DOI: 10.1478/AAPP.1001A3

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 100, No. 1, A3 (2022)

A THERMODYNAMICAL MODEL FOR POPULATION GROWTH WITH RELAXATION PHENOMENA

VINCENZO CIANCIO [a], ANTONELLA LUPICA [b*] AND ANNUNZIATA PALUMBO [c]

ABSTRACT. Reaction-diffusion models were used in dynamic fluid, population growth, pulse propagation in nerves and other biological phenomena. Some of these models have been expanded to describe memory effects in diffusion and therefore with the use of hyperbolic equations deriving from the generalization of the Fourier and Fick laws. These generalizations come from the theory of extended irreversible thermodynamics (EIT) which is based on kinetic theory arguments. Recently it has been shown that, using the procedures of the classical irreversible thermodynamics with internal variables (CIT-IV), we can obtain equations for the dissipative flows that generalize the laws of Fourier-Fick and Cattaneo-Vernotte. In this paper, using the methodology of CIT-IV, we propose a new model that includes the effect of memory in the diffusion highlighting the presence of two relaxation times. The diffusion flow obtained is characterized by the sum of a parabolic and a hyperbolic contribution which allows the formulation of a dynamic system. As example the traveling waves solutions in the case of the logistic growth are characterized.

1. Introduction

Generally, in mathematical biology, the Fisher equation is commonly used in dynamic population models with particular regard to the problems in population dispersal. Let $n(x,t)$ be the number density per unit lenght of biological particles (or individuals) at position x and time t in an one-dimesional system; then the Fisher equation is given by

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} + F(n); \quad (0 \leq n(x,t) \leq 1), \tag{1}$$

where D is the diffusion coefficient and $F(n)$ is the generating particle source function. Of course Eq. (1) is parabolic and therefore leads to the same paradox as the Fourier's equation for heat conduction (infinite propagation velocity). Some authors (Méndez and Camacho 1997; Méndez *et al.* 1999; Fort and Méndez 2002; Méndez *et al.* 2003) have proposed a generalization of the Fisher reaction-diffusion model in biological population by including memory effects. It was highlighted that this memory has as immediate consequence the delay in the appearance of the population flux and using the methods of the Extended Irreversible Termodynamics (EIT), proposed (Méndez and Camacho 1997; Méndez *et al.*

1999; Méndez and Casas-Vázquez 2008) a hyperbolic equation (finite propagation velocity) for diffusion processes:

$$\tau \frac{\partial^2 n}{\partial t^2} + \frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} + F(n) + \tau \frac{\partial F(n)}{\partial t}, \tag{2}$$

in which τ represents the relaxation time. The methodology in EIT is based on the considerations suggested by kinetic theory and assumes the hypothesis that the entropy depends on the fluxes beside on the classical thermodynamic variables.

Another point of view is offered by Classical Irreversible Thermodynamic with Internal Variables (CIT-IV) (Kluitenberg 1967; V. Ciancio *et al.* 1990). The flexibility of the methodology, used in CIT-IV, is due to the fact that the "a priori" physical meaning of the internal variables is not specified but only their influence on phenomena, occuring inside the material under consideration, is pointed out. This feature has enabled scientists to be able to apply the CIT-IV to wide-scale phenomena in continuous media. Thermodynamical theories of rheology (Kluitenberg 1967; Kluitenberg and V. Ciancio 1978; V. Ciancio and Kluitenberg 1979; Kluitenberg 1984; V. Ciancio and Verhás 1990, 1991; V. Ciancio and Restuccia 2016, 2019), thermal and diffusive effects in viscous fluid (V. Ciancio and Palumbo 2018a,b, 2019), dielectric and magnetic relaxation were developed (Kluitenberg 1981; V. Ciancio 1989; V. Ciancio and Kluitenberg 1989; V. Ciancio *et al.* 1990) with theoretical results confirmed by some experimental data (A. Ciancio *et al.* 2007; V. Ciancio *et al.* 2008; A. Ciancio 2011). In this framework (CIT-IV), we deduce a generalized model for population growth that admits the Fisher equation (1) and the hyperbolic diffusion reaction equation (2) as limiting case. The existence of traveling wave solutions is investigated using semi-analitical methods combining dynamical systems tecniques and numerical integration. Numerical simulations shows that the minimal speed of waves is lower and upper bounded by the minimal speeds in the hyperbolic and classical parabolic models, respectively.

2. Entropy balance

In one dimensional system we consider a medium composed of two subsystems, one formed by particles (animals, viruses, bacteria, etc.) subject to a diffusion process characterized by a production of entropy $\sigma_{syst} \geq 0$ and the other, characterized by an entropy production $\sigma_{gen} \geq 0$, in which new particles are generated. This means that the total production of entropy for the medium, $\sigma \geq 0$, is given by:

$$\sigma = \sigma_{syst} + \sigma_{gen} \geq 0. \tag{3}$$

The balance equation for number density $n(x,t)$ is given by (Méndez and Camacho 1997; Fort and Méndez 2002):

$$\frac{\partial n}{\partial t} = -\frac{\partial J}{\partial x} + F(n), \tag{4}$$

where $J(x,t)$ is the diffusive flux and $F(n)$ is the source function.

Let ξ be a hidden macroscopic variable of which we do not specify the physical nature. We assume that the temperature T is constant and that the local state of the medium is given by the n and the variable ξ, so the specific entropy s can be given as function of these variables, as well

$$s = s(n, \xi). \tag{5}$$

The balance equation for the specific entropy s, is given by (Méndez and Camacho 1997; Fort and Méndez 2002):

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{\mu}{T}J\right) = \sigma_{syst} \geq 0 \tag{6}$$

where σ_{syst} is the entropy production related to diffusion and

$$-\frac{\mu}{T}J \tag{7}$$

is the entropy flux (De Groot and Mazur 1984) with $\mu(n, \xi)$ the chemical potential per particle.

