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BAR CODE AND JANET-LIKE DIVISION

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Abstract. Bar Codes are combinatorial objects encoding many properties of monomial ideals. In this paper we employ these objects to study Janet-like divisions. Given a finite set of terms U, from its Bar Code we can compute the Janet-like non-multiplicative powers of its elements and detect completeness of the set.

1. Introduction

Bar Codes, first introduced by Ceria (2019c) and sistematically formalized by the same author (Ceria 2020), are combinatorial objects encoding many properties of monomial ideals (for the applications studied so far, see Ceria and Mora 2018; Ceria 2019a,b,c,d). Janet division dates back to the paper by Janet (1920) and has been first developed to study partial differential equations via algebraic methods, following and formalizing the approach by Riquier (1910). It has then been generalized by Gerdt and Blinkov (1998a,b, 2011), who defined the concept of *involutive division*. On the other hand, Janet-like division, introduced by Gerdt and Blinkov (2005a,b) to efficiently compute Groebner bases, though not being an involutive division, is strictly related to this concept, being a generalization of Janet division (Janet 1920) and preserving most of its properties.

In this note, that completes and concludes the discussion of Ceria (2019d), we show that Bar Codes can be successfully used as tools to describe Janet-like division.

2. Notation and preliminaries

2.1. Monomials and polynomials. Following Mora (2003, 2005, 2015, 2016), we denote the polynomial ring in n variables and coefficients in a field $\mathbf{k}$ by $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$, and we equip it with the lexicographical ordering $<$, while $\mathcal{T} := \{x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \mid \gamma := (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n\}$ is the semigroup of terms generated by the n variables of $\mathcal{P}$. Given a term $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ we define the *i-degree* of t, for $1 \leq i \leq n$, as $\deg_i(t) = \alpha_i$ and the *degree* of t as $\deg(t) = \sum_{i=1}^n \alpha_i$. A set $J \subseteq \mathcal{T}$ is called *semigroup ideal* if, for each $t \in J$, $s \in \mathcal{T}$, we have $st \in J$, while a set $\mathsf{N} \subseteq \mathcal{T}$ is an order ideal if $t \in \mathsf{N}$ implies $s \in \mathsf{N}$ for each s dividing t. Given $f \in \mathcal{P}$, we denote by $\mathsf{T}(f)$ its leading term, while for $G \subset \mathcal{P}$ we define the set $\mathsf{T}\{G\} := \{\mathsf{T}(g) : g \in G\}$ and we call $\mathsf{T}(G)$ the semigroup ideal of leading terms, namely $\mathsf{T}(G) := \{t\mathsf{T}(g) : t \in \mathcal{T}, g \in G\}$. For an ideal I of $\mathcal{P}$, the monomial basis of the semigroup ideal $\mathsf{T}(I) = \mathsf{T}\{I\}$ is named *monomial basis* of I and it is denoted by $\mathsf{G}(I)$. The order ideal $\mathsf{N}(I) := \mathcal{T} \setminus \mathsf{T}(I)$ takes the name of *Groebner escalier* of I.

Following Ceria (2019c, 2020), that we suggest as complete references on the topic, we recap the definiton of Bar Code (for an example see Fig. 1).

Definition 1. *A Bar Code* B *is a diagram composed by segments, (the* bars*), superimposed in horizontal rows, satisfying the Condition* a. *below. Denote by* $\mathsf{B}_j^{(i)}$ *the* j*-th bar (from left to right) of the* i*-th row (from top to bottom),* $1 \leq i \leq n$*, i.e. the* j-th i-bar *and by* $\mu(i)$ *the number of bars of the* i*-th row:*

a. $\forall i, j,\ 1 \leq i \leq n-1,\ 1 \leq j \leq \mu(i),\ \exists! \overline{j} \in \{1, ..., \mu(i+1)\}$ *s.t.* $\mathsf{B}_{\overline{j}}^{(i+1)}$ *lies under* $\mathsf{B}_j^{(i)}$.

