists a function $u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ such that:

(a) $-\Delta u(x)=\varphi(x,u(x))-\lambda e^{-\Delta u(x)}$ *and $u(x)\in\mathbf{R}\setminus(\mathbf{Q}-c)$for almost every $x\in\Omega$;*
(b) *$u(x)$ is irrational for almost every $x\in\Omega$.*

Proof. We argue as in Corollary **3.9**. Firstly, we observe that there exists a set $\Omega_0\subseteq\Omega$, with $m_n(\Omega_0)=0$, such that

$$\alpha\leq\varphi(x,z)\leq\|g\|_{L^\infty(\Omega)}+\mu(b+1)^\gamma+2\quad\text{for all}\quad x\in\Omega\setminus\Omega_0,\ z\in\mathbf{R}.$$

Since $\lim_{y\to 0^+}(y+\lambda e^y)=\lambda<\alpha$, there exists $y_0>0$ such that $y_0+\lambda e^{y_0}<\alpha$. Moreover, since $\lim_{y\to+\infty}(y+\lambda e^y)=+\infty$, there exists $y_1>y_0$ such that

$$y_1+\lambda e^{y_1}>\|g\|_{L^\infty(\Omega)}+\mu(b+1)^\gamma+2.$$

Now, we want to apply Theorem **3.1** with $h=1$, $Y=[y_0,y_1]$, $\xi(x)\equiv y_1$, $E=(\mathbf{Q}-c)\cup\mathbf{Q}$, $\psi(y)=y+\lambda e^y$, and φ defined as above. Of course, $Y\in\mathscr{G}_1$, and assumption (i) is satisfied since $\psi'(y)\geq 1$ for all $y\in\mathbf{R}$. Moreover, by the above construction we have that

$$\varphi(x,\mathbf{R})\subseteq[\alpha,\|g\|_{L^\infty(\Omega)}+\mu(b+1)^\gamma+2]\subseteq[\psi(y_0),\psi(y_1)]=\psi(Y)\quad\text{for all}\quad x\in\Omega\setminus\Omega_0,$$

hence assumption (iv) is satisfied. Finally, assumptions (ii), (iii) and (v) are obviously satisfied. Hence, by Theorem **3.1** there exists a function $u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ such that $-\Delta u(x)+\lambda e^{-\Delta u(x)}=\varphi(x,u(x))$ and $u(x)\notin\mathbf{Q}\cup(\mathbf{Q}-c)$ for almost every $x\in\Omega$, that is our conclusion. □

References

Adams, R. A. and Fournier, J. J. F., eds. (2003). *Sobolev Spaces*. Vol. 140. Pure and Applied Mathematics. Amsterdam: Elsevier. URL: https://www.sciencedirect.com/bookseries/pure-and-applied-mathematics/vol/140/suppl/C.

Ambrosetti, A. and Badiale, M. (1989). "The dual variational principle and elliptic problems with discontinuous nonlinearities". *Journal of Mathematical Analysis and Applications* **140**, 363–373. DOI: 10.1016/0022-247X(89)90070-X.

Ambrosetti, A. and Turner, R. E. L. (1988). "Some discontinuous variational problems". *Differential Integral Equations* **1**(3), 341–349. URL: https://projecteuclid.org/journals/differential-and-integral-equations/volume-1/issue-3/Some-discontinuous-variational-problems/die/1371669562.full?tab=ArticleLink.

Bielawski, R. and Górniewicz, L. (1986). "Some applications of the Leray-Schauder alternative to differential equations". In: *Nonlinear Functional Analysis and Its Applications*. Ed. by S. P. Singh. Vol. 173. NATO ASI Series (C: Mathematical and Physical Sciences). Dordrecht: Springer, pp. 187–194. DOI: 10.1007/978-94-009-4632-3_11.

Bielawski, R. and Górniewicz, L. (1989). "A fixed point index approach to some differential equations". In: *Topological Fixed Point Theory and Applications*. Ed. by B. Jiang. Vol. 1411. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, pp. 9–14. DOI: 10.1007/BFb0086436.

Bogachev, V. I. (2007). *Measure Theory*. Berlin, Heidelberg: Springer. DOI: 10.1007/978-3-540-34514-5.

Bonanno, G. and Marano, S. A. (1996). "Positive solutions of elliptic equations with discontinuous nonlinearities". *Topological Methods in Nonlinear Analysis* **8**(2), 263–273. URL: https://projecteuclid.org/journals/topological-methods-in-nonlinear-analysis/volume-8/issue-2/Positive-solutions-of-elliptic-equations-with-discontinuous-nonlinearities/tmna/1479265239.full.

Bonanno, G. and Marano, S. A. (2000). "Elliptic problems in $\mathbf{R}^N$ with discontinuous nonlinearities". *Proceedings of the Edinburgh Mathematical Society* **43**(3), 545–558. DOI: 10.1017/S0013091500021180.

Carl, S. and Heikkilä, S. (1992a). "An existence result for elliptic differential inclusions with discontinuous nonlinearity". *Nonlinear Analysis: Theory, Methods & Applications* **18**(5), 471–479. DOI: 10.1016/0362-546X(92)90014-6.