According to the maximum property of the entropy in an equilibrium state and to the Morse lemma (Morse 1925), we assume that the non-equilibrium entropy depends on the internal variable quadratically

$$s = s^{(eq)}(n) - \frac{\alpha}{2T}\xi^2, \tag{8}$$

where $s^{(eq)}(n)$ is the equilibrium entropy function which depends on the only equilibrium state variables n and $\alpha \geq 0$ is a constant material coefficient called *thermodynamic inductivity* (Gyarmati 1977; V. Ciancio and Palumbo 2018b). Introducing the following definition

$$\left(\frac{\partial s}{\partial n}\right)_{\xi} = -\frac{\mu}{T}, \tag{9}$$

from (8), we have the following Gibbs relation:

$$T ds = -\mu dn - \alpha\xi d\xi, \tag{10}$$

By virtue of (4) and (10), one obtains

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{\mu}{T}J\right) = -\frac{J}{T}\frac{\partial \mu}{\partial x} - \frac{\alpha}{T}\xi\frac{\partial \xi}{\partial t} - \frac{\mu}{T}F(n). \tag{11}$$

From the comparison of this equation and Eq. (6), one has

$$\sigma_{syst} = -\frac{J}{T}\frac{\partial \mu}{\partial x} - \frac{\alpha}{T}\xi\frac{\partial \xi}{\partial t} - \frac{\mu}{T}F(n). \tag{12}$$

Therefore Eq. (3) specializes in

$$\sigma = -\frac{J}{T}\frac{\partial \mu}{\partial x} - \frac{\alpha}{T}\xi\frac{\partial \xi}{\partial t} - \frac{\mu}{T}F(n) + \sigma_{gen} \geq 0. \tag{13}$$

This equation shows that the total entropy production is characterized by two contributions one for each irreversible process present. The first two terms are related to diffusion and the last two are associated to creation of particles by virtue of the source. So we have

$$\begin{cases} \sigma_{dif} = -\frac{1}{T}\left(J\frac{\partial \mu}{\partial x} + \alpha\xi\frac{\partial \xi}{\partial t}\right) \geq 0, \\ \sigma_{gen} - \frac{\mu}{T}F(n) \geq 0. \end{cases} \tag{14}$$

3. Phenomenological equations

In $(14)_1$ the entropy production σ_{dif} is determined as sum of inner products among the thermodynamic fluxes J, $\alpha\frac{\partial \xi}{\partial t}$ and the affinities $-\frac{\partial \mu}{\partial x}$, $-\xi$. According to the usual procedure of the non-equilibrium thermodynamics, by virtue of the form $(14)_1$ for σ_{dif}, the following phenomenological equations are obtained

$$\begin{cases} J = -\lambda_1\frac{\partial \mu}{\partial x} - \lambda_2\xi, \\ \alpha\frac{\partial \xi}{\partial t} = -\lambda_3\frac{\partial \mu}{\partial x} - \lambda_4\xi, \end{cases} \tag{15}$$

with λ_i, $(i = 1,...,4)$ constants. Substituting (15) in $(14)_1$, we obtain

$$\lambda_1\left(\frac{\partial \mu}{\partial x}\right)^2 + (\lambda_2+\lambda_3)\xi\frac{\partial \mu}{\partial x} + \lambda_4\xi^2 \geq 0, \tag{16}$$

which implies that coefficients λ_i, $i = 1,...,4$ must satisfy inequalities

$$\lambda_1 \geq 0,\ \lambda_4 \geq 0,\ 4\lambda_1\lambda_4 - (\lambda_2+\lambda_3)^2 \geq 0. \tag{17}$$

From $(15)_1$, we write the diffusive flux J as:

$$J = J_0 + J_1, \tag{18}$$

where

$$J_0 = -\lambda_1\frac{\partial \mu}{\partial x} = -\lambda_1\frac{\partial \mu}{\partial n}\frac{\partial n}{\partial x}, \tag{19}$$

and

$$J_1 = -\lambda_2\xi. \tag{20}$$

By virtue of $(15)_2$,we see that J_1 satisfies the following equation

$$\tau\frac{\partial J_1}{\partial t} + J_1 = \gamma\frac{\partial \mu}{\partial x}, \tag{21}$$

where

$$\begin{cases} \tau = \frac{\alpha}{\lambda_4} \geq 0 \\ \gamma = \frac{\lambda_2\lambda_3}{\lambda_4}. \end{cases} \tag{22}$$

From (18) we can see that the diffusive flux is split into two parts (V. Ciancio and Verhás 1990, 1991): the first one J_0 is governed by Fick's law (19), while the second part J_1, if $\gamma < 0$ (*i.e.*, $\lambda_2\lambda_3 < 0$), is governed by Maxwell-Cattaneo-Vernotte (MCV) type equation

(Cattaneo 1948; Vernotte 1958; Maxwell and Pesic 2001) in which a relaxation time τ is present. Using Eqs. (19) and (20), we obtain the following evolution equation for the total flux J:

$$\tau \frac{\partial J}{\partial t} + J = \beta \frac{\partial \mu}{\partial x} - \tau \lambda_1 \frac{\partial^2 \mu}{\partial t \partial x}, \tag{23}$$

where

$$\beta = \frac{\lambda_2 \lambda_3 - \lambda_1 \lambda_4}{\lambda_4}. \tag{24}$$

Of course, if the relaxation time τ, linked to the rheological properties of the medium (see $(22)_1$), can be neglected with respect to the diffusive flux time t ($\tau \ll t$) and $\lambda_1 \lambda_4 > \lambda_2 \lambda_3$, the total flux is governed by Fick's law.

Our results (18), together with (19) and (21), were taken as hypotheses by Zhou *et al.* (2001) in the study of the heat conduction in an anelastic media; they subsequently proved compatibility with the second law of thermodynamics. A similar approach was followed by Lupica and Palumbo (2021), who hypothesized coexistence of fast and slow diffusion processes in the life cycle of Aedes Aegypti mosquitoes.