We denote by $l_1(\mathsf{B}_j^{(1)}) := 1$, for each $j \in \{1, 2, ..., \mu(1)\}$, the $1-$*length* (or length for short) of the 1-bars and by $l_i(\mathsf{B}_j^{(k)})$, $2 \leq k \leq n$, $1 \leq i \leq k-1$, $1 \leq j \leq \mu(k)$ the i*-length* of $\mathsf{B}_j^{(k)}$, i.e. the number of i-bars lying over $\mathsf{B}_j^{(k)}$.

It is possible to associate to a given finite set of terms a Bar Code and to a given Bar Code a finite set of terms. The constructions to get it are described by Ceria (2019c, 2020). We recall that the main idea to construct a Bar Code from a finite set of terms U is, for $1 \leq i \leq n$, to define the projection maps $\pi_i : \mathbf{k}[x_1, ..., x_n] \to \mathbf{k}[x_1, ..., x_n]/(x_1, ..., x_{i-1})$. Then, U is ordered increasingly with respect to the lexicographic order, with $x_1 < x_2 < ... < x_n$. For $1 \leq i \leq n$, the ordered lists $U^{(i)} := [\pi_i(u) : u \in U]$ are computed; the i-bars correspond to the values of $\pi_i(u)$ for $u \in U$. We remark then that the $1-$bars correspond to the terms of U in increasing order with respect to the lexicographic order, induced by $x_1 < x_2 < ... < x_n$. The term corresponding to some bar is said to *lie over the bar.*

2.2. Janet and Janet-like division. In what follows, we revise Janet and Janet-like division, with their main properties.

Definition 2 (Janet 1920, pp. 75-79)**.** *Let* $U \subset \mathcal{T}$ *be a set of terms and* $u = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *be an element of* U*. A variable* x_j *is called* multiplicative *for* u *with respect to* U *if in* U *there is no term of the form* $u' = x_1^{\beta_1} \cdots x_j^{\beta_j} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$*, such that* $\beta_j > \alpha_j$*. We denote by* $M_J(u, U)$ *the set of multiplicative variables for* u *with respect to* U*, while the variables in* $NM_J(u, U) := \{x_1, ..., x_n\} \setminus M_J(u, U)$ *are called* non-multiplicative.

Janet division is defined as follows: for each $t \in \mathcal{T}$, we say that a term $u \in U$ Janet-divides t if $t = uv$ and each x_j dividing v, $j \in \{1, ..., n\}$, belongs to $M_J(u, U)$. In this case, u is called *Janet-divisor* of t and t is a *Janet-multiple* of u.

Definition 3. *The* cone *of* t *with respect to* U *is the set* $C_J(t, U) := \{t x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid$ *where* $\lambda_j \neq 0$ *only if* $x_j \in M_J(t, U)\}$.

It can be proved (see Janet 1920) that every $t \in \mathcal{T}$ has at most a Janet-divisor: the cones are disjoint. A priori, it may happen that a term $u \in \mathsf{T}(U)$ has no Janet-divisor; the notion of *completeness* characterizes the case in which this cannot happen.

Definition 4. *A set $U \subset \mathcal{T}$ is* complete *if* $\mathsf{T}(U) = \bigsqcup_{t \in U} C_J(t, U)$.

Janet division is used to construct a special kind of Groebner basis for an ideal $I = (G)$ called *Janet basis.* Roughly speaking, the complete set U is the set $\mathsf{T}\{G\}$ of all leading terms for the generators and any term $u \in \mathsf{T}(I)$ is reduced by means of the polynomial $f \in G$ such that $t := \mathsf{T}(f) \in U$ is the only Janet-divisor of u. Gerdt and Blinkov (1998a,b, 2011) give a generalization of Janet division/bases, defining *involutive divisions/bases* (Apel 1998; Albert *et al.* 2016). Gerdt and Blinkov (2005a,b) introduce Janet-like division/bases, with the aim to compute Groebner bases more efficiently.