Carl, S. and Heikkilä, S. (1992b). "On extremal solutions of an elliptic boundary value problem involving discontinuous nonlinearities". *Differential and Integral Equations* **5**(3), 581–589. URL: https://projecteuclid.org/journals/differential-and-integral-equations/volume-5/issue-3/On-extremal-solutions-of-an-elliptic-boundary-value-problem-involving/die/1370979321.full?tab=ArticleLink.

Carl, S. and Heikkilä, S. (1998). "Discontinuous implicit elliptic boundary value problems". *Differential and Integral Equations* **11**(6), 823–834. URL: https://projecteuclid.org/journals/differential-and-integral-equations/volume-11/issue-6/Discontinuous-implicit-elliptic-boundary-value-problems/die/1367329478.full.

Cerami, G. (1983a). "Metodi variazionali nello studio di problemi al contorno con parte non lineare discontinua". *Rendiconti del Circolo Matematico di Palermo* **32**, 336–357. DOI: 10.1007/BF02848538.

Cerami, G. (1983b). "Soluzioni positive di problemi con parte non lineare discontinua e applicazione a un problema di frontiera libera". *Bollettino della Unione Matematica Italiana. Serie VI. B* **2**, 321–338.

Cubiotti, P. and Yao, J. C. (2015). "On the two-point problem for implicit second-order ordinary differential equations". *Boundary Value Problems* **211**, 1–25. DOI: 10.1186/s13661-015-0475-5.

Cubiotti, P. and Yao, J. C. (2016). “On the Cauchy problem for k-th order discontinuous ordinary differential equations”. *Journal of Nonlinear and Convex Analysis* **17**(5), 853–863. URL: http://www.yokohamapublishers.jp/online2/jncav17-5.html.

Denkowski, Z., Migórski, S., and Papageorgiou, N. S. (2003). *An Introduction to Nonlinear Analysis: Theory*. New York: Springer. DOI: 10.1007/978-1-4419-9158-4.

Heikkilä, S. and Lakshmikantham, V. (1994). *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*. New York: Marcel Dekker. DOI: 10.1201/9780203746493.

Himmelberg, C. J. (1975). “Measurable relations”. *Fundamenta Mathematicae* **87**(1), 53–72. URL: https://eudml.org/doc/214809.

Klein, E. and Thompson, A. C. (1984). *Theory of Correspondences: Including Applications to Mathematical Economics*. Canadian Mathematical Society Series of Monographs and Advanced Texts. New York: John Wiley and Sons.

Kucia, A. (1991). “Scorza Dragoni type theorems”. *Fundamenta Mathematicae* **138**, 197–203. URL: https://www.impan.pl/en/publishing-house/journals-and-series/fundamenta-mathematicae/all/138/3/107249/scorza-dragoni-type-theorems.

Lupo, D. (1989). “A bifurcation result for a Dirichlet problem with discontinuous nonlinearity”. *Rendiconti del Circolo Matematico di Palermo* **38**, 305–318. DOI: 10.1007/BF02844002.

Marano, S. A. (1994). “Implicit elliptic differential equations”. *Set-Valued Analysis* **2**, 545–558. DOI: 10.1007/BF01033071.

Marano, S. A. (1995). “Elliptic boundary-value problems with discontinuous nonlinearities”. *Set-Valued Analysis* **3**, 167–180. DOI: 10.1007/BF01038598.

Marano, S. A. (1996). “Implicit elliptic boundary-value problems with discontinuous nonlinearities”. *Set-Valued Analysis* **4**, 287–300. DOI: 10.1007/BF00419370.

Marano, S. A. (1997a). “Elliptic eigenvalue problems with highly discontinuous nonlinearities”. *Proceedings of the American Mathematical Society* **125**, 2953–2961. DOI: 10.1090/S0002-9939-97-03969-5.

Marano, S. A. (1997b). “Elliptic equations and differential inclusions”. *Nonlinear Analysis: Theory, Methods & Applications* **30**(3), 1763–1770. DOI: 10.1016/S0362-546X(97)00252-6.

Naselli Ricceri, O. and Ricceri, B. (1990). “An existence theorem for inclusions of the type $\Psi(u)(t) \in F(t,\Phi(u)(t))$ and application to a multivalued boundary value problem”. *Applicable Analysis* **38**, 259–270. DOI: doi:10.1080/00036819008839966.

Ricceri, B. (1982). “Sur la semi-continuité inférieure de certaines multifonctions”. *Comptes Rendus des Séances de l’Académie des Sciences. Série I - Mathématique* **294**, 265–267. URL: https://gallica.bnf.fr/ark:/12148/bpt6k5532920h/f25.item.

Stuart, C. A. (1978). “Maximal and minimal solutions of elliptic differential equations with discontinuous nonlinearities”. *Mathematische Zeitschrift* **163**, 239–249. DOI: 10.1007/BF01174897.

Stuart, C. A. and Toland, J. F. (1980). “A variational method for boundary value problems with discontinuous nonlinearities”. *Journal of the London Mathematical Society* **21**, 319–328. DOI: 10.1112/jlms/s2-21.2.319.

Talenti, G. (1976). “Elliptic equations and rearrangements”. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Serie IV* **3**(4), 697–718. URL: http://www.numdam.org/item/ASNSP_1976_4_3_4_697_0/.

* Università degli Studi di Messina,
Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra,
Viale F. Stagno d'Alcontres 31, 98166 Messina, Italy

Email: pcubiotti@unime.it

Communicated 30 November 2021; manuscript received 27 December 2021; published online 9 March 2022