4. Basic equations

Now we define the constant diffusion coefficients D_0 e D_1:

$$D_0 = \lambda_1 \left(\frac{\partial \mu}{\partial n} \right)_{\xi}, \; D_1 = -\frac{\lambda_2 \lambda_3}{\lambda_4} \left(\frac{\partial \mu}{\partial n} \right)_{\xi}, \tag{25}$$

so, from Eq. (23), we obtain the following environment at rest equation:

$$\tau \frac{\partial J}{\partial t} + J = -(D_0 + D_1) \frac{\partial n}{\partial x} - \tau D_0 \frac{\partial^2 n}{\partial t \partial x}. \tag{26}$$

Therefore, the system of equations, governing the process considered in our model and proposed in this paper, is the following:

$$\begin{cases} \frac{\partial n}{\partial t} + \frac{\partial J}{\partial x} = F(n), \\ \tau \frac{\partial J}{\partial t} + J = -(D_0 + D_1) \frac{\partial n}{\partial x} - \tau D_0 \frac{\partial^2 n}{\partial t \partial x}. \end{cases} \tag{27}$$

After some algebraic manipulations, and introducing the coefficient $D = D_0 + D_1$, representing the total diffusivity, we obtain the following differential equation in $n(t)$:

$$\tau \frac{\partial^2 n}{\partial t^2} + \left(1 - \tau \frac{dF}{dn} \right) \frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} + \tau D_0 \frac{\partial^3 n}{\partial t \partial x^2} + F(n), \tag{28}$$

or, equivalently:

$$\tau \frac{\partial^2 n}{\partial t^2} + \left(1 - \tau \frac{dF}{dn} \right) \frac{\partial n}{\partial t} = D \left(\frac{\partial^2 n}{\partial x^2} + \tau_D \frac{\partial^3 n}{\partial t \partial x^2} \right) + F(n), \tag{29}$$

where $\tau_D = \frac{\tau D_0}{D}$ is the retardation time ($\tau_D \leq \tau$). Equation (29) contains parabolic and hyperbolic models for suitable values of the relaxation and retardation times; in fact, if $\tau = 0$

(therefore, $\tau_D = 0$), Eq. (29) reduces to the Fisher equation (1), whereas for $\tau_D = 0$, with $\tau \neq 0$, we recover the hyperbolic Eq. (2). In the population dynamics the non linear reaction term has often the form $F(n) = rf(n)$, where $r > 0$ represents the population growth rate and $f(n)$ is usually a non-linear polynomial. Standard nonlinear expressions of $f(n)$ are $f(n) = n(1-n)$ (quadratic case) and $f(n) = n(1-n)(n-1)$ (cubic case), with $0 \leq n \leq 1$. For further purposes it is convenient to rescale Eq. (28) by means of the adimensional coordinates:

$$t^* = rt, \quad x^* = \sqrt{\frac{r}{D}}x. \tag{30}$$

Then, omitting the asterisks for notational simplicity, Eq. (28) becomes:

$$a\frac{\partial^2 n}{\partial t^2} + (1 - af'(n))\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + aF_D\frac{\partial^3 n}{\partial t \partial x^2} + f(n) \tag{31}$$

where

$$a = r\tau, \quad F_D = \frac{D_0}{D} \equiv \frac{\tau_D}{\tau}, \tag{32}$$

with $0 \leq F_D \leq 1$ and $f'(n) = \dfrac{df}{dn}$. In the limit $a \to 0$, Eq. (31) reduces to the classical parabolic reaction-diffusion equation

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + f(n). \tag{33}$$

In the limit $F_D \to 0$, we have the following hyperbolic advenction-reaction-diffusion equation

$$a\frac{\partial^2 n}{\partial t^2} + (1 - af'(n))\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + f(n), \tag{34}$$

just present in literature (Méndez and Camacho 1997; Méndez *et al.* 1999, 2003).

5. Traveling Wave Solutions

In this section we are interested in finding traveling wave solutions for the non linear reaction-diffusion equation (31). As known in the literature (see, for example, Volpert *et al.* 1994), a traveling wave is a wave that travels without changing the shape with a constant speed of propagation. We denote by c the dimensionless front velocity, related to dimensional front wave speed v by the relation $c = \dfrac{v}{\sqrt{rD}}$. Since Eq. (31) is invariant under the change $x \longrightarrow -x$, we need only consider, without loss of generality, right-traveling waves. Thus, we seek solution of the specific form $n(x,t) = n(x - ct) = N(z)$, where $z = x - ct$ represents the wave variable. In terms of the z variable we rewrite Eq. (31) as:

$$acF_D\frac{d^3N}{dz^3} + (ac^2 - 1)\frac{d^2N}{dz^2} - c(1 - af'(N))\frac{dN}{dz} - f(N) = 0. \tag{35}$$

Traveling wave solutions of (31) correspond to wave fronts, solutions of (35), which describe the local evolution of the population density from an unstable state to a stable one. Therefore the following boundary conditions must be associated to Eq. (35):

$$\lim_{z\to-\infty} N(z) = N_-, \quad \lim_{z\to+\infty} N(z) = N_+, \tag{36}$$

being N_- and N_+ equilibrium states, solutions of Eq. $f(N) = 0$. Moreover the function $N(z)$ will satisfy the condition

$$\frac{dN}{dz} < 0, \text{ in } (N_-, N_+) \text{ and } \lim_{z\to\pm\infty} \frac{dN}{dz} = 0.$$

Now we rewrite Eq. (35) as a dynamical system:

$$\begin{cases} \dfrac{dN}{dz} = X, \\ \dfrac{dX}{dz} = Y, \\ \dfrac{dY}{dz} = \dfrac{1}{acF_D}\left[(1-ac^2)Y + c(1-af'(N))X + f(N)\right]. \end{cases} \tag{37}$$

Of course, a necessary condition for the existence of the solution is that the first equilibrium point must have an unstable (departing) manifold and the second one is a stable (incoming) manifold. As usual, in order to do the required stability analysis, we linearize the dynamical system (37) around the equilibrium points in the phase space (N, X, Y) and we determine the eigenvalues of the Jacobian matrix for each singular point. The characteristic equation related to the generic equilibrium state $(N^*, 0, 0)$ is the following cubic equation:

$$p(\zeta, c) = \zeta^3 + a_2\zeta^2 + a_1\zeta + a_0 = 0, \tag{38}$$

where

$$a_2 = \frac{ac^2-1}{acF_D}, \quad a_1 = \frac{af'(N^*)-1}{aF_D}, \quad a_0 = -\frac{f'(N^*)}{acF_D}. \tag{39}$$

Owing the biological meaning of our state variables, we have to impose that the polynomial $p(\zeta, c)$ has all the roots with real values. We know that $p(0,c) = -\dfrac{f'(N^*)}{acF_D}$ and we have $\lim_{\zeta\to\pm\infty} p(\zeta, c) = \pm\infty$, so we must distinguishes different cases corresponding to $f'(N^*)$.