Definition 5 (Gerdt and Blinkov 2005a,b). *Let $U \subset \mathcal{T}$ be a finite set of terms. For each $u \in U$, $1 \leq i \leq n$ consider*
$h_i(u, U) = \max\left\{\deg_i(v) : v \in U, \deg_j(v) = \deg_j(u), \forall j \in \{i+1, ..., n\}\right\} - \deg_i(u) \in \mathbb{N}$.
If $h_i(u, U) > 0$, define

$$k_i := \min_{v \in U}\left\{\deg_i(v) - \deg_i(u) : \deg_j(v) = \deg_j(u), \forall j \in \{i+1, ..., n\}, \deg_i(v) > \deg_i(u)\right\};$$

then $x_i^{k_i}$ is called non-multiplicative power *of $u \in U$. $NMP(u, U)$ is the set of non-multiplicative powers of $u \in U$.*

Definition 6 (Gerdt and Blinkov 2005a,b). *Let $U \subset \mathcal{T}$ be a finite set of terms and $u \in U$. The elements in the monoid ideal*

$$NM(u, U) = \{v \in \mathcal{T} \mid \exists w \in NMP(u, U) : w \mid v\}$$

are called Janet-like non-multipliers *for u, whereas the elements in $M(u, U) = \mathcal{T} \setminus NM(u, U)$ are called Janet-like* multipliers *for u. A term $u \in U$ is the* Janet-like divisor *of $w \in \mathcal{T}$ if $w = uv$ with $v \in M(u, U)$. In particular, if $u \in U$ and $x_i^{k_i} \in NMP(u, U)$, we denote by $j_i(u)$ the Janet-like divisor of $ux_i^{k_i}$, if it exists.*

We remark that, though Janet-like division preserves many properties of Janet division, it is *not an involutive division* (Gerdt and Blinkov 2005a,b). The concept of *completeness* w.r.t. Janet-like division is analogous to the one of Definition 4.

Definition 7 (Gerdt and Blinkov 2005a,b). *A set $U \subset \mathcal{T}$ is called* complete *w.r.t. Janet-like division if for the sets $C_J(U) := \{uv : u \in U, v \in M(u, U)\}$ and $C(U) := \{uv : u \in U, v \in \mathcal{T}\}$ holds $C(U) = C_J(U)$. A complete set $U' \supseteq U$ such that $C(U) = C_J(U')$ is called* completion *of U.*

Proposition 8 (Gerdt and Blinkov 2005a,b). *A set $U \subset \mathcal{T}$ is* complete *w.r.t. Janet-like division if and only if for each $u \in U$, for each $p := x_i^{k_i} \in NMP(u, U)$, there is $j_i(u) \in U$.*

3. Bar Codes and Janet-like division

In this section, we see that a Bar Code can be used as a tool for studying Janet-like division, completing the treatment of Ceria (2019d), where Janet division is examined. Let us start recalling (see Ceria 2019d) that the construction of a Bar Code is a very fast preprocessing to assign to each element t of a finite set of terms $U \subset \mathcal{T}$ its multiplicative variables, according to Definition 2. Let $U \subset \mathcal{T}$ be a finite set of terms and suppose the variables ordered as $x_1 < x_2 < \dots < x_n$. We associate a Bar Code B to U and modify it as follows:

a) $\forall 1 \leq i \leq n$, we place a star symbol $*$ on the right of the bar $\mathsf{B}^{(i)}_{\mu(i)}$;
b) $\forall 1 \leq i \leq n-1$, $\forall 1 \leq j \leq \mu(i)-1$ let $\mathsf{B}^{(i)}_j$ and $\mathsf{B}^{(i)}_{j+1}$ be two consecutive bars such that they do not lay over the same $(i+1)$-bar. We place then a star symbol $*$ between them.

Proposition 9 (Ceria 2019d, Prop. 19)**.** *Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by B_U its Bar Code. For each $t \in U$, x_i, $1 \leq i \leq n$, is multiplicative for t with respect to U if and only if, in B_U, the i-bar $\mathsf{B}^{(i)}_j$, over which t lies, is followed by a star.*

Now we are ready to focus on how to study Janet-like division by means of Bar Codes.