Case A $f'(N^*) > 0$.

The condition $f'(N^*) > 0$ implies that $p(0,c) < 0$; therefore, each ζ - polynomial of this c - family $p(\zeta, c)$ has always one positive real root. Now, as for the other two root, we compute the extrema of the polynomial $p(\zeta, c)$, solutions of the equation

$$\frac{\partial p(\zeta, c)}{\partial \zeta} = 3\zeta^2 + 2a_2\zeta + a_1 = 0, \tag{40}$$

whose discriminant is given by

$$\frac{\Delta}{4} = a_2^2 - 3a_1 = \left(\frac{ac^2-1}{acF_D}\right)^2 - 3\frac{af'(N^*)-1}{aF_D}. \tag{41}$$

Two subcases should be considered.

Case $\mathbf{A}_1$ $0 < f'(N^*) < \dfrac{1}{a}$

We have $\frac{\partial p(\zeta,c)}{\partial \zeta}\,|_{\zeta=0}= \dfrac{af'(N^*)-1}{aF_D} < 0$. Moreover, the extrema of the polynomial $p(\zeta,c)$ are

$$\zeta_{\pm} = \frac{1}{3}\left(-\frac{ac^2-1}{acF_D} \pm \sqrt{\left(\frac{ac^2-1}{acF_D}\right)^2 - 3\frac{af'(N^*)-1}{aF_D}}\right), \tag{42}$$

where $\zeta_- < 0$ is the local maximum and $\zeta_+ > 0$ is the local minimum. If we impose that $p(\zeta_-,c)$ assume values greater than or equal to zero, that is $p(\zeta_-,c) \geq 0$, then the roots of cubic (38) have real values. Moreover, requiring that the cubic (38) evaluated in the unique local maximum is null, *i.e.*, $p(\zeta_-,c) = 0$, we determine the minimal velocity of the waves. So the double negative real root determines the critical value $c^{(1)}_{min}$ (see Fig. 1(a)). For $c < c^{(1)}_{min}$ there are complex solutions for $p(\zeta,c)$, and for $c > c^{(1)}_{min}$ there are two different negative roots and one positive root.

Case $\mathbf{A}_2$ $f'(N^*) > \dfrac{1}{a}$

In this case it turns out $\frac{\partial p(\zeta,c)}{\partial \zeta}\,|_{\zeta=0}= \dfrac{af'(N^*)-1}{aF_D} > 0$. Moreover, Eq. (40) admits real roots ζ_- and ζ_+ if the wave speed satisfy the inequality

$$a^2c^4 - a[2+3F_D(af'(N^*)-1)]c^2 + 1 = a^2(c^2-\alpha_1)(c^2-\alpha_2) > 0, \tag{43}$$

with

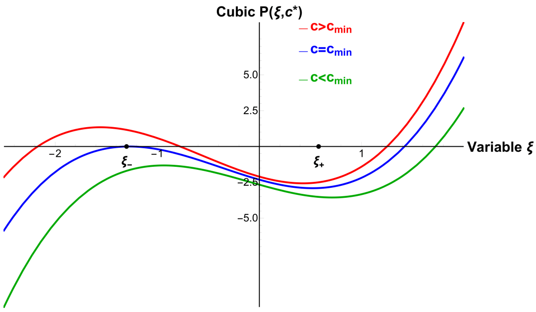
$$\alpha_{1,2} = \frac{1}{2a}\left[2+3F_D(af'(N^*)-1) \mp \sqrt{9F_D^2(af'(N^*)-1)^2 + 12(af'(N^*)-1)}\right]. \tag{44}$$

Firstly we observe that $\zeta_-\zeta_+ > 0$, so ζ_- and ζ_+ have the same sign, both positive if $c < \sqrt{\alpha_1}$, both negative if $c > \sqrt{\alpha_2}$. We also remark that the following inequality holds:

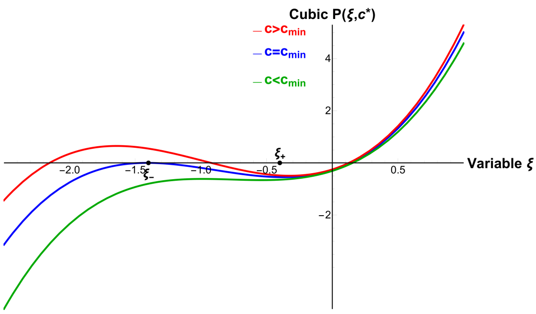
$$\frac{\partial p(\zeta,c)}{\partial c} = \frac{1}{ac^2F_D}\left[(1+ac^2)\zeta^2 + f'(N^*)\right] > 0, \tag{45}$$

for each fixed value of ζ.

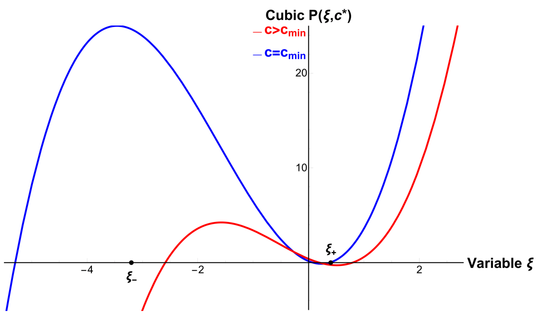
It is possible to determine the minimal value of the wave velocity compatible with the inequality (43) only in the case shown in Fig. 1(b). Then, also in this case the double negative roots of the polynomial $p(\zeta,c)$ determines $c^{(1)}_{min}$ from $p(\zeta_-,c) = 0$. From the result $p(\zeta_-,\sqrt{\alpha_2}) < 0$, we also have $c^{(1)}_{min} > \sqrt{\alpha_2}$. For $c < c^{(1)}_{min}$ there are complex solutions for $p(\zeta,c)$, and for $c > c^{(1)}_{min}$ there are two different negative roots and one positive root.



(a)



(b)



(c)

FIGURE 1. Generic graph of the c - family of ζ - polynomial $p(\zeta,c)$ corresponding to: $0 < f'(N^*) < \frac{1}{a}$ with $\zeta_- < 0$ and $\zeta_+ > 0$, ((a) panel), $f'(N^*) > \frac{1}{a}$ with $\zeta_- < 0$, $\zeta_+ < 0$ and $c > \sqrt{\alpha_2}$, ((b) panel), $f'(N^*) < 0$ with $\zeta_- < 0$ and $\zeta_+ > 0$ ((c) panel).

Case B $f'(N^*) < 0$**.**

The condition $f'(N^*) < 0$ implies that the family of polynomial $p(\zeta, c)$ admits all real roots and we have $p(\zeta, c) > 0$ together to $\frac{\partial p(\zeta,c)}{\partial \zeta} \mid_{\zeta=0} = \dfrac{af'(N^*)-1}{aF_D} < 0$. Then, we conclude immediately that the polynomial $p(\zeta, c)$ has always one negative real root. With respect the two remaining roots, we impose that at the local minimum ζ_+ the polynomial $p(\zeta, c)$ is at most zero, *i.e.*, $p(\zeta_+, c) \leq 0$. The minimal velocity $c^{(2)}_{min}$ is determined by the condition that the polynomial evaluated at the unique local minimum must be zero, that is, $p(\zeta_+, c) = 0$ (see Fig. 1(c)). Numerical simulations suggest that the inequality $p(\zeta_+, c) \leq 0$ is held for every value of the speed c and therefore $c^{(2)}_{min} = 0$.

For $c \geq c^{(2)}_{min}$ there are three real roots for the polynomial $p(\zeta, c)$. We observe that in all cases the equilibrium state $(N^*, 0, 0)$ is a saddle point in the phases space and we report below the results obtained:

- if $f'(N^*) > 0$, then the polynomial $p(\zeta, c)$ admits one positive and two negatives roots for $c \geq c^{(1)}_{min}$ such that $p\left(\zeta_-, c^{(1)}_{min}\right) = 0$;
- if $f'(N^*) < 0$, then the polynomial $p(\zeta, c)$ admits two positives and one negative roots for $c \geq c^{(2)}_{min}$ such that $p\left(\zeta_+, c^{(2)}_{min}\right) = 0$.

We observe that in the limit $F_D \to 0$, we recover the results obtained by Méndez and Camacho (1997) and Méndez *et al.* (2003) for the hyperbolic reaction-diffusion equation (34) : the possible wave speed must satisfy the inequalities

$$c_h = \frac{2\sqrt{f'(N^*)}}{1 + af'(N^*)} \leq c < \frac{1}{\sqrt{a}}; \quad f'(N^*) > 0. \tag{46}$$

and for $a \to 0$, we recover the classical parabolic diffusion-reaction case (see Eq. (33) (Zhou *et al.* 2001), with

$$c \geq c_p = 2\sqrt{f'(N^*)}. \tag{47}$$

6. Example: the Logistic Growth

As an illustrative example, we apply the previous analysis to the logistic growth: thus the source term becomes

$$f(n) = n\left(1 - \frac{n}{K}\right), \tag{48}$$

where K is a positive parameter that represents the carrying capacity. Thus, Eq. (31) becomes:

$$a\frac{\partial^2 n}{\partial t^2} + \left[1 - a\left(1 - 2\frac{n}{K}\right)\right]\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + aF_D\frac{\partial^3 n}{\partial t \partial x^2} + n\left(1 - \frac{n}{K}\right), \tag{49}$$

while Eq. (35) reads as:

$$acF_D \frac{d^3N}{dz^3} + (ac^2-1)\frac{d^2N}{dz^2} - c\left[1 - a\left(1 - 2\frac{N}{K}\right)\right]\frac{dN}{dz} - N\left(1 - \frac{N}{K}\right) = 0, \qquad (50)$$

with the boundary conditions (36). In this case, Eq. (49) admits two equilibium states $N_1^* = 0$ and $N_2^* = K$. This implies that equilibria admitted by (49), are given by $\mathbf{E}_1 = (0,0,0)$ and $\mathbf{E}_2 = (K,0,0)$. As has already been proven, neither of the two steady states is stable, with respect to Eq. (50). The derivative $f'(n)$ evaluated at the two singular points provides $f'(N_1^*) = 1 > 0$ and $f'(N_2^*) = -1 < 0$. Therefore, Eq. (49) admits two homogeneous in space steady states, one of which is stable and the other unstable.

(1) Equilibrium state $\mathbf{E}_1 = (0,0,0)$.

Since $f'(N_1^*) = f'(0) = 1 > 0$, then $p(\xi)$ has one positive and two negatives real solutions, whenever $c > c_{min}^{(1)}$. If the speed c does not satisfy the restrictions listed above, then $p(\xi)$ admits complex roots and we obtain an oscillatory and unstable behavior of the solution around $\mathbf{E}_1$.

(2) Equilibrium state $\mathbf{E}_2 = (K,0,0)$.

Since $f'(N_2^*) = f'(K) < 0$, the characteristic polynomial (38) evaluated at $\mathbf{E}_2$ has two positives and one negative solutions, whenever $c > c_{min}^{(2)} = 0$. Thus, Eq. (38) evaluated at $\mathbf{E}_2$ admits two positive and one negative real solutions for any positive c value.

Therefore, we can deduce that a travelling wave solution of positive velocity c, $\forall\ c \in [c_{min}^{(1)}, +\infty[$, connecting equilibrium $\mathbf{E}_2$ and equilibrium $\mathbf{E}_1$ may exist, *i.e.*, a function $n(x-ct)$, solution of Eq. (50) on the real line $]-\infty, +\infty[$ and satisfying Eq. (36) with $N_- = K$ and $N_+ = 0$. This solution is monotone and decreasing.