As remarked by Gerdt and Blinkov (2005b), *every non-multiplicative power is nothing else then the power of Janet-non-multiplicative variable.* This reflects on the Bar Code associated to U, since trivially the absence of stars after some bar is equivalent to the presence of a non-multiplicative power of the corresponding variable for the terms over that bar. Moreover, Janet divisibility implies Janet-like divisibility, whereas the viceversa does not hold (see again Gerdt and Blinkov 2005b for a proof). We prove now the analogous of Proposition 9 for Janet-like division.

Proposition 10. *Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by B_U its Bar Code. Let $t \in U$, $x_i \in NM_J(t,U)$ a Janet-non-multiplicative variable, $\mathsf{B}^{(i)}_l$ the i-bar under t and t' any term over $\mathsf{B}^{(i)}_{l+1}$. Then $k_i = \deg_i(t') - \deg_i(t)$.*

Proof. Since $x_i \in NM_J(t,U)$, by Proposition 9, $\mathsf{B}^{(i)}_l$ is not followed by a star. Now since $x_i \in NM_J(t,U)$, there is a term $v \in U$ such that $\deg_j(v) = \deg_j(t)$, $i+1 \leq j \leq n$ and $\deg_i(v) > \deg_i(t)$. To find the value k_i, we should find the minimal exponent of a term with the same j-degree as t, $i+1 \leq j \leq n$, and bigger i-degree. All terms over $\mathsf{B}^{(i)}_l$ have the same ι–degree as t, $i \leq \iota \leq n$; considering $\mathsf{B}^{(i)}_{l+1}$ we have terms which have the same ι–degree as t, $i+1 \leq \iota \leq n$ (if $\mathsf{B}^{(i)}_{l+1}$ was not over the same $(i+1)$-bar as $\mathsf{B}^{(i)}_l$ we would have a star after $\mathsf{B}^{(i)}_l$). Moreover, their i-degree is bigger than $\deg_i(t)$ and it is the minimum with this property due to the Lex ordering of the terms in the Bar Code. □

Bar Codes can help us to detect completeness of a finite set of terms, as it is shown in the theorem below.

Theorem 11. *Let $U \subset \mathcal{T}$ be a finite set of terms, $\mathsf{B} := \mathsf{B}_U$ its Bar Code, $t \in U$, $p = x_i^{k_i} \in NMP(t, U)$ a non-multiplicative power and $\mathsf{B}_j^{(i)}$ the i-bar under t and $s \in U$. Then $s = j_i(t)$ if and only if the following conditions hold:*

1. *s divides pt;*
2. *s lies over $\mathsf{B}_{j+1}^{(i)}$;*
3. *$\forall l$ such that x_l divides $\frac{pt}{s}$ either there is a star after the l-bar under s or the non-multiplicative power $x_l^{k_l}$ of s w.r.t. U has greater degree than $deg_l(\frac{pt}{s})$.*

Proof. "$\Leftarrow$" In order to have $s = j_i(t)$, we need that s divides pt and no non-multipicative power of s with respect to U divides pt/s. Condition 1. ensures that s divides pt, therefore we can actually write $pt = sw$, with $w := pt/s$. Therefore, we only need to show that no non-multipicative power of s in U divides w. We first observe that x_i does not divide w. This comes from Proposition 10. Indeed s lies over $\mathsf{B}_{j+1}^{(i)}$, so $\deg_i(s) = k_i + \deg_i(t)$. For what concerns the other variables, we can deduce the assertion by Condition 3. Those variables whose corresponding bars are followed by stars are multiplicative variables. For a variable x_l that is a non-multiplicative variable, that is the corresponding bar is not followed by a star, Condition 3. says that the maximal power of x_l dividing w is smaller that the non-multiplicative power $x_l^{k_l}$ of s with respect to U, that is, $x_l^{k_l}$ does not divide w. Therefore w is a Janet-like multiplier for s and we can conclude that $s = j_i(u)$.
"$\Rightarrow$" Let $s \in U$, $s = j_i(t)$. By definition of Janet-like division s divides pt. If s was a term lying over $\mathsf{B}_j^{(i)}$, then it would be $\deg_m(s) = \deg_m(t)$ for $m = i, ..., n$, i.e. in s and t the variables $x_i, ..., x_n$ would appear with the same exponent. Then, being $s = j_i(t)$ and $\deg_i(s) = \deg_i(t)$, $x_i^{k_i}$ would divide $w := \frac{pt}{s}$.