In Fig. 2, a numerical simulation shows the propagation of traveling wave with minimal speed $c = c_{min}^{(1)}$: the numerical solution starts from $N_2^* = K$ and reaches the equilibrium $N_1^* = 0$. Some numerical simulations are conducted in order to compare Eq. (49) obtained

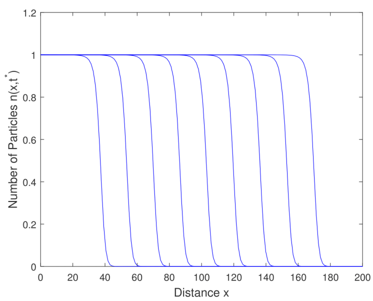


FIGURE 2. Numerical solution of Eq. (49) showing the stable connection (traveling wave) from $N_2^* = K$ to $N_1^* = 0$. The parameters used in this simulation are $D_0 = D_1 = 1$, $\tau = 2$, $k = 1$ and $K = 1$. The wave propagates at the minimal speed $c_{min}^{(1)} = 1.54563$.

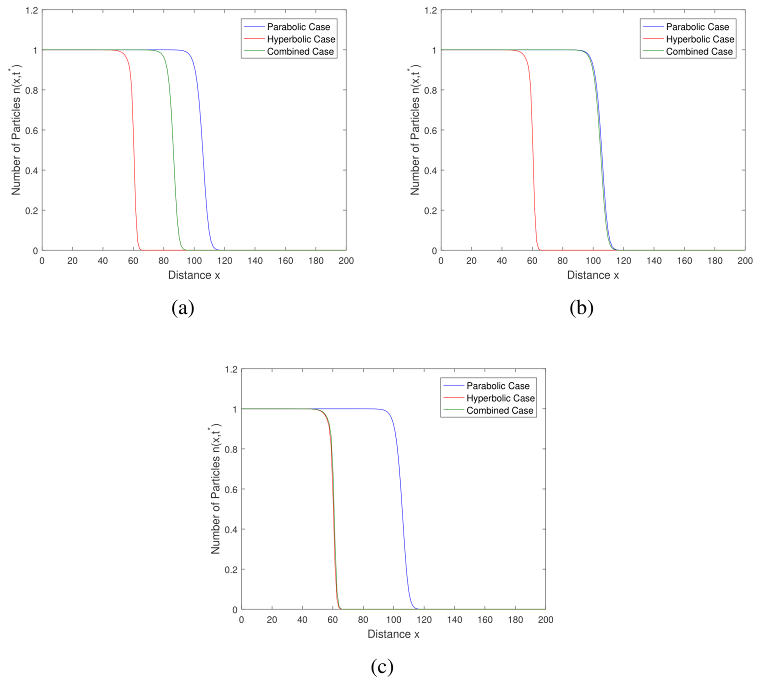


FIGURE 3. Numerical solution of Eq. (51) (blue line), numerical solution of Eq. (52) (red line) and numerical solution of Eq. (49) (green line), with minimal speeds. Parameters value are listed in Fig. 2, with $D_0 = 10^3$ in (b) panel and $D_1 = 10^3$ (c) panel.

assuming the coexistence of both diffusion processes, with the parabolic corresponding equation, obtained from (49) passing to the limit $a \to 0$, and hyperbolic correponding equations, obtained from (49) passing to the limit $F_D \to 0$. Thus, from Eq. (49), if $a \to 0$, we get the classic parabolic equation

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + n\left(1 - \frac{n}{K}\right). \tag{51}$$

While if $F_D \to 0$, then Eq. (49) becomes hyperbolic, *i.e.*:

$$a\frac{\partial^2 n}{\partial t^2} + \left[1 - a\left(1 - 2\frac{n}{K}\right)\right]\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + n\left(1 - \frac{n}{K}\right). \tag{52}$$

Figure 3(a) depicts a comparison between the numerical solution obtained from Eq. (49), numerical solution obtained from parabolic equation (51) and solution obtained from hyperbolic equation (52). We see the propagation of the wave in the positive direction of the x-axis. All the trajectories travel at their minimal speed: in the parabolic case (equation (51)), $c_p = 2\sqrt{f'(0)} = 2$, in the hyperbolic case (Eq. (52)) $c_h = 2\dfrac{\sqrt{f'(0)}}{1 + af'(0)} = 0.667$ and for the combined case (Eq. (49)) $c_{min}^{(1)} = 1.54563$. As we expect, Eq. (49) appears to be a compromise between the parabolic case and the hyperbolic case. In the simulation shown in Fig. 3, we assume that the diffusion coefficient D_0 (connected to the fickian propagation) is much greater with respect to the diffusion coefficient D_1 (*i.e.*, the coefficient linked to the flux J_1, satisfiyng the Cattaneo evolution equation). Figure 3(b) shows that solution of Eq. (49) is superimposed on the solution of Eq. (51). All solutions travel at the minimum speed: $c_p = 2\sqrt{f'(0)} = 2$, $c_h = 2\dfrac{\sqrt{f'(0)}}{1+af'(0)} = 0.6667$ and $c_{min}^{(1)} = 1.9992$. Note that c_p and $c_{min}^{(1)}$ are very close. Similarly, suppose now that the diffusion coefficient D_1 (linked to the Cattaneo equation) is much greater than the coefficient D_0 (related to the Fickian diffusion). Figure 3(c) shows that solution of Eq. (49) is superimposed on the solution of Eq. (52). All solutions travel at the their minimal speed: $c_p = 2$, $c_h = 0.6667$ and $c_{min}^{(1)} = 0.730476$. Note that c_h and $c_{min}^{(1)}$ are very close.

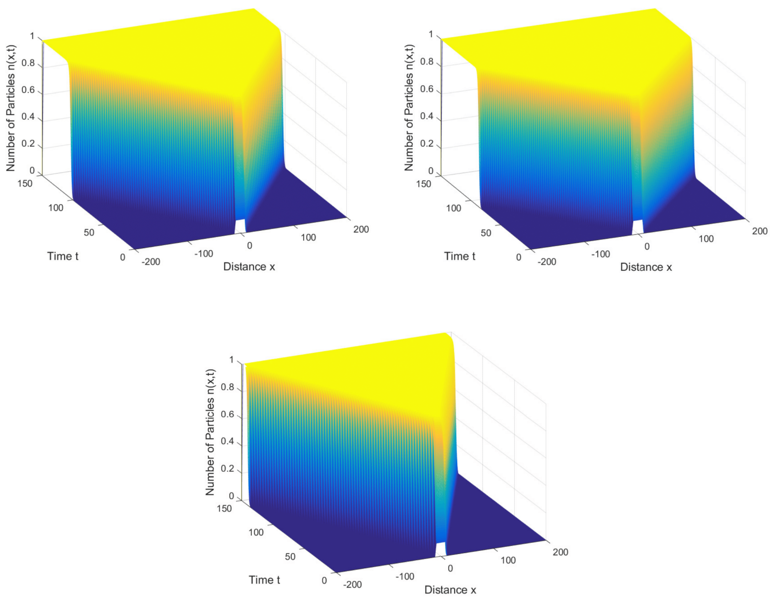


FIGURE 4. Numerical solution of Eq. (49) (combined model) (a), numerical solution of Eq. (51) (Parabolic model) (b) and numerical solution of Eq. (52) (Hyperbolic model) (c). Parameters value are listed in Fig. 2.

Figure 4 shows the comparison of the numerical solutions of Eq. (49) (the combined model), (51) (Parabolic model) and (52) (Hyperbolic model) obtained with zero flux boundary conditions. The state variables start from initial conditions close to 0 and evolve in time approaching to the stable equilibrium N_2^*. Note that the time required to reach equilibrium is different depending on the model: in fact, as already observed, using the parabolic model

(Fig. 4(b)), it takes less time to reach equilibrium, while using the hyperbolic one (Fig. 4(c)) takes the longest possible time. Instead, using the combined model (Fig. 4(a)), the time required to reach equilibrium represents an average between the two borderline cases.

7. Discussion and conclusions

The main feature of our model is the capability of reproducing both Fickian and non-Fickian behavior according to the values of the parameters: indeed both the numerical solution of the hyperbolic model corresponding to Eq. (52) (Méndez *et al.* 1999) and the solution of the classical parabolic model (Méndez and Camacho 1997; Méndez *et al.* 1999; Méndez and Casas-Vázquez 2008) corresponding to Eq. (51) can be obtained as limiting cases. As can be seen from Fig. 3(c), when F_D is very small (*i.e.*, D_1 is much larger than D_0) a non-Fickian transport phenomenon is actually observed: slow diffusion process and relaxation appear. In this case, the solution of the model (49) approaches the solution of the limiting hyperbolic model corresponding to Eq. (52). For intermediate values of F_D, the model (49) represented a compromise between the classical parabolic model and the hyperbolic model, since both diffusion processes coexist and overlap simultaneously. When the parameter F_D reaches its maximum value ($F_D = 1$; for instance, D_0 is much larger than D_1) and $a \to 0$, the model reproduces Fickian diffusion. In this case, the solution of the model (49) is very close to the solution of the parabolic equation (51). As we have already observed in the case of trajectories, even the minimal speed $c_{min}^{(1)}$ (for which can exist monotone traveling wave solutions of Eq. (50)) has the characteristic of reproducing, for appropriate values of the parameters, the classic parabolic case c_p given by Volpert *et al.* (1994) and the hyperbolic case c_h given by Méndez *et al.* (1999). The minimal speed $c_{min}^{(1)}$ deduced for Eq. (50) seems to recover the two minimal speeds c_h, defined by Méndez *et al.* (1999), and c_p, recovered by Volpert *et al.* (1994), deduced for the hyperbolic and classical cases, respectively.

Acknowledgments

This work was supported by GNFM and INdAM.

References

Cattaneo, C. (1948). “Sulla conduzione del calore”. *Atti del Seminario Matematico e Fisico della Università di Modena* **3**, 83–101.

Ciancio, A. (2011). “An approximate evaluation of the phenomenological and state coefficients for visco-anelastic media with memory”. *UPB Scientific Bulletin A* **73**(4), 3–14. URL: https://www.scientificbulletin.upb.ro/rev_docs_arhiva/full17925.pdf.

Ciancio, A., Ciancio, V., and Farsaci, F. (2007). “Wave propagation in media obeying a thermoviscoanelastic model”. *UPB Scientific Bullettin A* **69**(4), 69–79. URL: https://www.scientificbulletin.upb.ro/rev_docs_arhiva/full73995.pdf.

Ciancio, V. (1989). “On the generalized Debye equation for media with dielectric relaxation phenomena described by vectorial internal thermodynamic variables”. *Journal of Non-Equilibrium Thermodynamics* **14**(3), 239–250. DOI: 10.1515/jnet.1989.14.3.239.

Ciancio, V., Ciancio, A., and Farsaci, F. (2008). "On general properties of phenomenological and state coefficients for isotropic viscoanelastic media". *Physica B: Condensed Matter* **403**(18), 3221–3227. DOI: 10.1016/j.physb.2008.04.021.

Ciancio, V. and Kluitenberg, G. A. (1979). "On linear dynamical equations of state for isotropic media II: some cases of special interest". *Physica A: Statistical Mechanics and its Applications* **99**(3), 592–600. DOI: 10.1016/0378-4371(79)90074-8.

Ciancio, V. and Kluitenberg, G. A. (1989). "On electromagnetic waves in isotropic media with dielectric relaxation". *Acta Physica Hungarica* **66**(1-4), 251–276. DOI: 10.1007/BF03155797.

Ciancio, V. and Palumbo, A. (2018a). "A thermodynamic theory on the interaction of thermal and diffusive effects in viscous fluids". *BSG Proceedings* **25**, 17–29. URL: http://www.mathem.pub.ro/proc/bsgp-25/K25-cj-ZE98.pdf.

Ciancio, V. and Palumbo, A. (2018b). "A Thermodynamical Theory with Internal Variables Describing Thermal Effects in Viscous Fluids". *Journal of Non-Equilibrium Thermodynamics* **43**(2), 171–184. DOI: 10.1515/jnet-2017-0048.

Ciancio, V. and Palumbo, A. (2019). "A thermodynamic theory with hidden vectorial variables on possible interactions among heat conduction, diffusion phenomena, viscous flow and chemical reactions in fluid mixtures". *Atti della Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matematiche e Naturali* **97**(S1), A4 [13 pages]. DOI: 10.1478/AAPP.97S1A4.