Since $s = j_i(t)$, there cannot be non-multipliers in w, so either x_i is multiplicative for s, or the non-multiplicative power of x_i for s has degree greater than k_i. We show that both these alternatives are impossible. If x_i was multiplicative for s then there would be a star after $\mathsf{B}_j^{(i)}$, which is impossible by hypothesis, since $p = x_i^{k_i}$ is a non-multiplicative power for t and they lay over the same i-bar. It is also impossible that the non-multiplicative power of x_i for s has degree greater than k_i since $\deg_m(s) = \deg_m(t)$ for $m = i, ..., n$, and by definition of non-multiplicative power. Therefore, we have shown that it is impossible for s to lie over $\mathsf{B}_j^{(i)}$.

If s was a term lying over $\mathsf{B}_l^{(i)}$, $l > j+1$, there would exist $h \in \{i, ..., n\}$ s.t. $\deg_h(s) > \deg_h(pt)$, so s would not divide pt, which is again a contradiction.

If s was a term lying over $\mathsf{B}_l^{(i)}$, $l < j$, then it would be $s <_{Lex} t$. It cannot happen that $\deg_{l'}(s) = \deg_{l'}(t)$ for $l' = i, ..., n$ since otherwise s would have been over $\mathsf{B}_j^{(i)}$. Let $x_k := \max\{x_h,\ h = 1, ..., n | \deg_h(s) < \deg_h(t)\}$; it is clear that $k \geq i$.

Since $t \in U$ and $\deg_n(t) = \deg_n(s), ..., \deg_{k+1}(t) = \deg_{k+1}(s)$ and $\deg_k(t) > \deg_k(s)$, x_k cannot be a multiplicative variable for s. Now, let $x_k^{h_k}$ the non-multiplicative power of s w.r.t. the variable x_k. Being $\deg_n(t) = \deg_n(s), ..., \deg_{k+1}(t) = \deg_{k+1}(s)$ and $\deg_k(t) > \deg_k(s)$, $h_k \leq \deg_k(t) - \deg_k(s)$, so $\deg_k(sx_k^{h_k}) \leq \deg_k(t) \leq \deg_k(tp)$, and this is again a contradiction. Then s must lie over $\mathsf{B}_{j+1}^{(i)}$ and this proves Condition 2.

Finally, since $s = j_i(t)$, all the variables appearing with nonzero exponent in $\frac{pt}{s}$ must be either multiplicative for s or, if they are non-multiplicative, with an exponent that is smaller than that of the non-multiplicative powers of s with respect to U. This implies that Condition 3. holds, because of Propositions 9 and 10. □

Example 12. *Let us consider the set* $U = \{x_1^5, x_2x_1^2, x_2^4x_1, x_3^2x_1^2, x_3^2x_2^2x_1, x_3^5\} \subset \mathcal{T}$ *of Gerdt and Blinkov (2005b, Example 1.) and suppose*[1] $x_1 < x_2 < x_3$. *The associated Bar Code is displayed in Fig. 1, while Table 1 reports the non-multiplicative powers, the computation to get them according to Proposition 10 (where, for brevity's sake, we call* $\deg_i(\mathsf{B}_j^{(i)})$, $1 \le i \le n$, $1 \le j \le \mu(i)$, *the* i*-th degree of the terms over* $\mathsf{B}_j^{(i)}$*) and the Janet-like divisors of the terms* $tx_i^{k_i}$, $t \in U$, $x_i^{k_i} \in NMP(t, U)$, *showing the completeness of* U.

Figure 1. Bar Code equipped with stars

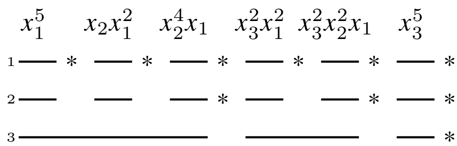


Table 1. Non-multiplicative powers and Janet-like divisors

t	$NMP(t,U)$	*Why?*	$j_1(t)$	$j_2(t)$	$j_3(t)$
x_1^5	x_2, x_3^2	$\deg_2(\mathsf{B}_2^{(2)}) - \deg_2(\mathsf{B}_1^{(2)}) = 1$ $\deg_3(\mathsf{B}_2^{(3)}) - \deg_3(\mathsf{B}_1^{(3)}) = 2$	—	$x_1^2x_2$	$x_3^2x_1^2$
$x_2x_1^2$	x_2^3, x_3^2	$\deg_2(\mathsf{B}_3^{(2)}) - \deg_2(\mathsf{B}_2^{(2)}) = 3$ $\deg_3(\mathsf{B}_2^{(3)}) - \deg_3(\mathsf{B}_1^{(3)}) = 2$	—	$x_2^4x_1$	$x_3^2x_1^2$
$x_2^4x_1$	x_3^2	$\deg_3(\mathsf{B}_2^{(3)}) - \deg_3(\mathsf{B}_1^{(3)}) = 2$	—	—	$x_3^2x_2^2x_1$
$x_3^2x_1^2$	x_2^2, x_3^3	$\deg_2(\mathsf{B}_5^{(2)}) - \deg_2(\mathsf{B}_4^{(2)}) = 2$ $\deg_3(\mathsf{B}_3^{(3)}) - \deg_3(\mathsf{B}_2^{(3)}) = 3$	—	$x_1x_2^2x_3^2$	x_3^5
$x_3^2x_2^2x_1$	x_3^3	$\deg_3(\mathsf{B}_3^{(3)}) - \deg_3(\mathsf{B}_2^{(3)}) = 3$	—	—	x_3^5
x_3^5	$\emptyset$	—	—	—	—

◊

Example 13. *Consider the set* $U = \{x_1^3x_2x_3, x_1^5x_2^2x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, *supposing fixed the lexicographical ordering induced by* $x_1 < x_2 < x_3$*; its Bar Code is displayed in Fig. 2.*

[1]In the example we exchanged the role of x_1 and x_3 to stay consistent with our notation.

Figure 2. Bar Code of U

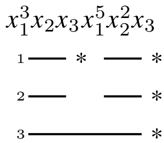
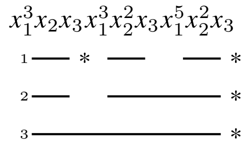


Figure 3. Bar Code of U'



The only term having non-multipicative powers is $x_1^3x_2x_3$, and in particular, its only non-multiplicative power is x_2. Now, we have $(x_1^3x_2x_3)x_2 = x_1^3x_2^2x_3$, which does not belong to the cone of any element of U and therefore U is not complete with respect to Janet-like division. We add to U the term $x_1^3x_2^2x_3$, making a step to complete U and we get $U' = \{x_1^3x_2x_3, x_1^3x_2^2x_3, x_1^5x_2^2x_3\}$, whose Bar Code is displayed in Fig. 3. The only non-multiplicative power of $x_1^3x_2x_3$ is again x_2 and $(x_1^3x_2x_3)x_2 = x_1^3x_2^2x_3 \in U'$. The term $x_1^3x_2^2x_3$ has x_1^2 as its only non-multiplicative power. Since $(x_1^3x_2^2x_3) \cdot x_1^2 = x_1^5x_2^2x_3 \in U'$, then U' is complete and so it is a completion *of U.* ◊