Ciancio, V. and Restuccia, L. (2016). "On heat equation in the framework of classic irreversible thermodynamics with internal variables". *International Journal of Geometric Methods in Modern Physics* **13**(08), 1640003. DOI: 10.1142/S021988781640003X.

Ciancio, V. and Restuccia, L. (2019). "A derivation of a Guyer-Krumhansl type temperature equation in classical irreversible thermodynamics with internal variables". *Atti della Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matematiche e Naturali* **97**(S1), A5 [16 pages]. DOI: 10.1478/AAPP.97S1A5.

Ciancio, V., Restuccia, L., and Kluitenberg, G. A. (1990). "A thermodynamic derivation of equations for dielectric relaxation phenomena in anisotropic polarizable media". *Journal of Non-Equilibrium Thermodynamics* **15**(2), 157–171. DOI: 10.1515/jnet.1990.15.2.157.

Ciancio, V. and Verhás, J. (1990). "A thermodynamic theory for radiating heat transfer". *Journal of Non-Equilibrium Thermodynamics* **15**(1), 33–43. DOI: 10.1515/jnet.1990.15.1.33.

Ciancio, V. and Verhás, J. (1991). "A thermodynamic theory for heat radiation through the atmosphere". *Journal of Non-Equilibrium Thermodynamics* **16**(1), 57–66. DOI: 10.1515/jnet.1991.16.1.57.

De Groot, S. R. and Mazur, P. (1984). "Non-Equilibrium Thermodynamics". *Dover Publications, New York*.

Fort, J. and Méndez, V. (2002). "Wavefronts in time-delayed reaction-diffusion systems. Theory and comparison to experiment". *Reports on Progress in Physics* **65**(6), 895. DOI: 10.1088/0034-4885/65/6/201.

Gyarmati, I. (1977). "On the wave approach of thermodynamics and some problems of non-linear theories". *Journal of Non-Equilibrium Thermodynamics* **2**(4), 233–260. DOI: 10.1515/jnet.1977.2.4.233.

Kluitenberg, G. A. (1967). "On heat dissipation due to irreversible mechanical phenomena in continuous media". *Physica* **35**(2), 177–192. DOI: 10.1016/0031-8914(67)90064-X.

Kluitenberg, G. A. (1981). "On vectorial internal variables and dielectric and magnetic relaxation phenomena". *Physica A: Statistical Mechanics and its Applications* **109**(1-2), 91–122. DOI: 10.1016/0378-4371(81)90039-X.

Kluitenberg, G. A. (1984). "Plasticity and Nonequilibrium Thermodynamics". In: *The Constitutive Law in Thermoplasticity*. Ed. by T. Lehmann. Vol. 281. CISM International Centre for Mechanical Sciences (Courses and Lectures). Vienna: Springer, pp. 157–250. DOI: 10.1007/978-3-7091-2636-3_4.

Kluitenberg, G. A. and Ciancio, V. (1978). "On linear dynamical equations of state for isotropic media I: general formalism". *Physica A: Statistical Mechanics and its Applications* **93**(1-2), 273–286. DOI: 10.1016/0378-4371(78)90221-2.

Lupica, A. and Palumbo, A. (2021). "The coexistence of fast and slow diffusion processes in the life cicle of Aedes Aegypti mosquitoes". *International Journal of Biomathematics* **14**(01), 2050087. DOI: 10.1142/S1793524520500874.

Maxwell, J. C. and Pesic, P. (2001). *Theory of Heat*. Mineola, New York: Dover Publications.

Méndez, V. and Camacho, J. (1997). "Dynamics and thermodynamics of delayed population growth". *Physical Review E* **55**(6), 6476. DOI: 10.1103/PhysRevE.55.6476.

Méndez, V. and Casas-Vázquez, J. (2008). "Hyperbolic reaction-diffusion model for virus infections". *International Journal of Thermodynamics* **11**(1), 35–38. URL: https://dergipark.org.tr/en/pub/ijot/issue/5767/76750.

Méndez, V., Fort, J., and Farjas, J. (1999). "Speed of wave-front solutions to hyperbolic reaction-diffusion equations". *Physical Review E* **60**(5), 5231. DOI: 10.1103/PhysRevE.60.5231.

Méndez, V., Fort, J., Rotstein, H. G., and Fedotov, S. (2003). "Speed of reaction-diffusion fronts in spatially heterogeneous media". *Physical Review E* **68**(4), 041105. DOI: 10.1103/PhysRevE.68.041105.

Morse, M. (1925). "Relations between the critical points of a real function of n independent variables". *Transactions of the American Mathematical Society* **27**(3), 345–396. DOI: 10.2307/1989110.

Vernotte, P. (1958). "Les paradoxes de la théorie continue de l'équation de la chaleur". *Comptes Rendus Hebdomadaires des Séances de L'Academie des Sciences* **246**, 3154–3155. URL: https://gallica.bnf.fr/ark:/12148/bpt6k723q/f1108.item.

Volpert, A. I., Volpert, V. A., and Volpert, V. A. (1994). *Traveling Wave Solutions of Parabolic systems*. Vol. 140. American Mathematical Society. DOI: 10.1090/mmono/140.

Zhou, X., Tamma, K., and Anderson, C. V. D. R. (2001). "On a new c-and f-processes heat conduction constitutive model and the associated generalized theory of dynamic thermoelasticity". *Journal of Thermal Stresses* **24**(6), 531–564. DOI: 10.1080/014957301300158094.

[a] Accademia Peloritana dei Pericolanti,
Classe di Scienze Fisiche, Matematiche e Naturali,
Piazza S. Pugliatti 1, 98121 Messina, Italy

[b] Università degli Studi della Tuscia,
Dipartimento di Economia, Ingegneria, Società e Impresa,
Scuola di Ingegneria,
Largo dell'Università, Viterbo 01100, Italy

[c] Università degli Studi di Messina,
Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra,
Viale Ferdinando Stagno d'Alcontres 31, 98166 Messina, Italy

* To whom correspondence should be addressed | email: alupica@unitus.it

Communicated 28 May 2021; manuscript received 11 June 2021; published online 2 March 2022