Remark 14. *It is possible to set a connection between Janet-like multiplicative powers and previous results on decomposition of ideals in irreducible primary components. The first result in this framework dates back to Macaulay (Macaulay 1913, 1927; Gröbner 1970; Macaulay 1994; Alonso* et al. *2006), who gave an irreducible primary decomposition of a zerodimensional ideal within a fixed coordinates' system, using, as a tool, the* corner set *also for non necessarily zerodimensional ideals. Such a result has been generalized by Alonso, Marinari and Mora, who gave the definition of* infinite corner *(Alonso* et al. *2020):*

Let I be an ideal of $\mathbf{k}[x_1, ..., x_n]$. If I is zerodimensional, its *corner set* is defined as $\mathbf{C}(\mathsf{I}) := \{\tau \in \mathbf{N}(\mathsf{I}) : \forall\, 1 \leq i \leq n, X_i\tau \in \mathbf{T}(\mathsf{I})\} \subset \mathbf{N}(\mathsf{I})$ and has the following property:

$$\omega \in \mathbf{N}(\mathsf{I}) \iff \exists \tau \in \mathbf{C}(\mathsf{I}) : \omega \mid \tau.$$

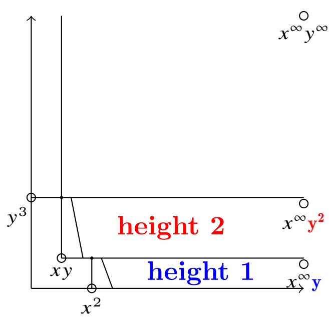
In the non zerodimensional case, the corner set can be generalized (Macaulay 1913) considering also elements $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \alpha_i \in \mathbb{N} \cup \{\infty\}$ and setting

$$\omega \mid \tau \iff \beta_i \leq \alpha_i \forall \omega = x_1^{\beta_1} \cdots x_n^{\beta_n}.$$

It is then easy to see that there is a finite set $\mathbf{C}^\infty(\mathsf{I}) \subset \{x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ \alpha_i \in \mathbb{N} \cup \{\infty\}\}$ which satisfies

$$\omega \in \mathbf{N}(\mathsf{I}) \iff \exists \tau \in \mathbf{C}^\infty(\mathsf{I}) : \omega \mid \tau.$$

Figure 4. Infinite corners



The ideas of Alonso et al. *(2020) can be interpreted in the language by Gerdt and Blinkov in the sense that* non-multiplicative powers arise from infinite corners.

The idea behind this connection is to take a generating set $U = \{t_1, ..., t_m\} \subset \mathcal{T}$ *for a monomial ideal* J *and consider it ordered in* decreasing order with respect to Lex, *so* $t_1 > t_2 > ... > t_m$. *First of all we consider the term* t_1*: all multiples of* t_1 *are in* J *and all the variables are multiplicative for* t_1 *so we say that its infinite corner is* $x_1^\infty ... x_n^\infty$. *Taken then* t_2*, we want to consider all the multiples of* t_2 *not divided by* t_1. *The infinite corner of* t_2 *with respect to* t_1*, i.e. the element* $t \in \mathcal{T}$ *such that* $\{w : w \mid t\} = \{w : t_1 \nmid wt_2\}$ *gives the non-multiplicative powers of* t_2. *In particular, the non-multiplicative powers are the finite exponents of the corresponding variables in the infinite corner, while the infinite ones represent the multipicative variables. Continuing in this fashion with* $t_3, ..., t_m$*, and defining the infinite corner of* t_i *with respect to* $\{t_1, ..., t_{i-1}\}$, $3 \leq i \leq m$*, as the element* $t \in \mathcal{T}$ *such that*

$$\{w : w \mid t\} = \{w : wt_i \notin \mathsf{T}(\{t_1, ..., t_{i-1}\})\}$$

we get all the non-multiplicative powers of the remaining terms. As a simple example, if $U = \{y^3, xy, x^2\} \subset \mathbf{k}[x, y]$*, we have that the corner of* xy *with respect to* y^3 *is* $x^\infty y^2$ *and the corner of* x^2 *with respect to* $\{y^3, xy\}$ *is* $x^\infty y$*, as shown in Fig. 4.*

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